MECA0525 VEHICLE DYNAMICS

Pierre Duysinx

Research Center in Sustainable Automotive
Technologies of University of Liege

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Lesson 3:
Dynamic vehicle stability

References

- G. Sander « Véhicules Automobiles», Lecture notes, 1983, Université de Liège
- G. Genta. « Motor Vehicle Dynamics: Modeling and Simulation ». World Scientific. 1997.
- J.R. Ellis. Vehicle Dynamics. London Business Book Limited. 1969

Outline

- Bicycle or single track model
- Equations of dynamic behaviour of the single track model
  - Equilibrium equations
  - Compatibility equations
  - Differential equations of vehicle dynamics
  - Stability derivatives
  - Canonical form of equations
- Investigation of the vehicle dynamics stability
  - Sign of real parts
  - Investigation of the discriminant
- Steady state particular case
- Trajectory description

Single track model

Single track model

- Stiff vehicle
  - Pitch motion (q=0)
  - Pumping motion (w=0)
- No body roll : p=0
- One can neglect any lateral load transfer leading to a reduction of the lateral cornering stiffness when lateral accelerations remain below 0.5 g (L. Segel, *Theoretical Prediction and Experimental Substantiation of the Response of Automobile Steering Control*, Cornell Aer. Lab. Buffalo. NY.)
- Constant speed forward motion: V
- Symmetry plane y=0: $J_{yx} = 0$ and $J_{yz} = 0$

Single track model

- Small angles and perturbations
  - Small steering angles (at wheel) $\delta \ll 1$
  - Small side slip angles $\alpha_i \ll 1$

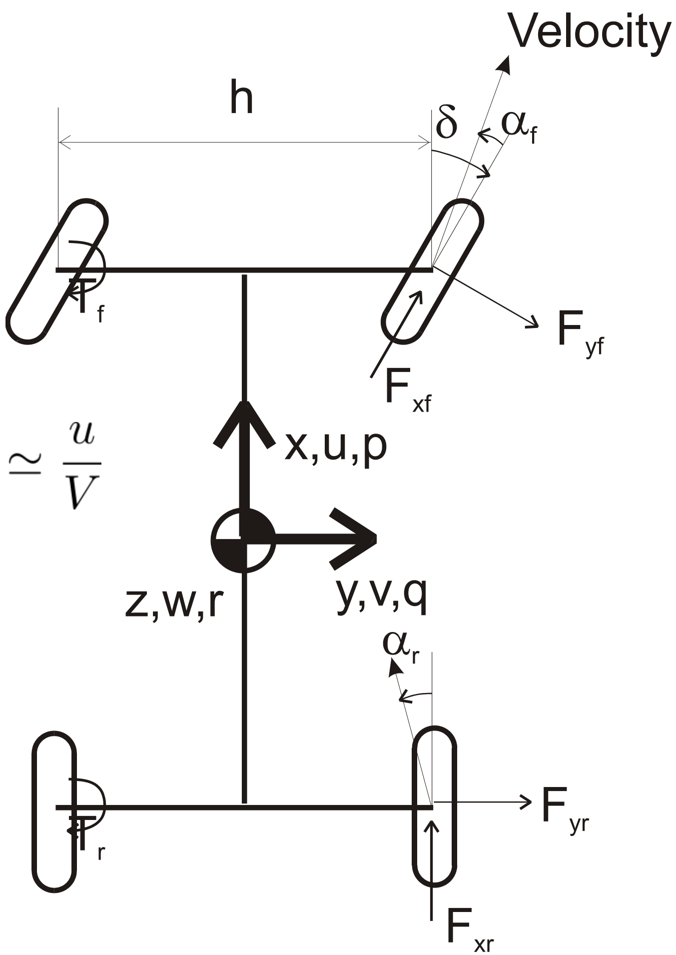
  $\Rightarrow$ Linearized theory $\sin\delta \simeq \delta \quad \cos\delta \simeq 1$

