MECA0525 : Vehicle dynamics

Pierre Duysinx

Research Center in Sustainable Automotive Technologies of University of Liege

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Exercise 1:
Steady State Cornering

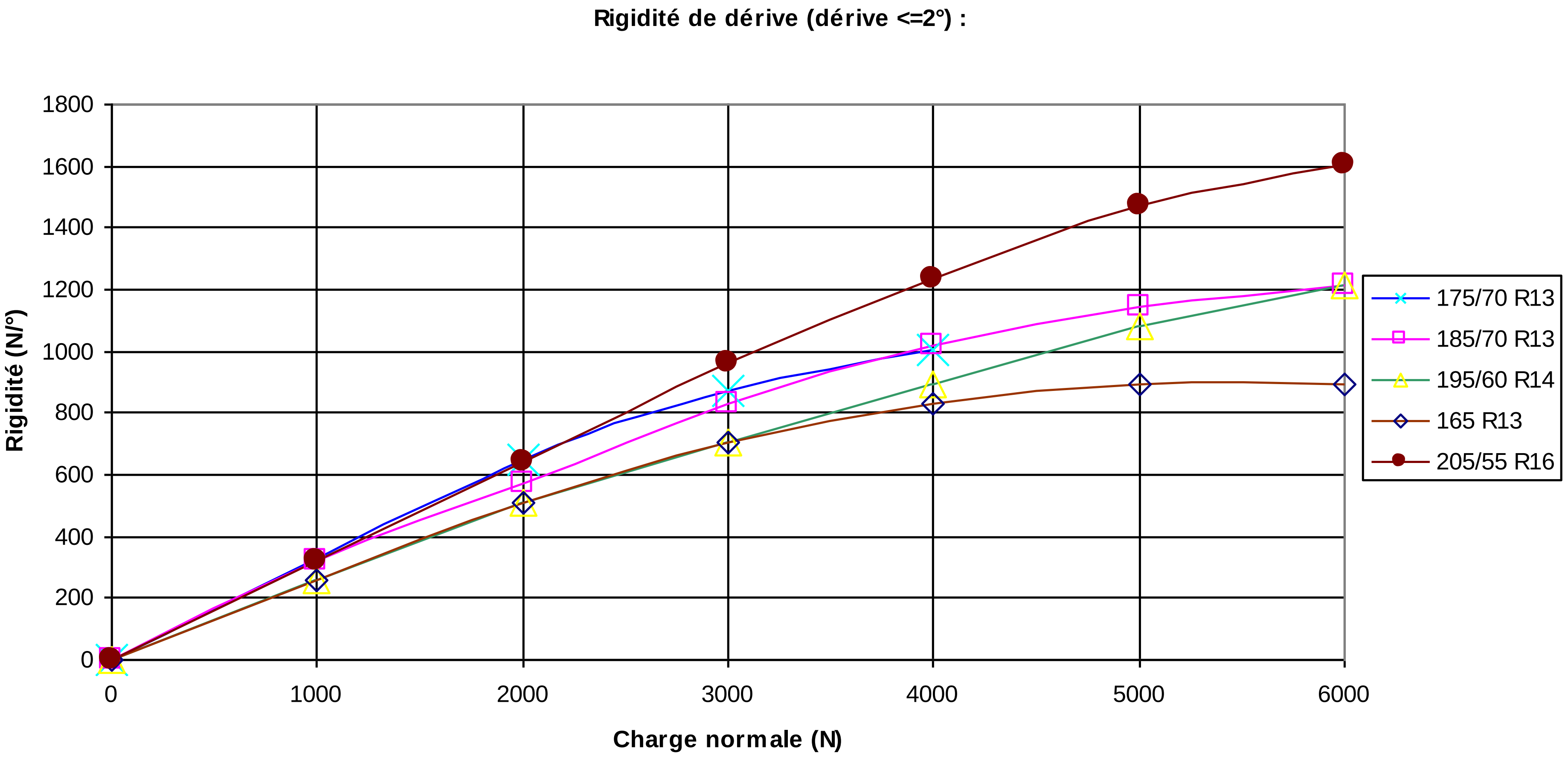
Understeer Gradient Computation

Exercise

- Let a vehicle A with the following characteristics:
  - Wheelbase L=2,522m
  - Position of CG w.r.t. front axle b=0,562m
  - Mass=1431 kg
  - Tires: 205/55 R16 (see Figure)
  - Radius of the turn R=110 m at speed V=80 kph

- Let a vehicle B with the following characteristics :
  - Wheelbase L=2,605m
  - Position of CG w.r.t. front axle b=1,146m
  - Mass=1510 kg
  - Tires: 205/55 R16 (see Figure)
  - Radius of the turn R=110 m at speed V=80 kph

Exercise



Rigidité de dérive (dérive <=2°) :
Rigidité (N/°)
1800
1600
1400
1200
1000
800
600
400
200
0
0
1000
2000
3000
4000
5000
6000
Charge normale (N)
175/70 R13
185/70 R13
195/60 R14
165 R13
205/55 R16

Exercise

- Compute:
  - The Ackerman angle (in °)
  - The cornering stiffness (N/°) of front and rear wheels and axles
  - The sideslip angles under front and rear tires (in °)
  - The side slip of the vehicle at CG (in °)
  - The steering angle at front wheels (in °)
  - The understeer gradient (in °/g)
  - Depending on the case: the characteristic or the critical speed (in kph)
  - The lateral acceleration gain (in g/°)
  - The yaw speed velocity gain (in s^{-1})
  - The vehicle static margin (%)

Exercise 1

- Data

$$b = 0,562\ m$$

$$\frac{b}{L} = \frac{0,562}{2,522} = 0,2228$$

$$c = L - b = 2,522 - 0,562 = 1,960\ m$$

$$\frac{c}{L} = \frac{1,960}{2,522} = 0,7772$$

$$m = 1431\ kg$$

$$W_f = mg\frac{c}{L} = 10.909,8714\ N$$

$$W_r = mg\frac{b}{L} = 3.127,6909\ N$$

$$V = 80\ km/h = 22,2222\ m/s$$

$$R = 110\ m$$

$$a_y = \frac{V^2}{R} = \frac{22,2222^2}{110} = 4,4893\ m/s^2$$

Exercise 1

- Ackerman angle

$$\delta = \arctan\frac{L}{R} = \arctan\frac{2,522}{110} = 0,0229\ rad = 1,3134^\circ$$

- Tire cornering stiffness of front wheels

$$W_f = mg\frac{c}{L} = 10.909,8714\ N \longrightarrow C_{\alpha f}^{(1)} = 1550\ N/deg$$

