MECA0525 : Vehicle dynamics

Pierre Duysinx
Research Center in Sustainable Automotive Technologies of University of Liege
Academic Year 2019-2020
Exercise 1:
Steady State Cornering

Understeer Gradient Computation
Exercise

- Let a vehicle A with the following characteristics:
  - Wheelbase L=2,522m
  - Position of CG w.r.t. front axle b=0,562m
  - Mass=1431 kg
  - Tires: 205/55 R16 (see Figure)
  - Radius of the turn R=110 m at speed V=80 kph

- Let a vehicle B with the following characteristics:
  - Wheelbase L=2,605m
  - Position of CG w.r.t. front axle b=1,146m
  - Mass=1510 kg
  - Tires: 205/55 R16 (see Figure)
  - Radius of the turn R=110 m at speed V=80 kph
Exercise

Rigidité de dérive (dérive <=2°) :

<table>
<thead>
<tr>
<th>Charge normale (N)</th>
<th>Rigidité (N/°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>175/70 R13</td>
</tr>
<tr>
<td>1000</td>
<td>185/70 R13</td>
</tr>
<tr>
<td>2000</td>
<td>195/60 R14</td>
</tr>
<tr>
<td>3000</td>
<td>165 R13</td>
</tr>
<tr>
<td>4000</td>
<td>205/55 R16</td>
</tr>
</tbody>
</table>
Compute:

- The Ackerman angle (in °)
- The cornering stiffness (N/°) of front and rear wheels and axles
- The sideslip angles under front and rear tires (in °)
- The side slip of the vehicle at CG (in °)
- The steering angle at front wheels (in °)
- The understeer gradient (in °/g)
- Depending on the case: the characteristic or the critical speed (in kph)
- The lateral acceleration gain (in g/°)
- The yaw speed velocity gain (in s⁻¹)
- The vehicle static margin (%)
Exercise 1

- **Data**

  \[ b = 0,562 \text{ m} \]
  \[ \frac{b}{L} = \frac{0,562}{2,522} = 0,2228 \]
  \[ c = L - b = 2,522 - 0,562 = 1,960 \text{ m} \]
  \[ \frac{c}{L} = \frac{1,960}{2,522} = 0,7772 \]

  \[ m = 1431 \text{ kg} \]

  \[ W_f = mg \frac{c}{L} = 10,909,8714 \text{ N} \]
  \[ W_r = mg \frac{b}{L} = 3,127,6909 \text{ N} \]

  \[ V = 80 \text{ km/h} = 22,2222 \text{ m/s} \]
  \[ R = 110 \text{ m} \]

  \[ a_y = \frac{V^2}{R} = \frac{22,222^2}{110} = 4,4893 \text{ m/s}^2 \]
Exercise 1

- **Ackerman angle**
  \[ \delta = \arctan \frac{L}{R} = \arctan \frac{2,522}{110} = 0,0229 \text{ rad} = 1,3134^\circ \]

- **Tire cornering stiffness of front wheels**
  \[ W_f = mg \frac{c}{L} = 10.909,8714 \text{ N} \quad \Rightarrow \quad C_{\alpha f}^{(1)} = 1550 \text{ N/deg} \]
  \[ F_{zf}^{(1)} = 5.454,93 \text{ N} \quad \Rightarrow \quad C_{\alpha f} = 2C_{\alpha f}^{(1)} = 3110 \text{ N/deg} = 177.616 \text{ N/\text{rad}} \]

- **Tire cornering stiffness of rear wheels**
  \[ W_r = mg \frac{b}{L} = 3.127,6909 \text{ N} \quad \Rightarrow \quad C_{\alpha r}^{(1)} = 500 \text{ N/deg} \]
  \[ F_{zr}^{(1)} = 1.563,84 \text{ N} \quad \Rightarrow \quad C_{\alpha r} = 2C_{\alpha r}^{(1)} = 1000 \text{ N/deg} = 57.295,8 \text{ N/\text{rad}} \]
Exercise