  $\sin\alpha \simeq \alpha \quad \cos\alpha \simeq 1$

CONCLUSION

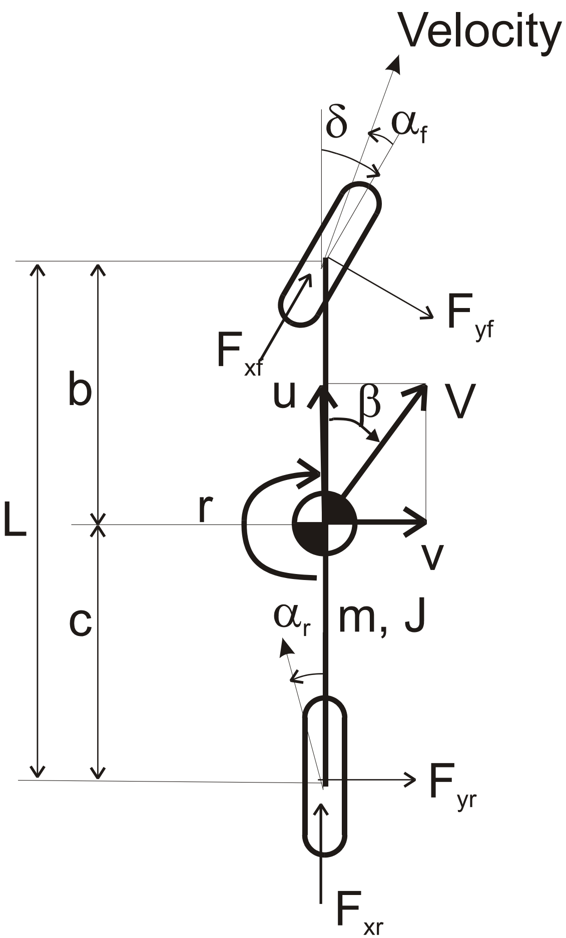
- Linearized model with two degrees of freedom:
  - Body side slip angle β (v)
  - Yaw velocity r

Single track model



$$\beta \simeq \tan\beta = \frac{v}{u} \simeq \frac{u}{V}$$

$$V = \sqrt{u^2 + v^2}$$



Dynamic equations in vehicle body axes

- Newton-Euler dynamic equations

$$\sum \overrightarrow{F} = \frac{d}{dt}(m\,\overrightarrow{V})$$

$$\sum \overrightarrow{T} = \frac{d}{dt}(J\,\overrightarrow{\omega})$$

- Time differentiation in non inertial frame

$$\frac{d}{dt}\overrightarrow{V}\bigg|_{absolu} = \frac{d}{dt}\overrightarrow{V}\bigg|_{relatif} + \overrightarrow{\omega} \times \overrightarrow{V}$$

- Equilibrium equations

$$\sum \overrightarrow{F} = m\frac{d}{dt}\overrightarrow{V} + m\overrightarrow{\omega} \times \overrightarrow{V}$$

$$\sum \overrightarrow{T} = \frac{d}{dt}(J\overrightarrow{\omega}) + \overrightarrow{\omega} \times (J\overrightarrow{\omega})$$

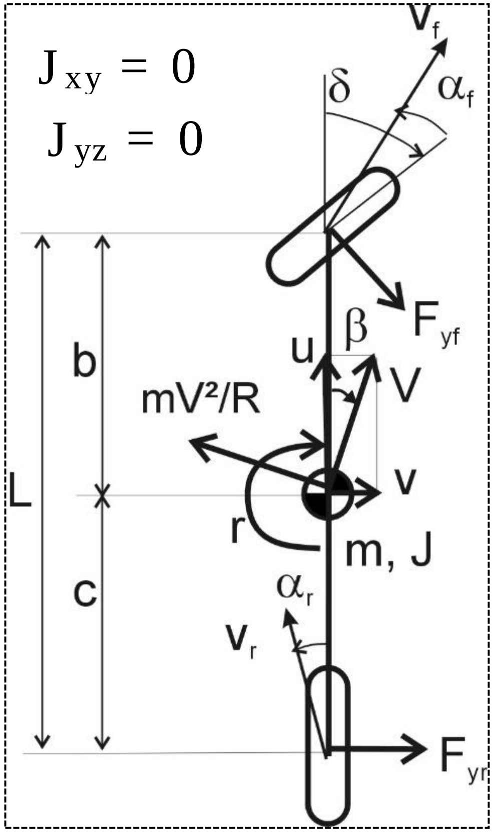
Dynamics equations of the vehicle motion

- Model with 2 dof β & r

$$V = \sqrt{u^2 + v^2} \qquad \tan\beta \simeq \beta \simeq \frac{v}{V}$$

$$\vec{V} = [u\; v\; 0]^T \qquad \vec{\omega} = [0\; 0\; r]^T$$

$$\vec{V} = u\vec{e}_x + v\vec{e}_y \qquad \vec{\omega} = \omega_z \vec{e}_z$$



- Inertia tensor

$$J = \begin{pmatrix} J_{xx} & 0 & J_{xz} \\ 0 & J_{yy} & 0 \\ J_{xz} & 0 & J_{zz} \end{pmatrix}$$