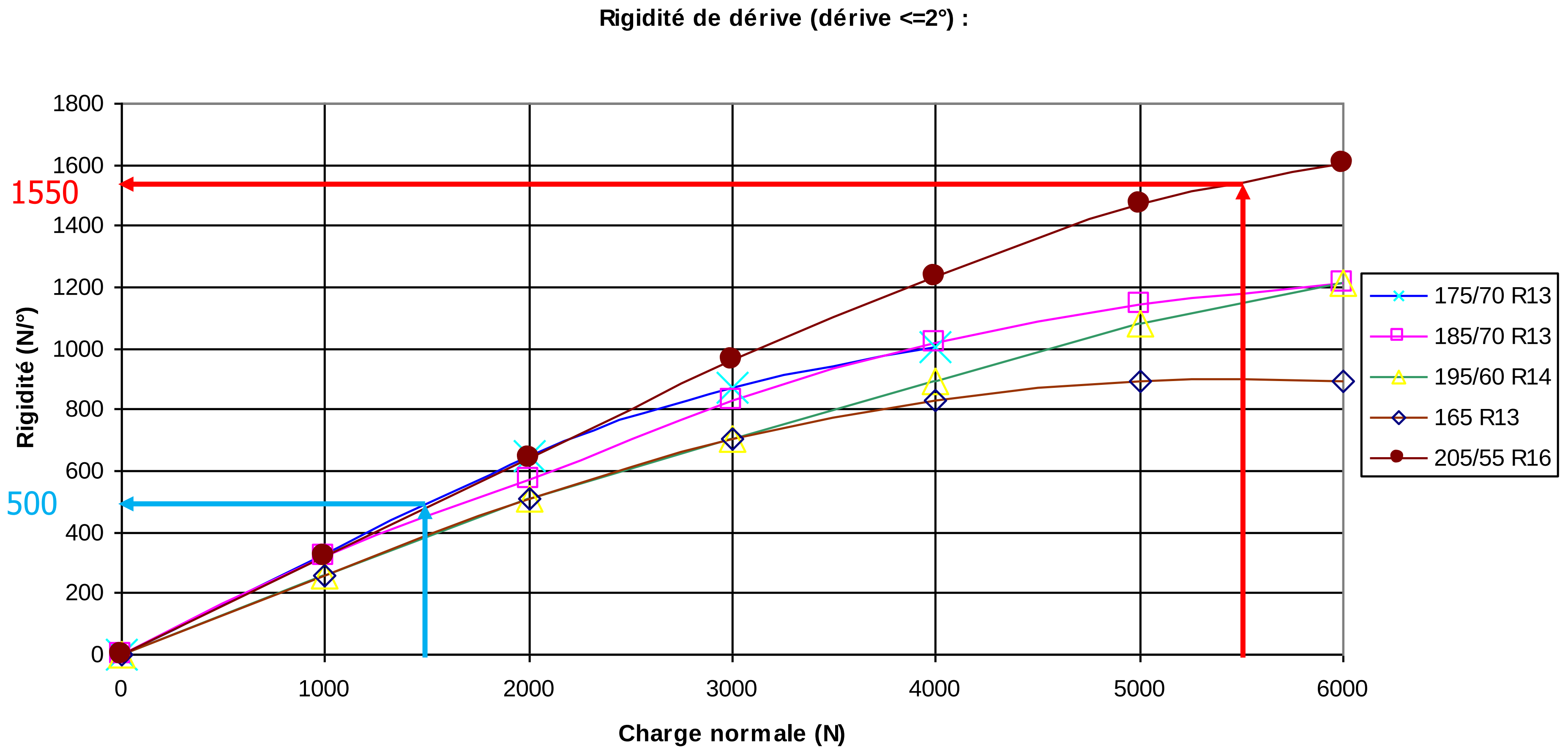
$$F_{zf}^{(1)} = 5.454,93\ N \qquad C_{\alpha f} = 2C_{\alpha f}^{(1)} = 3110\ N/deg = 177.616\ N/rad$$

- Tire cornering stiffness of rear wheels

$$W_r = mg\frac{b}{L} = 3.127,6909\ N \longrightarrow C_{\alpha r}^{(1)} = 500\ N/deg$$

$$F_{zr}^{(1)} = 1.563,84\ N \qquad C_{\alpha r} = 2C_{\alpha r}^{(1)} = 1000\ N/deg = 57.295,8\ N/rad$$

Exercise


Rigidité de dérive (dérive <=2°) :
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Charge normale (N)
175/70 R13
185/70 R13
195/60 R14
165 R13
205/55 R16

Exercise 1

- Side slip angles under the front tires

$$a_y = \frac{V^2}{R} = 4,4893\ m/s^2 \qquad\qquad ma_y = 6.424,1883\ N$$

$$F_{yf} = m\frac{V^2}{R}\frac{c}{L} = 4.992,879\ N$$

$$C_{\alpha f}\alpha_f = F_{yf} \qquad\qquad C_{\alpha f} = 3110\ N/deg$$

$$\alpha_f = \frac{F_{yf}}{C_{\alpha f}} = \frac{4.492,879}{3110} = 1,6106^\circ = 0,00281\ rad$$

Exercise 1

- Side slip angles under the rear tires

$$F_{yr} = m\frac{V^2}{R}\frac{b}{L} = 1431,3092\ N$$

$$C_{\alpha r}\alpha_r = F_{yr} \qquad\qquad C_{\alpha r} = 1000\ N/deg$$

$$\alpha_r = \frac{F_{yr}}{C_{\alpha r}} = \frac{1431,3092}{1000} = 1,4313^\circ = 0,00250\ rad$$

- Side slip angle at CG

$$\beta = \frac{cr}{V} - \alpha_r = \frac{c}{R} - \alpha_r$$

$$\beta = \frac{1,960}{110} - 0,0250 = -0,0072\ rad = -0,4105^\circ$$

Exercise 1

- Steering angle at front wheels

$$\delta = \frac{L}{R} + \left(\frac{mc/L}{C_{\alpha f}} - \frac{mb/L}{C_{\alpha r}} \right) \frac{V^2}{R} \qquad \delta = \frac{L}{R} + \alpha_f - \alpha_r$$

$$\delta = 1,3134^\circ + 1,6106^\circ - 1,4313^\circ = 1,4927^\circ$$

- Understeer gradient

$$K = \left(\frac{mc/L}{C_{\alpha f}} - \frac{mb/L}{C_{\alpha r}} \right) = \frac{1112,1732}{3110} - \frac{318,8268}{1000} = 0,0399^\circ/ms^{-2}$$

$$K' = K \cdot g = 0,3918\ {}^\circ/g$$

Exercise 1

- Understeer gradient: check!

$$\delta = \frac{L}{R}\frac{180}{\pi} + K\frac{V^2}{R} \qquad K = \left(\frac{mc/L}{C_{\alpha f}} - \frac{mb/L}{C_{\alpha r}}\right) = 0,0399^\circ/ms^{-2}$$

$$\delta = 1,3134^\circ + 0,0399 \cdot 4,4893 = 1,4925^\circ$$

- Characteristic speed

$$\delta = 2\frac{L}{R} \qquad V_{\text{Charac}} = \sqrt{\frac{L}{K}}$$

$$K = 0,0399^\circ/ms^{-2} = 6,9639E{-}4\ rad/m/s^2$$

$$V_{\text{Charac}} = \sqrt{\frac{2,522}{6,9639E{-}4}} = 60,1793\ m/s = 216,64\ km/h$$

Exercise 1

- Lateral acceleration gain

$$G_{a_y} = \frac{a_y}{\delta} = \frac{4,4893/9,81}{1,4927°} = 0,3066\ g/deg$$

- Yaw speed gain

$$r = \frac{V}{R} = 0,2020\ rad/s = 0,2020\frac{\pi}{180} = 11,5749\ deg/s$$

$$G_r = \frac{r}{\delta} = \frac{11,5749}{1,4927} = 7,7543\ deg/s/deg$$

Exercise 1

- Neutral maneuver point

$$e = \frac{bC_{\alpha f} - cC_{\alpha r}}{C_{\alpha f} + C_{\alpha r}} = \frac{0,562 \cdot 3100 - 1,960 \cdot 1000}{3100 + 1000} = -0,0531\ m$$