Rigidité de dérive (dérive ≤2°):

Charge normale (N)

Rigidité (N/°)

175/70 R13
185/70 R13
195/60 R14
165 R13
205/55 R16
Exercise 1

- Side slip angles under the front tires

\[ a_y = \frac{V^2}{R} = 4,4893 \text{ m/s}^2 \quad ma_y = 6.424,1883 \text{ N} \]

\[ F_{yf} = m \frac{V^2}{R} \frac{c}{L} = 4.992,879 \text{ N} \]

\[ C_{\alpha f} \alpha_f = F_{yf} \quad C_{\alpha f} = 3110 \text{ N/deg} \]

\[ \alpha_f = \frac{F_{yf}}{C_{\alpha f}} = \frac{4.492,879}{3110} = 1,6106^\circ = 0,00281 \text{ rad} \]
Exercise 1

- Side slip angles under the rear tires

\[ F_{yr} = m \frac{V^2 b}{R L} = 1431,3092 \text{ N} \]

\[ C_{\alpha r} \alpha_r = F_{yr} \quad C_{\alpha r} = 1000 \text{ N/deg} \]

\[ \alpha_r = \frac{F_{yr}}{C_{\alpha r}} = \frac{1431,3092}{1000} = 1,4313^\circ = 0,00250 \text{ rad} \]

- Side slip angle at CG

\[ \beta = \frac{c r}{V} - \alpha_r = \frac{c}{R} - \alpha_r \]

\[ \beta = \frac{1,960}{110} - 0,0250 = -0,0072 \text{ rad} = -0,4105^\circ \]
Exercise 1

- Steering angle at front wheels

\[ \delta = \frac{L}{R} + \left( \frac{mc/L}{C_{\alpha f}} - \frac{mb/L}{C_{\alpha r}} \right) \]

\[ \delta = \frac{L}{R} + \alpha_f - \alpha_r \]

\[ \delta = 1,3134^\circ + 1,6106^\circ - 1,4313^\circ = 1,4927^\circ \]

- Understeer gradient

\[ K = \left( \frac{mc/L}{C_{\alpha f}} - \frac{mb/L}{C_{\alpha r}} \right) = \frac{1112,1732}{3110} - \frac{318,8268}{1000} = 0,0399^\circ/\text{m/s}^{-2} \]

\[ K' = K \cdot g = 0,3918^\circ/g \]
Exercise 1

- Understeer gradient: check!

\[ \delta = \frac{L}{R} \frac{180}{\pi} + K \frac{V^2}{R} \quad K = \left( \frac{mc/L}{C_{\alpha f}} - \frac{mb/L}{C_{\alpha r}} \right) = 0.0399^\circ /ms^{-2} \]

\[ \delta = 1.3134^\circ + 0.0399 \times 4.4893 = 1.4925^\circ \]

- Characteristic speed

\[ \delta = 2 \frac{L}{R} \quad V_{\text{Charac}} = \sqrt{\frac{L}{K}} \]

\[ K = 0.0399^\circ /ms^{-2} = 6.9639 \times 10^{-4} \text{ rad/m/s}^2 \]

\[ V_{\text{Charac}} = \sqrt{\frac{2.522}{6.9639 \times 10^{-4} - 4}} = 60.1793 \text{ m/s} = 216.64 \text{ km/h} \]
Exercise 1

- Lateral acceleration gain

\[ G_{a_y} = \frac{a_y}{\delta} = \frac{4,4893/9,81}{1,4927}\, 0,3066 \, g/deg \]

- Yaw speed gain

\[ r = \frac{V}{r} = 0,2020 \, rad/s = 0,2020 \frac{\pi}{180} = 11,5749 \, deg/s \]

\[ G_r = \frac{r}{\delta} = \frac{11,5749}{1,4927} = 7,7543 \, deg/s/deg \]
Exercise 1

- Neutral maneuver point

\[ e = \frac{bC_{\alpha f} - cC_{\alpha r}}{C_{\alpha f} + C_{\alpha r}} = \frac{0.562 \cdot 3100 - 1.960 \cdot 1000}{3100 + 1000} = -0.0531 \text{ m} \]

- Static margin

\[ \frac{e}{L} = -2.11\% \]
Exercise 2: Steady State Cornering

Developing analytical models: Baby kart
Baby kart

- Dr Watson bought a baby carriage with 3 wheels for their baby. The three baby carriage wheels are the same. Front wheel is equipped with a hand brake. All wheels are fixed. None of them can be steered.