- It comes

$$\vec{\omega} \times \vec{v} = \omega_z \vec{e}_z \times (u\vec{e}_x + v\vec{e}_y) = \omega_z u\, \vec{e}_y - \omega_z v\, \vec{e}_x$$

$$J\vec{\omega} = [J_{xy}\omega_z \;\; 0 \;\; J_{zz}\omega_z]^T$$

$$\vec{\omega} \times J\vec{\omega} = \omega_z \vec{e}_z \times (\omega_z J_{xz}\vec{e}_x + \omega_z J_{zz}\vec{e}_z) = \omega_z^2 J_{xz}\vec{e}_y$$

Dynamics equations of the vehicle motion

- Finally, dynamics equations write

$$X = \sum F_x \; = \; m\left(\dot{u} - \omega_z v\right) \qquad L = \sum M_x \; = \; 0$$
$$Y = \sum F_y \; = \; m\left(\dot{v} + \omega_z u\right) \qquad M = \sum M_y \; = \; J_{xy}\omega_z^2$$
$$Z = \sum F_z \; = 0 \qquad N = \sum M_z \; = \; J_{zz}\dot{\omega}_z$$

- The only nontrivial equations are

$$Y = \sum F_y \; = \; m\left(\dot{v} + \omega_z u\right)$$
$$N = \sum M_z \; = \; J_{zz}\dot{\omega}_z$$