- Static margin

$$\frac{e}{L} = -2,11\%$$

Exercise 2: Steady State Cornering

Developing analytical models:
Baby kart

Baby kart

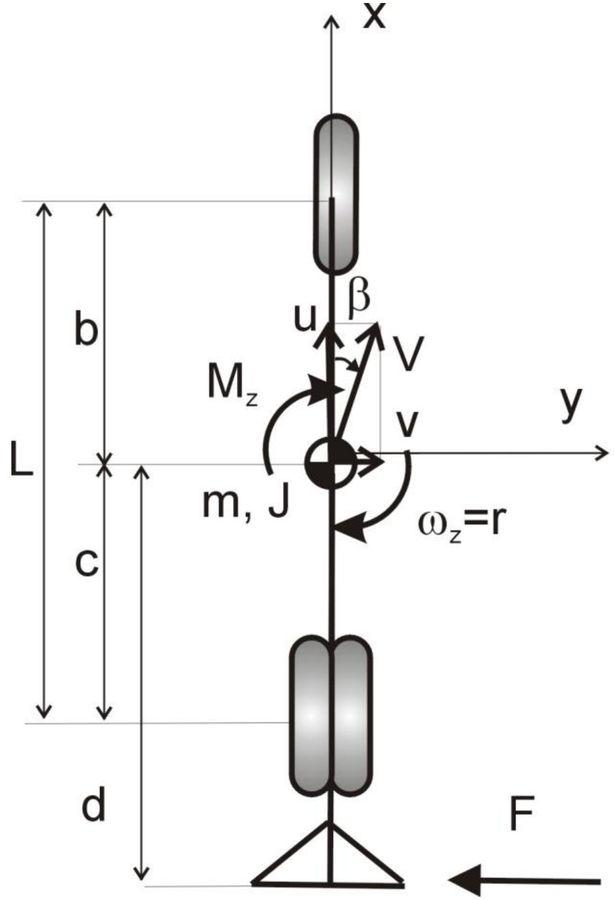
- Dr Watson bought a baby carriage with 3 wheels for their baby. The three baby carriage wheels are the same. Front wheel is equipped with a hand brake. All wheels are fixed. None of them can be steered.

- The wheelbase is L=1 meter and the gross weight is 20 kg. The center of gravity is located 0.3 m in front of the rear axle. The handlebar is located 0.2 m behind the rear wheels.

Baby kart

- It is asked to solve the following questions:
  - Is the baby carriage understeer, oversteer or neutral? Justify your answer. Where is located the neutral steer point.
  - Complete the modelling sketch that is provided in Figure 2. Write the important parameters that are necessary to carry out the vehicle steady state cornering study a given constant speed V.
  - Revisiting the procedure developed in the lectures to investigate steady state cornering, derive the expression of the external yaw moment M to be developed around vertical axis by acting on the steering bar to follow a turn at speed V.
  - Describe at least two methods to reduce the yaw steering moment calculated at point 3. We only consider actions that are compatible with a regular usage of the baby carriage. No structural or hardware modifications are allowed.

Baby kart



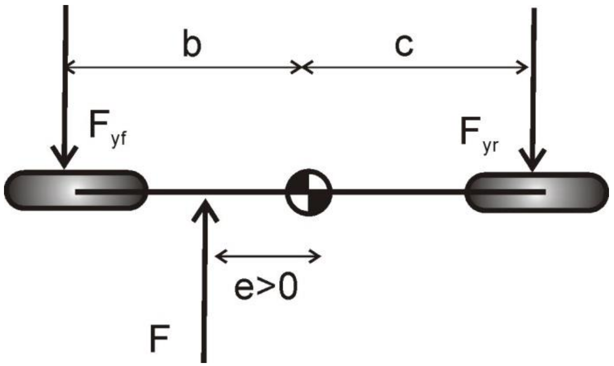
$$M_z = F \cdot d$$

Oversteer / neutral /oversteer?

- The question can be solved in different ways. One natural approach is to consider the neutral point position of the babykart.

- The static margin writes

$$\boxed{e = \frac{b\,C_{\alpha_f} - c\,C_{\alpha_r}}{C_{\alpha_f} + C_{\alpha_r}}}$$



- A vehicle is
  - Neutral steer if e = 0
  - Under steer (K>0) if e<0 (behind the CG)
  - Over steer (K<0) if e>0 (in front of the CG)

Oversteer / neutral /oversteer?

- Neutral point does not call for any consideration of steering system but is basically founded on the steering capacity of front and rear wheels.

$$\boxed{e = \frac{b\,C_{\alpha_f} - c\,C_{\alpha_r}}{C_{\alpha_f} + C_{\alpha_r}}}$$

- Front axle cornering stiffness

$$C_{\alpha f} = 1 \times C_{\alpha}^{(1)} \qquad b = L - c = 1 \; - \; 0.3 = 0.7\ m$$

- Rear axle cornering stiffness

$$C_{\alpha r} = 2 \times C_{\alpha}^{(1)} \qquad c = 0.3\ m$$

Oversteer / neutral /oversteer?

- The neutral point position is

$$e = \frac{b\,C_{\alpha_f} - c\,C_{\alpha_r}}{C_{\alpha_f} + C_{\alpha_r}}$$

$$= \frac{0.7\,C_{\alpha}^{(1)} - 0.3\,2\,C_{\alpha}^{(1)}}{3\,C_{\alpha}^{(1)}} = \frac{0.7 - 0.6}{3} = \frac{1}{30} > 0$$

- The baby kart is oversteer!

Oversteer / neutral /oversteer?

- Alternatively, one could have noticed that the understeer gradient does not depend on the existence of any steering system. So one can evaluate the understeer gradient of the baby kart