- The wheelbase is $L=1$ meter and the gross weight is 20 kg. The center of gravity is located 0.3 m in front of the rear axle. The handlebar is located 0.2 m behind the rear wheels.
Baby kart

- It is asked to solve the following questions:
  - Is the baby carriage understeer, oversteer or neutral? Justify your answer. Where is located the neutral steer point.
  - Complete the modelling sketch that is provided in Figure 2. Write the important parameters that are necessary to carry out the vehicle steady state cornering study a given constant speed $V$.
  - Revisiting the procedure developed in the lectures to investigate steady state cornering, derive the expression of the external yaw moment $M$ to be developed around vertical axis by acting on the steering bar to follow a turn at speed $V$.
  - Describe at least two methods to reduce the yaw steering moment calculated at point 3. We only consider actions that are compatible with a regular usage of the baby carriage. No structural or hardware modifications are allowed.
Baby kart

\[ M_z = F \cdot d \]
Oversteer / neutral / oversteer?

- The question can be solved in different ways. One natural approach is to consider the neutral point position of the babykart.

- The static margin writes

\[ e = \frac{b C_{\alpha f} - c C_{\alpha r}}{C'_{\alpha f} + C'_{\alpha r}} \]

- A vehicle is
  - Neutral steer if \( e = 0 \)
  - Under steer (\( K > 0 \)) if \( e < 0 \) (behind the CG)
  - Over steer (\( K < 0 \)) if \( e > 0 \) (in front of the CG)
Oversteer / neutral / oversteer?

- Neutral point does not call for any consideration of steering system but is basically founded on the steering capacity of front and rear wheels.

\[
e = \frac{b \, C_{\alpha_f} - c \, C_{\alpha_r}}{C_{\alpha_f} + C_{\alpha_r}}
\]

- Front axle cornering stiffness

\[
C_{\alpha_f} = 1 \times C_{\alpha}^{(1)} \quad b = L - c = 1 - 0.3 = 0.7 \, m
\]

- Rear axle cornering stiffness

\[
C_{\alpha_r} = 2 \times C_{\alpha}^{(1)} \quad c = 0.3 \, m
\]
Oversteer / neutral / oversteer?

- The neutral point position is

\[ e = \frac{b \, C_{\alpha f} - c \, C_{\alpha r}}{C_{\alpha f} + C_{\alpha r}} \]

\[ = \frac{0.7 \, C_{\alpha}^{(1)} - 0.3 \, 2 \, C_{\alpha}^{(1)}}{3 \, C_{\alpha}^{(1)}} \]

\[ = \frac{0.7 - 0.6}{3} = \frac{1}{30} > 0 \]

- The baby kart is oversteer!
Oversteer / neutral / oversteer?

- Alternatively, one could have noticed that the understeer gradient does not depend on the existence of any steering system. So one can evaluate the understeer gradient of the baby kart

\[ K = \frac{m c}{C_{\alpha f} L} - \frac{m b}{C_{\alpha r} L} \]

- It comes

\[ K = \frac{m c}{C_{\alpha f} L} - \frac{m b}{C_{\alpha r} L} = \frac{m}{L} \left( \frac{0.3}{C_{\alpha}^{(1)}} - \frac{0.7}{2 C_{\alpha}^{(1)}} \right) \]

\[ = \frac{m}{L C_{\alpha}^{(1)}} \left( \frac{0.3 \cdot 2 - 0.7}{2} \right) = \frac{m}{L C_{\alpha}^{(1)}} (-0.05) < 0 \]

- And the vehicle is understeer
Bicycle model of the baby kart

Side slip $\alpha_f$ angle becomes positive because of positive lateral velocity. So front lateral forces are pointing to the left of the vehicle.