Equation of motion

Dynamic equations in vehicle body axes

- 2 dof model

$$\vec{V} = [u\ v\ 0]^T$$
$$\vec{\omega} = [0\ 0\ r]^T$$

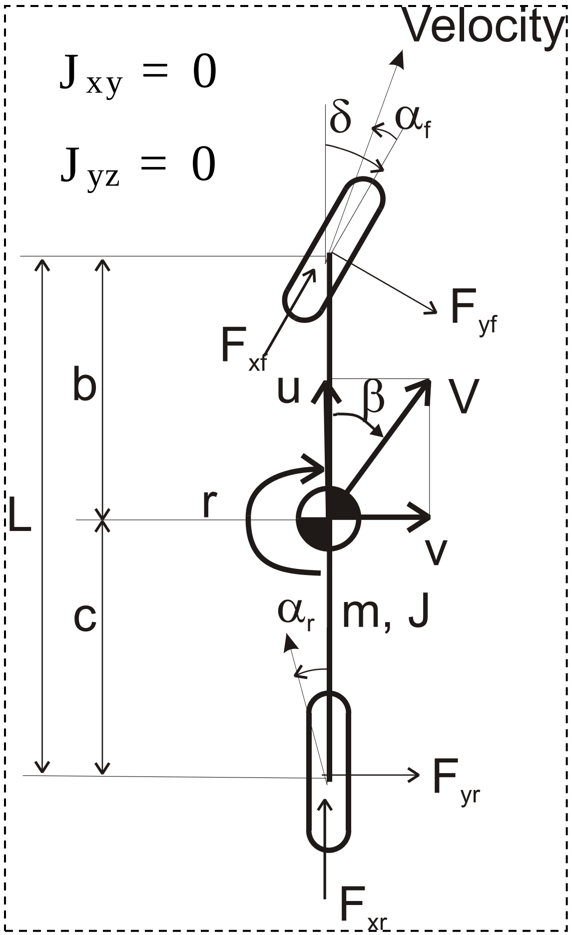
- Dynamic equations of motion

$$F_y = m(\dot{v} + ru)$$
$$N = J_{zz}\dot{r}$$

- Equations related to the fixed dof

→ Reaction forces / moment

$$F_x = -m\,r\,v$$
$$F_z = 0$$
$$L = J_{xz}\dot{r}$$
$$M = J_{xz}r^2$$



Dynamic equations in vehicle body axes

- Explanation

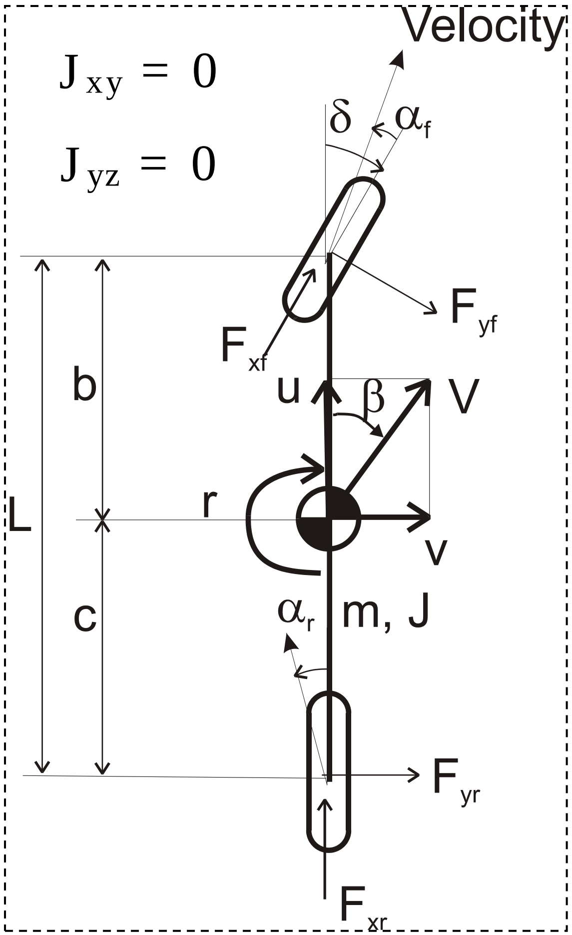
$$F_x = -m\,r\,v$$

Circular motion $\quad r = \dfrac{V}{R}$

$$\begin{aligned} F_x &= -m\frac{V^2}{R}\sin\beta \\ &= -m\frac{V}{R}V\,\sin\beta = -m\,r\,v \end{aligned}$$

- Major working forces:
  - Tyre forces
  - Other forces (ex aerodynamic forces)

  → Neglected because they don't depend on perturbations (in a first approximation)



Dynamic equations in vehicle body axes

- Equilibrium along F_y and M_z

$$
\begin{aligned}
m(\dot{v} + ru) &= F_{yr} + F_{xf} \sin\delta + F_{yf} \cos\delta \\
J_{zz}\dot{r} &= -F_{yr}\, c + F_{xf} \sin\delta\, b + F_{yf} \cos\delta\, b + T_{zf} + T_{zr}
\end{aligned}
$$

- Small angles assumption

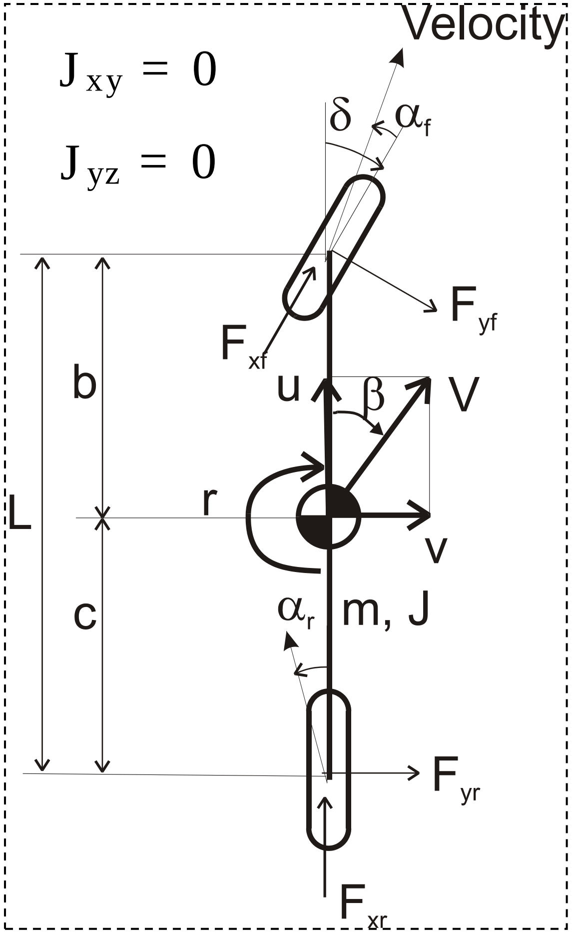
$$\beta \in [0^\circ, 15^\circ] \quad \beta \simeq v/u \simeq v/V$$

$$\sin\delta \simeq \delta \qquad \cos\delta \simeq 1$$

$$
\begin{aligned}
u &= V\cos\beta \simeq V \\
v &= V\sin\beta \simeq V\ \beta
\end{aligned}
$$

- Linearized equilibrium

$$
\begin{aligned}
mV(\dot{\beta} + r) &= F_{yr} + F_{yf} + F_{xf}\ \delta \\
J_{zz}\dot{r} &= -F_{yr}\, c + F_{yf}\, b + F_{xf}\delta\, b + T_{zf} + T_{zr}
\end{aligned}
$$