$$\boxed{K = \frac{m\,c}{C_{\alpha f}\,L} - \frac{m\,b}{C_{\alpha r}\,L}}$$

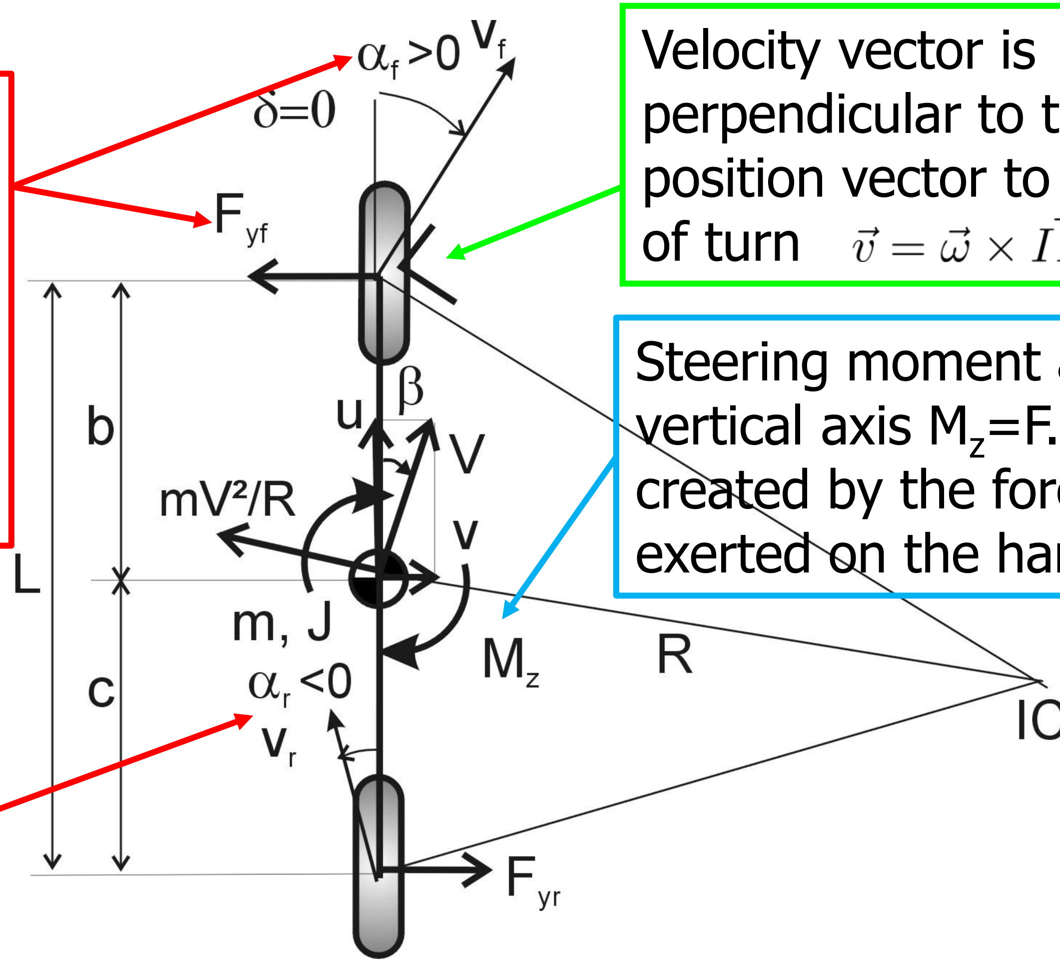
- It comes

$$\begin{aligned} K &= \frac{m\,c}{C_{\alpha f}\,L} - \frac{m\,b}{C_{\alpha r}\,L} = \frac{m}{L}\left(\frac{0.3}{C_\alpha^{(1)}} - \frac{0.7}{2\,C_\alpha^{(1)}}\right) \\ &= \frac{m}{L\,C_\alpha^{(1)}}\left(\frac{0.3 \cdot 2 - 0.7}{2}\right) = \frac{m}{L\,C_\alpha^{(1)}}(-0.05) < 0 \end{aligned}$$

- And the vehicle oversteer

Bicycle model of the baby kart

Side slip α_f angle becomes positive because of positive lateral velocity. So front lateral forces are pointing to the left of the vehicle



Velocity vector is perpendicular to the position vector to center of turn $\vec{v} = \vec{\omega} \times \vec{IP}$

Steering moment about vertical axis M_z=F.d created by the force exerted on the handlebar

Side slip α_r angle is supposed to be negative ($\alpha<0$) to counteract the centrifugal forces

Equation of the baby kart

- The bicycle model will be revisited. One has to adapt the three sets of equations
  - Equilibrium equations
  - Behavior equations
  - Compatibility equations

- Then they can be combined to predict the steering moment to follow a circular motion with a given radius and at a given speed.

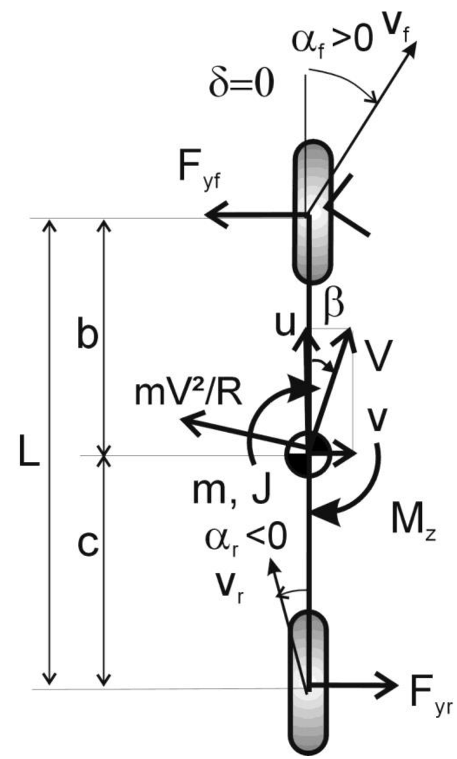
Equilibrium equations of the vehicle

- Equilibrium equations in lateral direction and rotation about z axis

$$- F_{yf} + F_{yr} = m\frac{V^2}{R} cos\beta \qquad - F_{yf} + F_{yr} = m\frac{V^2}{R}$$

$$- F_{yf}\, b - F_{yr}\, c + M_z = 0 \qquad F_{yf}\, b + F_{yr}\, c = M_z$$

$$\beta \ll 1 \qquad \cos\beta \simeq 1$$



- Solutions (see next slide)

$$F_{yf} = \frac{M}{L} - m\,\frac{V^2}{R}\,\frac{c}{L}$$

$$F_{yr} = \frac{M}{L} + m\,\frac{V^2}{R}\,\frac{b}{L}$$

The lateral forces are in the same ratio as the vertical forces under the wheel sets

Equilibrium equations of the vehicle

- Solving

$$-F_{yf} + F_{yr} = m\frac{V^2}{R}$$

$$F_{yf}\,b + F_{yr}\,c = M_z$$

- Can be made by using Cramer's rule

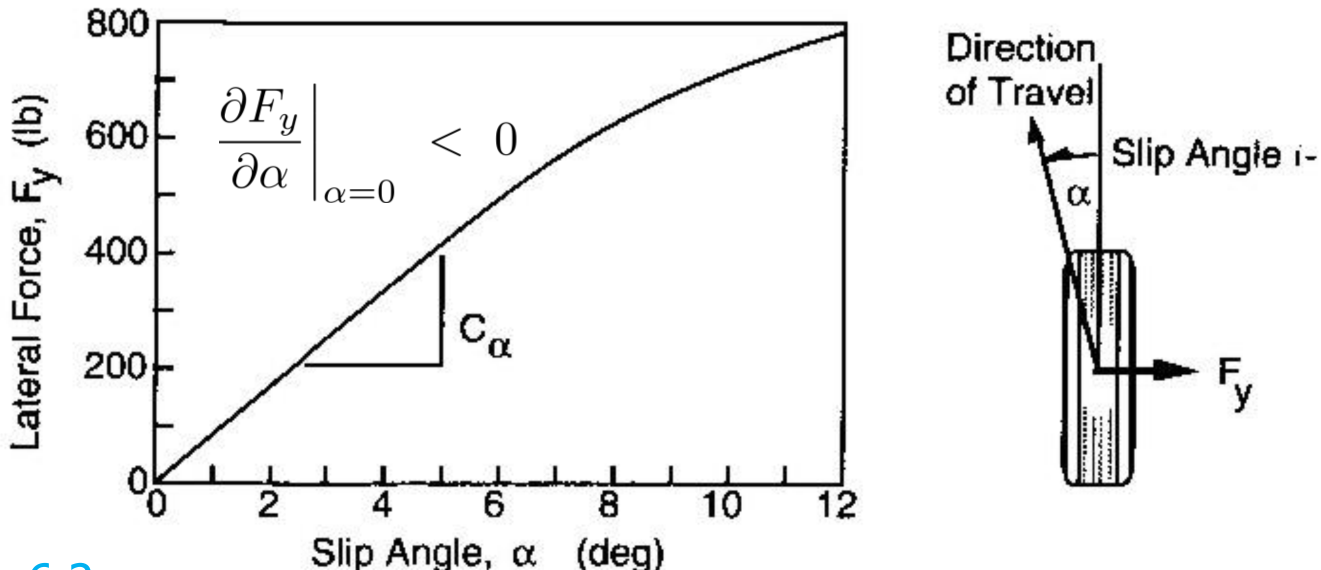
$$F_{yf} = \frac{\begin{vmatrix} mV^2/R & 1 \\ M & c \end{vmatrix}}{\begin{vmatrix} -1 & 1 \\ b & c \end{vmatrix}} = \frac{c\,mV^2/R - M}{-c-b} = \frac{M}{L} - m\,\frac{V^2}{R}\,\frac{c}{L}$$

$$F_{yr} = \frac{\begin{vmatrix} -1 & mV^2/R \\ b & M \end{vmatrix}}{\begin{vmatrix} -1 & 1 \\ b & c \end{vmatrix}} = \frac{-M - b\,mV^2/R}{-c-b} = \frac{M}{L} + m\,\frac{V^2}{R}\,\frac{b}{L}$$