Side slip $\alpha_r$ angle is supposed to be negative ($\alpha<0$) to counteract the centrifugal forces.

Velocity vector is perpendicular to the position vector to center of turn $\vec{v} = \vec{\omega} \times \vec{IP}$

Steering moment about vertical axis $M_z = F.d$ created by the force exerted on the handlebar.
Equation of the baby kart

- The bicycle model will be revisited. One has to adapt the three sets of equations
  - Equilibrium equations
  - Behavior equations
  - Compatibility equations

- Then they can be combined to predict the steering moment to follow a circular motion with a given radius and at a given speed.
Equilibrium equations of the vehicle

- Equilibrium equations in lateral direction and rotation about z axis

\[-F_{yf} + F_{yr} = m \frac{V^2}{R} \cos \beta\]
\[-F_{yf} b - F_{yr} c + M_z = 0\]

\[\beta \ll 1\]
\[\cos \beta \simeq 1\]

- Solutions (see next slide)

\[F_{yf} = \frac{M}{L} - m \frac{V^2}{R} \frac{c}{L}\]
\[F_{yr} = \frac{M}{L} + m \frac{V^2}{R} \frac{b}{L}\]

The lateral forces are in the same ratio as the vertical forces under the wheel sets.
Equilibrium equations of the vehicle

- Solving

\[- F_{yf} + F_{yr} = m \frac{V^2}{R}\]
\[F_{yf} \cdot b + F_{yr} \cdot c = M_z\]

- Can be made by using Horner rule

\[F_{yf} = \frac{m V^2 / R \quad 1}{M \quad c} = \frac{c m V^2 / R - M}{-c - b} = \frac{M}{L} - m \frac{V^2}{R} \frac{c}{L}\]
\[F_{yr} = \frac{-1 \quad m V^2 / R}{b \quad M} = \frac{-M - b m V^2 / R}{-c - b} = \frac{M}{L} + m \frac{V^2}{R} \frac{b}{L}\]
Behaviour equations of the tires

- Cornering force for small slip angles

\[ F_y = C_\alpha \alpha \]

\[ C_\alpha = -\left. \frac{\partial F_y}{\partial \alpha} \right|_{\alpha=0} > 0 \]

\[ C_\alpha = \sum_{i \in \text{axle}} C_{\alpha i} \]

Gillespie, Fig. 6.2
Compatibility equations

- Compatibility equation consists in evaluating the side slip angles in terms of the velocities

Because of assumption $\alpha_r < 0$!

$$\tan \alpha_r = \frac{-v_r}{u_r}$$

Because of assumption $\alpha_f > 0$

$$\tan \alpha_f = \frac{v_f}{u_f}$$
Compatibility equations

The velocity under the rear wheels are given by

\[ u_r = u \approx V \]
\[ v_r = v - c r \]

The compatibility of the velocities yields the slip angle under the rear wheels

\[ \tan \alpha_r = \frac{-v_r}{u_r} = \frac{r + c r}{V} \]
\[ V = r R \]
\[ \alpha_r = -\beta + \frac{c}{R} \]
Compatibility equations

- The velocity under the front wheels are given by

\[ u_f = u \simeq V \]
\[ v_f = v + b r \]

- The compatibility of the velocities yields the slip angle under the front wheels

\[ \tan \alpha_f = \frac{v_f}{u_f} = \frac{v + b r}{V} \]
\[ V = r R \]
\[ \alpha_f = \beta + \frac{b}{R} \]
Steering angle as a function of the slip angles under front and rear wheels

\[ \alpha_f = \beta + \frac{b}{R} \]
\[ \alpha_r = -\beta + \frac{c}{R} \]

\[ \alpha_f + \alpha_r = 0 + \frac{b + c}{R} \]

This gives relation between the steering angles and the

\[ \frac{L}{R} = \alpha_f + \alpha_r \]

Ackerman angle

Correction due to side slip
Steering angle

- Steering angle as a function of the slip angles under front and rear wheels
  \[ \frac{L}{R} = \alpha_f + \alpha_r \]

- Let’s insert the expression of the side slip angles in terms of lateral forces and of the cornering stiffness

\[
\alpha_f = \frac{F_{yf}}{C_{\alpha f}}
\]
\[
\alpha_r = \frac{F_{yr}}{C_{\alpha r}}
\]
\[
\alpha_f = \frac{M}{L} - m \frac{V^2}{R} \frac{c}{L}
\]
\[
= \frac{M/L}{C_{\alpha f}} - \frac{m c/L}{C_{\alpha f}} \frac{V^2}{R}
\]
\[
\alpha_r = \frac{M}{L} + m \frac{V^2}{R} \frac{b}{L}
\]
\[
= \frac{M/L}{C_{\alpha r}} + \frac{m b/L}{C_{\alpha r}} \frac{V^2}{R}
\]
Steering angle

- It comes
\[
\frac{L}{R} = \alpha_f + \alpha_r
\]
\[
= \frac{M/L}{C_{\alpha_f}} - \frac{m c/L}{C_{\alpha_f}} \frac{V^2}{R} + \frac{M/L}{C_{\alpha_r}} + \frac{m b/L}{C_{\alpha_r}} \frac{V^2}{R}
\]
\[
= \frac{M/L}{C_{\alpha_f}} + \frac{M/L}{C_{\alpha_r}} - \frac{m c/L}{C_{\alpha_f}} \frac{V^2}{R} + \frac{m b/L}{C_{\alpha_r}} \frac{V^2}{R}
\]

- We finally end up with a similar expression to the steering system where the command M is related to Ackerman angle and to the understeer gradient
\[
\frac{M}{L} \left( \frac{1}{C_{\alpha_f}} + \frac{1}{C_{\alpha_r}} \right) = \frac{L}{R} + \left( \frac{m c}{C_{\alpha_f} L} - \frac{m b}{C_{\alpha_r} L} \right) \frac{V^2}{R}
\]
Understeer gradient

- The steering angle is expressed in terms of the centrifugal acceleration

\[ M \frac{C_{\alpha f} + C_{\alpha r}}{L C_{\alpha f} C_{\alpha r}} = \frac{L}{R} + \left( \frac{mc}{C_{\alpha f} L} - \frac{mb}{C_{\alpha r} L} \right) \frac{V^2}{R} \]

- If we define an effective cornering stiffness \( C_\alpha^* \) that results from the serial layout of the tyres

\[ \frac{1}{C_\alpha^*} = \frac{1}{C_{\alpha f}} + \frac{1}{C_{\alpha r}} \]

- We have

\[ M \frac{1}{C_\alpha^* L} = \frac{L}{R} + \left( \frac{mc}{C_{\alpha f} L} - \frac{mb}{C_{\alpha r} L} \right) \frac{V^2}{R} \]
Understeer gradient

- The steering torque is expressed in terms of the centrifugal acceleration

\[ M = (C_\alpha^* L) \left( \frac{L}{R} + \left( \frac{m c}{C_{\alpha f} L} - \frac{m b}{C_{\alpha r} L} \right) \frac{V^2}{R} \right) \]

- So

\[ M = (C_\alpha^* L) \left( \frac{L}{R} + K \frac{V^2}{R} \right) \]

\[ M = (C_\alpha^* L) \left( \frac{L}{R} + K' \frac{V^2}{g R} \right) \]

- With the understeer gradient \( K \) of the vehicle

\[ K = \frac{m c}{C_{\alpha f} L} - \frac{m b}{C_{\alpha r} L} \]

\[ K' = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} \]