Behaviour equations of the tires

- Cornering force for small slip angles

$$F_y \;=\; C_\alpha\,\alpha \qquad C_\alpha = -\left.\frac{\partial F_y}{\partial \alpha}\right|_{\alpha=0} \;>\; 0 \qquad C_\alpha = \sum_{i\in \mathrm{axle}} C_{\alpha i}$$



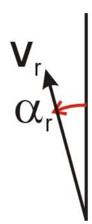
Gillespie, Fig. 6.2

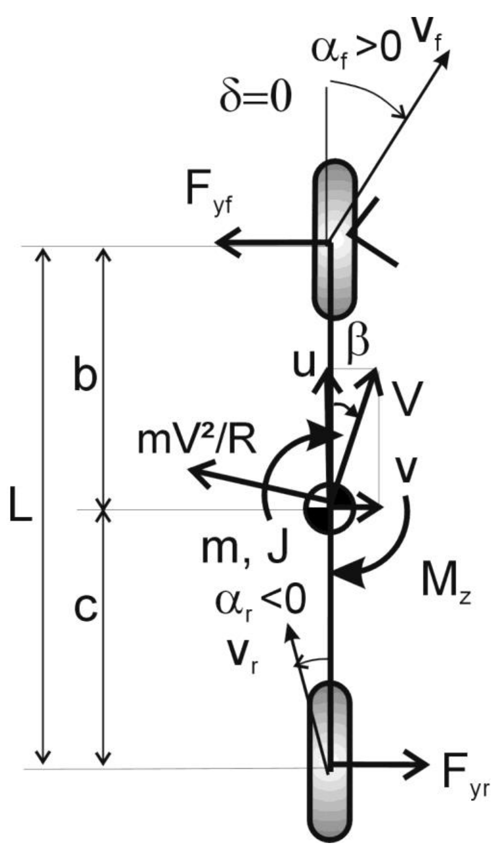
Compatibility equations

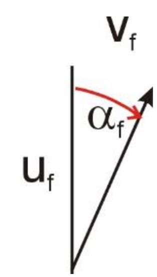
- Compatibility equation consists in evaluating the side slip angles in terms of the velocities

Because of assumption α_r<0!

$$\tan \alpha_r = \frac{-v_r}{u_r}$$







$$\tan \alpha_f = \frac{v_f}{u_f}$$

Because of assumption α_f>0

Compatibility equations

- The velocity under the rear wheels are given by

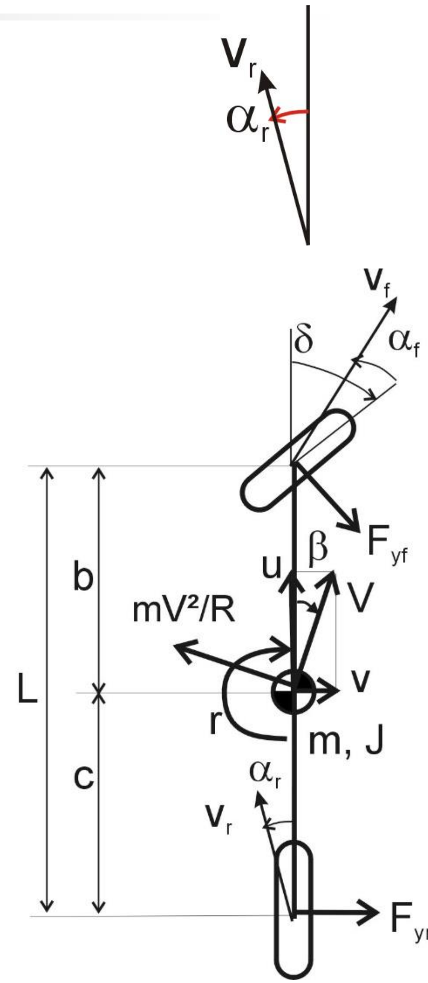
$$u_r = u \simeq V$$
$$v_r = v - c\,r$$

- The compatibility of the velocities yields the slip angle under the rear wheels

$$\tan\alpha_r = \frac{-v_r}{u_r} = \frac{-v + c\,r}{V}$$

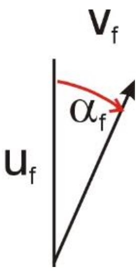
$$V = r\,R$$

$$\boxed{\alpha_r = -\beta + \frac{c}{R}}$$



Compatibility equations

- The velocity under the front wheels are given by



$$u_f = u \simeq V$$
$$v_f = v + b\, r$$

- The compatibility of the velocities yields the slip angle under the front wheels

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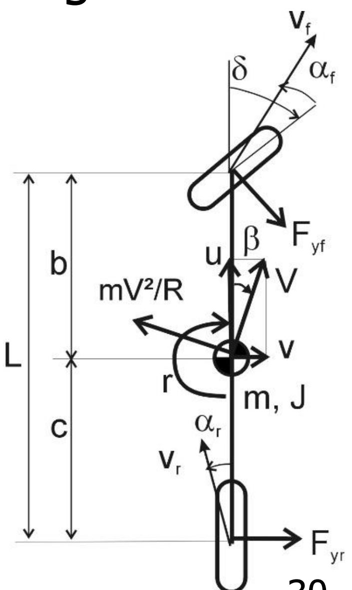
$$\boxed{\alpha_f = \beta + \frac{b}{R}}$$



Steering angle

- Steering angle as a function of the slip angles under front and rear wheels

$$\alpha_f = \beta + \frac{b}{R}$$
$$\alpha_r = -\beta + \frac{c}{R}$$

$$\alpha_f + \alpha_r = 0 \ + \ \frac{b+c}{R}$$

- This gives relation between the steering angles and the

$$\boxed{\frac{L}{R} = \underbrace{\alpha_f + \alpha_r}}$$

Ackerman angle

Correction due to side slip

Steering angle

- Steering angle as a function of the slip angles under front and rear wheels

$$\frac{L}{R} = \alpha_f + \alpha_r$$

- Let's insert the expression of the side slip angles in terms of lateral forces and of the cornering stiffness

$$\alpha_f = \frac{F_{yf}}{C_{\alpha f}} \qquad \alpha_r = \frac{F_{yr}}{C_{\alpha r}}$$

$$\begin{aligned}\alpha_f &= \frac{\frac{M}{L} - m\,\frac{V^2}{R}\,\frac{c}{L}}{C_{\alpha f}} \\ &= \frac{M/L}{C_{\alpha f}} - \frac{m\,c/L}{C_{\alpha f}}\,\frac{V^2}{R}\end{aligned} \qquad \begin{aligned}\alpha_r &= \frac{\frac{M}{L} + m\,\frac{V^2}{R}\,\frac{b}{L}}{C_{\alpha r}} \\ &= \frac{M/L}{C_{\alpha r}} + \frac{m\,b/L}{C_{\alpha r}}\,\frac{V^2}{R}\end{aligned}$$

Steering angle

- It comes

$$
\begin{aligned}
\frac{L}{R} &= \alpha_f + \alpha_r \\
&= \frac{M/L}{C_{\alpha f}} - \frac{m\,c/L}{C_{\alpha f}} \frac{V^2}{R} + \frac{M/L}{C_{\alpha r}} + \frac{m\,b/L}{C_{\alpha r}} \frac{V^2}{R} \\
&= \frac{M/L}{C_{\alpha f}} + \frac{M/L}{C_{\alpha r}} - \frac{m\,c/L}{C_{\alpha f}} \frac{V^2}{R} + \frac{m\,b/L}{C_{\alpha r}} \frac{V^2}{R}
\end{aligned}
$$

- We finally end up with a similar expression to the steering system where the command M is related to Ackerman angle and to the understeer gradient

$$
\frac{M}{L}\left(\frac{1}{C_{\alpha f}} + \frac{1}{C_{\alpha r}}\right) = \frac{L}{R} + \left(\frac{m\,c}{C_{\alpha f}\,L} - \frac{m\,b}{C_{\alpha r}\,L}\right)\frac{V^2}{R}
$$

Understeer gradient

- The steering angle is expressed in terms of the centrifugal acceleration

$$M \frac{C_{\alpha f} + C_{\alpha r}}{L\, C_{\alpha f}\, C_{\alpha r}} = \frac{L}{R} + \left(\frac{m\, c}{C_{\alpha f}\, L} - \frac{m\, b}{C_{\alpha r}\, L}\right) \frac{V^2}{R}$$

- If we define an *effective cornering stiffness* $C_\alpha{}^*$ that results from the serial layout of the tyres

$$\frac{1}{C_\alpha^*} = \frac{1}{C_{\alpha f}} + \frac{1}{C_{\alpha r}}$$

- We have

$$M \frac{1}{C_\alpha^*\, L} = \frac{L}{R} + \left(\frac{m\, c}{C_{\alpha f}\, L} - \frac{m\, b}{C_{\alpha r}\, L}\right) \frac{V^2}{R}$$

Understeer gradient

- The steering torque is expressed in terms of the centrifugal acceleration

$$M = (C_\alpha^* L)\left[\frac{L}{R} + (\frac{m\,c}{C_{\alpha f}\,L} - \frac{m\,b}{C_{\alpha r}\,L})\frac{V^2}{R}\right]$$

- So

$$\boxed{M = (C_\alpha^* L)\left[\frac{L}{R} + K\,\frac{V^2}{R}\right]} \qquad \boxed{M = (C_\alpha^* L)\left[\frac{L}{R} + K'\,\frac{V^2}{g\,R}\right]}$$

- With the understeer gradient K of the vehicle

$$\boxed{K = \frac{m\,c}{C_{\alpha f}\,L} - \frac{m\,b}{C_{\alpha r}\,L}} \qquad \boxed{K' = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}}$$

- Which is the same as for the steered wheel vehicle because it is a characteristic of the vehicle and not of the steering/command system!

Neutralsteer, understeer and oversteer vehicles

- If K=0, the vehicle is said to be *neutralsteer*:

$$K = 0 \quad \Leftrightarrow \quad c\,C_{\alpha r} = b\,C_{\alpha f} \qquad \alpha_f = \alpha_r$$

  The front and rear wheels sets have the same directional ability

- If K>0, the vehicle is *understeer* :

$$K > 0 \quad \Leftrightarrow \quad c\,C_{\alpha r} > b\,C_{\alpha f} \qquad \alpha_f > \alpha_r$$

  Larger directional factor of the rear wheels

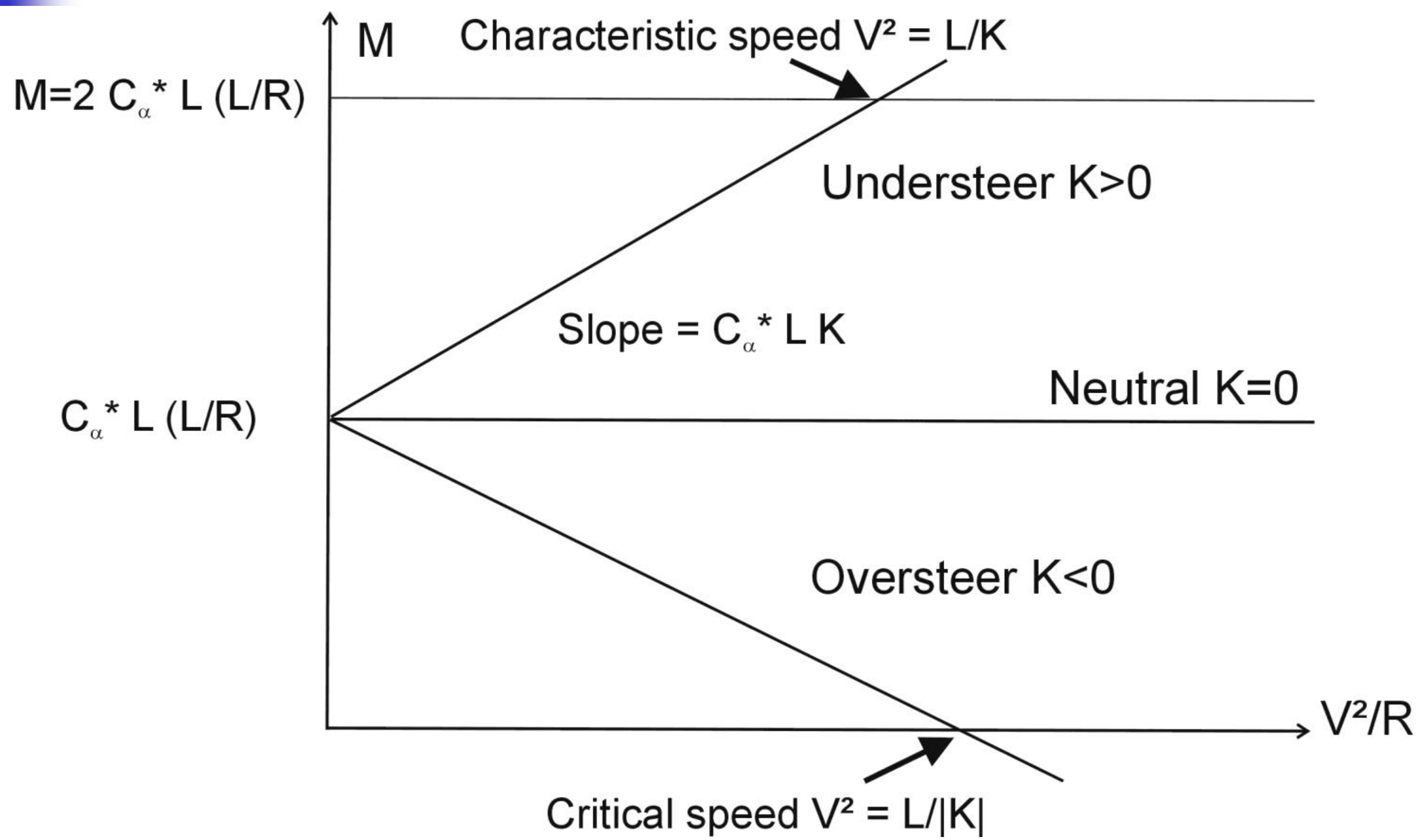
- If K<0, the vehicle is *oversteer*:

$$K < 0 \quad \Leftrightarrow \quad c\,C_{\alpha r} < b\,C_{\alpha f} \qquad \alpha_f < \alpha_r$$