- Which is the same as for the steered wheel vehicle because it is a characteristic of the vehicle and not of the steering/command system!
Neutralsteer, understeer and oversteer vehicles

- If $K=0$, the vehicle is said to be **neutralsteer**:
  \[ K = 0 \Leftrightarrow c \ C_{\alpha_r} = b \ C_{\alpha_f} \quad \alpha_f = \alpha_r \]
  The front and rear wheels sets have the same directional ability

- If $K>0$, the vehicle is **understeer**:
  \[ K > 0 \Leftrightarrow c \ C_{\alpha_r} > b \ C_{\alpha_f} \quad \alpha_f > \alpha_r \]
  Larger directional factor of the rear wheels

- If $K<0$, the vehicle is **oversteer**:
  \[ K < 0 \Leftrightarrow c \ C_{\alpha_r} < b \ C_{\alpha_f} \quad \alpha_f < \alpha_r \]
  Larger directional factor of the front wheels
Steering angle as a function of $V$

Modification of the steering torque as a function of the speed

Characteristic and critical speeds

\[ M = (C^*_{\alpha} L) \left[ \frac{L}{R} + K \frac{V^2}{R} \right] \]
Characteristic and critical speeds

For an understeer vehicle, the understeer level may be quantified by a parameter known as the characteristic speed. It is the speed that requires a steering angle that is twice the Ackerman angle (turn at V=0)

\[ M = 2C^*_\alpha L L/R \quad V_{\text{characteristic}} = \sqrt{\frac{L}{K}} \]

For an oversteer vehicle, there is a critical speed above which the vehicle will be unstable

\[ M = 0 \quad V_{\text{critical}} = \sqrt{\frac{L}{|K|}} \]

\[ M = (C^*_\alpha L) \left[ \frac{L}{R} + K \frac{V^2}{R} \right] = 0 \quad \Leftrightarrow \quad V^2 = -\frac{L}{K} \]
Methods to reduce the steering torque

- Can we suggest methods to reduce the steering torque size?

\[ M = (C^*_{\alpha} L) \left[ \frac{L}{R} + K \frac{V^2}{R} \right] \]

- Method 1: reduce the wheelbase \( L \)

- Method 2: reduce the \( C^*_{\alpha} \)

\[ \frac{1}{C^*_{\alpha}} = \frac{1}{C^*_{\alpha f}} + \frac{1}{C^*_{\alpha r}} \quad C^*_{\alpha} = \frac{C^*_{\alpha f} + C^*_{\alpha r}}{C^*_{\alpha f} C^*_{\alpha r}} \]

- For our babykart

\[ C^*_{\alpha} = \frac{C^*_{\alpha f} + C^*_{\alpha r}}{C^*_{\alpha f} C^*_{\alpha r}} = \frac{C^*_{\alpha} + 2 C^*_{\alpha}}{C^*_{\alpha} 2 C^*_{\alpha}} = \frac{3}{2} C^*_{\alpha} \]
Methods to reduce the steering torque

- **Method 2:** to reduce the $C_{\alpha}^*$

$$C_{\alpha}^* = \frac{C_{\alpha f} + C_{\alpha r}}{C_{\alpha f} C_{\alpha r}} = \frac{C_{\alpha} + 2 C_{\alpha}}{C_{\alpha} 2 C_{\alpha}} = \frac{3}{2} C_{\alpha}$$

which suggests to take tires with a lower $C_{\alpha}$

- To reduce the $C_{\alpha}^*$ as it is the serial arrangement of cornering stiffnesses, we can also take very low $C_{\alpha f}$ or very low $C_{\alpha r}$

$$\frac{1}{C_{\alpha}^*} = \frac{1}{C_{\alpha f}} + \frac{1}{C_{\alpha r}}$$
Methods to reduce the steering torque

- If we are not allowed to modify the structure of the kart

\[ M = (C^*_\alpha L) \left( \frac{L}{R} + K \frac{V^2}{R} \right) \]
Methods to reduce the steering torque