  Larger directional factor of the front wheels

$$M = (C^*_\alpha L) \left[\frac{L}{R} + K \frac{V^2}{R} \right]$$

Steering angle as a function of V



Modification of the steering torque as a function of the speed
Characteristic and critical speeds

Characteristic and critical speeds

- For an understeer vehicle, the understeer level may be quantified by a parameter known as the characteristic speed. It is the speed that requires a steering angle that is twice the Ackerman angle (turn at V=0)

$$M = 2C^*_\alpha \, L \, L/R \qquad V_{\text{characteristic}} = \sqrt{\frac{L}{K}}$$

- For an oversteer vehicle, there is a critical speed above which the vehicle will be unstable

$$M = 0 \qquad V_{\text{critical}} = \sqrt{\frac{L}{|K|}}$$

$$M = (C^*_\alpha \, L) \left[\frac{L}{R} + K \frac{V^2}{R} \right] = 0 \quad \Leftrightarrow \quad V^2 = -\frac{L}{K}$$

Methods to reduce the steering torque

- Can we suggest methods to reduce the steering torque size?

$$M = (C_\alpha^* L)\left[\frac{L}{R} + K\frac{V^2}{R}\right]$$

- Method 1: reduce the wheelbase L
- Method 2: reduce the C_α*

$$\frac{1}{C_\alpha^*} = \frac{1}{C_{\alpha f}} + \frac{1}{C_{\alpha r}} \qquad C_\alpha^* = \frac{C_{\alpha f} + C_{\alpha r}}{C_{\alpha f}\, C_{\alpha r}}$$

  - For our babykart

$$C_\alpha^* = \frac{C_{\alpha f} + C_{\alpha r}}{C_{\alpha f}\, C_{\alpha r}} = \frac{C_\alpha + 2\, C_\alpha}{C_\alpha\, 2\, C_\alpha} = \frac{3}{2} C_\alpha$$

Methods to reduce the steering torque

- Method 2: to reduce the C_α*

$$C_\alpha^* = \frac{C_{\alpha f} + C_{\alpha r}}{C_{\alpha f}\, C_{\alpha r}} = \frac{C_\alpha + 2\, C_\alpha}{C_\alpha\, 2\, C_\alpha} = \frac{3}{2} C_\alpha$$

  which suggests taking tires with a lower C_α

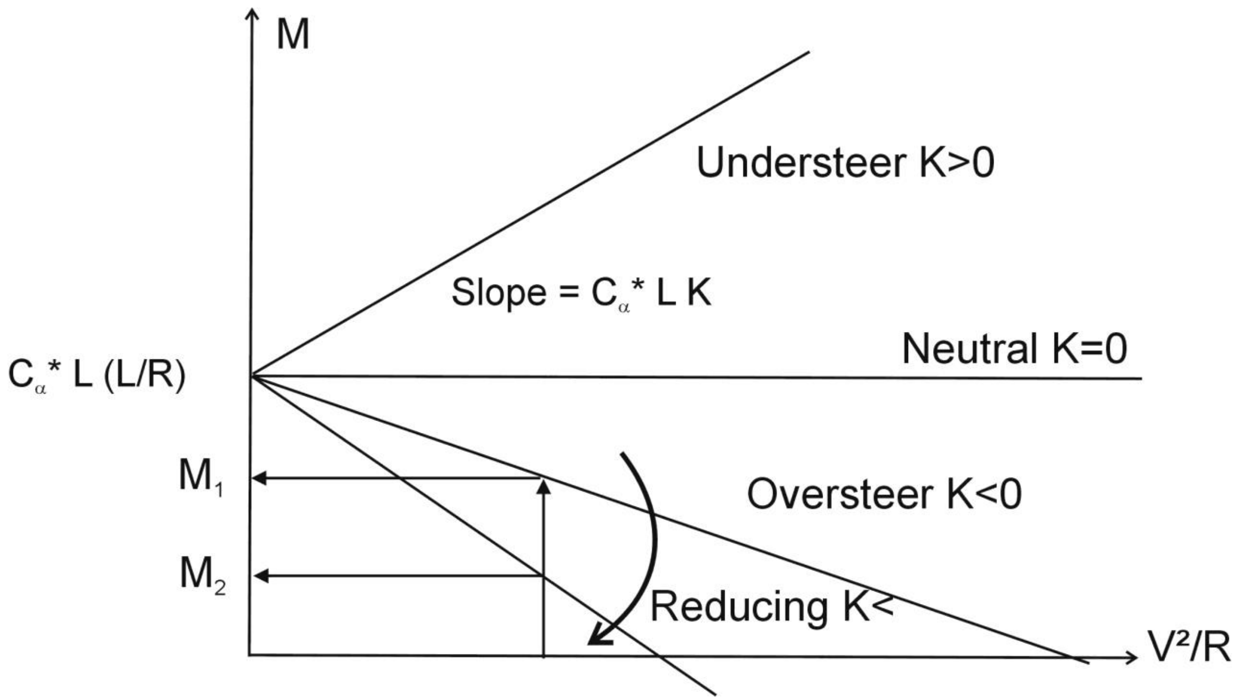
- To reduce the C_α* as it is the serial arrangement of cornering stiffnesses, we can also take very low $C_{\alpha f}$ or very low $C_{\alpha r}$

$$\frac{1}{C_\alpha^*} = \frac{1}{C_{\alpha f}} + \frac{1}{C_{\alpha r}}$$

Methods to reduce the steering torque

- If we are not allowed to modify the structure of the kart

$$M = (C_\alpha^* L)\left[\frac{L}{R} + K\frac{V^2}{R}\right]$$



Methods to reduce the steering torque

- If we are not allowed to modify the structure of the kart

$$M = (C_\alpha^* L)\left[\frac{L}{R} + K\frac{V^2}{R}\right]$$

  - Method 1: Reinforce the oversteer character of the vehicle. For a given speed this will reduce the steering torque magnitude

$$K' = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}$$

  - Parents can push downward on the handlebar to transfer the weight from front to rear axles

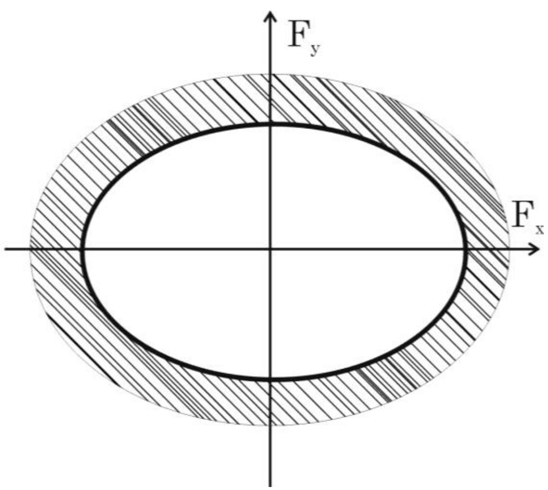
$$W_f \searrow \quad W_r \nearrow \quad \Longrightarrow \quad K' \ll 0$$

Methods to reduce the steering torque

- If we are not allowed to modify the structure of the kart

$$M = (C_\alpha^* L)\left[\frac{L}{R} + K\frac{V^2}{R}\right]$$

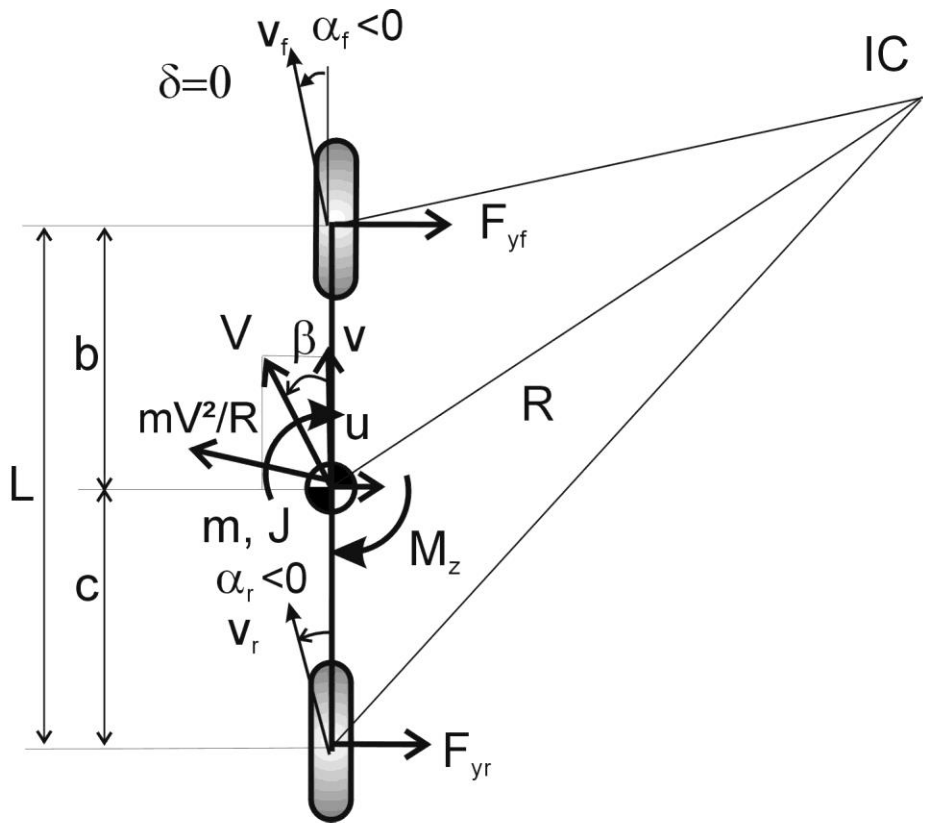
  - Method 2: Parents can use the brake to create a large braking force on the rear wheels. Due to the friction ellipsis, the cornering coefficients will be reduced thus leading to an increase of the oversteer character



$$F_{Brr} \nearrow \implies C_{\alpha r} \searrow$$
$$\implies K' = \frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} \searrow$$

REMARK

- One could have derived all equations by assuming a different sign of the side slip angles



REMARK

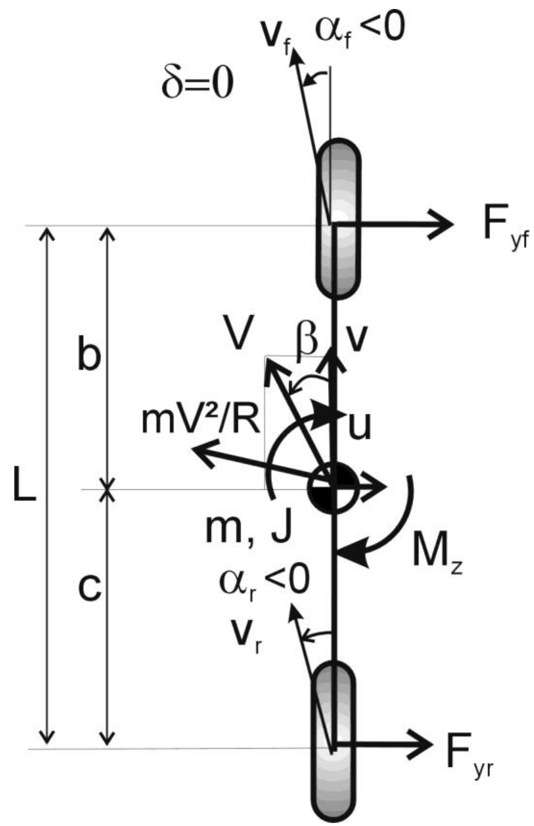
- In the alternative assumption, one would get the following results if being fully consistent with his/her assumptions

- Equilibrium

$$F_{yf} + F_{yr} = m\frac{V^2}{R}$$
$$F_{yf}\, b - F_{yr}\, c = -M$$

$$F_{yf} = -\frac{M}{L} + m\,\frac{V^2}{R}\,\frac{c}{L}$$

$$F_{yr} = \frac{M}{L} + m\,\frac{V^2}{R}\,\frac{b}{L}$$



REMARK

- Constitutive / behavior equations

$$F_y \;=\; C_\alpha\,\alpha \qquad\qquad C_\alpha = \sum_{i\in \text{axle}} C_{\alpha i}$$

- Compatibility equations

$$\tan\alpha_r = \frac{-v_r}{u_r} = \frac{-v + c\,r}{V} \qquad\qquad \boxed{\alpha_r = -\beta + \frac{c}{R}}$$

$$\tan\alpha_f = -\frac{v_f}{u_f} = -\frac{v + b\,r}{V} \qquad\qquad \boxed{\alpha_f = -\beta - \frac{b}{R}}$$

REMARK

- Deriving the steering torque relation

$$\alpha_f = -\beta - \frac{b}{R}$$
$$\alpha_r = -\beta + \frac{c}{R}$$

$$\alpha_f - \alpha_r = 0 \ - \ \frac{b+c}{R}$$

$$\boxed{\frac{L}{R} + \alpha_f - \alpha_r = 0}$$

REMARK

- Inserting the equilibrium and behavior equations

$$\frac{L}{R} + \alpha_f - \alpha_r = \frac{L}{R} + \frac{-M/L}{C_{\alpha f}} + \frac{m\,c/L}{C_{\alpha f}}\,\frac{V^2}{R} - \frac{M/L}{C_{\alpha r}} - \frac{m\,b/L}{C_{\alpha r}}\,\frac{V^2}{R}$$

$$= \frac{L}{R} - \frac{M}{L}\left(\frac{1}{C_{\alpha f}} + \frac{1}{C_{\alpha r}}\right) + \left(\frac{m\,c/L}{C_{\alpha f}} - \frac{m\,b/L}{C_{\alpha r}}\right)\frac{V^2}{R}$$

- This yields

$$\frac{M}{L}\left(\frac{1}{C_{\alpha f}} + \frac{1}{C_{\alpha r}}\right) = \frac{L}{R} + \left(\frac{m\,c}{C_{\alpha f}\,L} - \frac{m\,b}{C_{\alpha r}\,L}\right)\frac{V^2}{R}$$

$$\boxed{M = (C_\alpha^*\,L)\left[\frac{L}{R} + K\,\frac{V^2}{R}\right]}$$