- If we are not allowed to modify the structure of the kart

\[ M = (C^*_\alpha L) \left[ \frac{L}{R} + K \frac{V^2}{R} \right] \]

- Method 1: Reinforce the oversteer character of the vehicle. For a given speed this will reduce the steering torque magnitude

\[ K' = \frac{W_f}{C_{\alpha_f}} - \frac{W_r}{C_{\alpha_r}} \]

- Parents can push downward on the handlebar to transfer the weight from front to rear axles

\[ W_f \downarrow \quad W_r \uparrow \implies K' \ll 0 \]
Methods to reduce the steering torque

- If we are not allowed to modify the structure of the kart

\[ M = (C^*_\alpha L) \left[ \frac{L}{R} + K \frac{V^2}{R} \right] \]

- Method 2: Parents can use the brake to create a large braking force on the rear wheels. Due to the friction ellipsis, the cornering coefficients will be reduced thus leading to an increase of the oversteer character.

\[ F_{Brr} \quad \Rightarrow \quad C_{\alpha r} \downarrow \]

\[ \Rightarrow \quad K' = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} \downarrow \]
REMARK

- One could have derived all equations by assuming a different sign of the side slip angles.
In the alternative assumption, one would get the following results if being fully consistent with his/her assumptions.

Equilibrium

\[ F_{yf} + F_{yr} = m \frac{V^2}{R} \]
\[ F_{yf} b - F_{yr} c = -M \]
\[ F_{yf} = -\frac{M}{L} + m \frac{V^2}{R} \frac{c}{L} \]
\[ F_{yr} = \frac{M}{L} + m \frac{V^2}{R} \frac{b}{L} \]
REMARK

- Constitutive / behavior equations

\[ F_y = C_\alpha \alpha \quad \text{and} \quad C_\alpha = \sum_{i \in \text{axle}} C_{\alpha i} \]

- Compatibility equations

\[ \tan \alpha_r = \frac{-v_r}{u_r} = \frac{-v + c r}{V} \]

\[ \alpha_r = -\beta + \frac{c}{R} \]

\[ \tan \alpha_f = \frac{-v_f}{u_f} = \frac{-v + b r}{V} \]

\[ \alpha_f = -\beta - \frac{b}{R} \]
Deriving the steering torque relation

\[
\begin{align*}
\alpha_f &= -\beta - \frac{b}{R} \\
\alpha_r &= -\beta + \frac{c}{R} \\
\hline
\alpha_f - \alpha_r &= 0 - \frac{b + c}{R}
\end{align*}
\]

\[
\frac{L}{R} + \alpha_f - \alpha_r = 0
\]
REMARK

- Inserting the equilibrium and behavior equations

\[
\frac{L}{R} + \alpha_f - \alpha_r = \frac{L}{R} - \frac{-M/L}{C_{\alpha_f}} + \frac{m c/L}{C_{\alpha_f}} \frac{V^2}{R} - \frac{M/L}{C_{\alpha_r}} - \frac{m b/L}{C_{\alpha_r}} \frac{V^2}{R}
\]

\[
= \frac{L}{R} - \frac{M}{L} \left( \frac{1}{C_{\alpha_f}} + \frac{1}{C_{\alpha_r}} \right) + \left( \frac{m c/L}{C_{\alpha_f}} - \frac{m b/L}{C_{\alpha_r}} \right) \frac{V^2}{R}
\]

- This yields

\[
\frac{M}{L} \left( \frac{1}{C_{\alpha_f}} + \frac{1}{C_{\alpha_r}} \right) = \frac{L}{R} + \left( \frac{m c}{C_{\alpha_f} L} - \frac{m b}{C_{\alpha_r} L} \right) \frac{V^2}{R}
\]

\[
M = (C^*_{\alpha} L) \left[ \frac{L}{R} + K \frac{V^2}{R} \right]
\]