Dynamic equations in vehicle body axes

- If we neglect the self aligning torques and the tractive forces in a first step

$$\left\{ \begin{array}{ccc} mV(\dot{\beta} + r) & = & F_{yr} + F_{yf} \\ J_{zz}\dot{r} & = & -F_{yr}\, c + F_{yf}\, b \end{array} \right.$$

Compatibility equations

- Compatibility = relations between velocities and angles

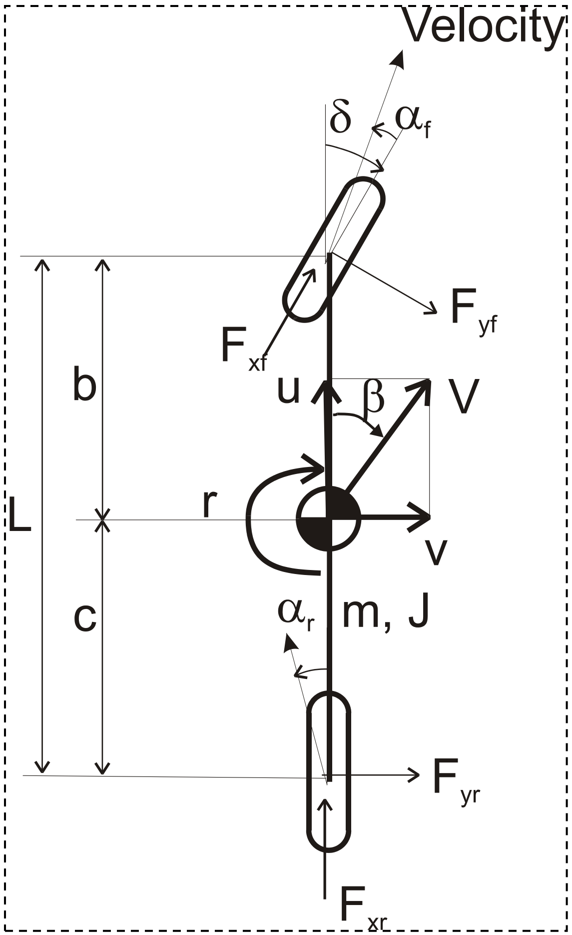
$$\tan(\delta - \alpha_f) = \frac{br + v}{u} \qquad \tan \alpha_r = \frac{cr - v}{u}$$

- Small side slip and steering angles

$$\alpha_f \simeq \delta - \frac{br + v}{u} \qquad \beta \simeq v/u$$
$$\alpha_r \simeq \frac{cr - v}{u}$$

- If u=V

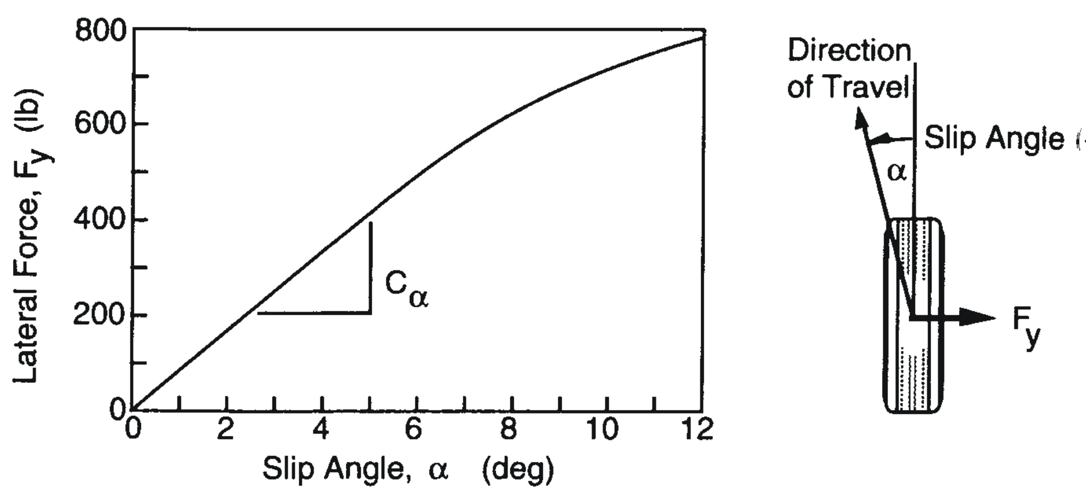
$$\alpha_f \simeq \delta - \frac{br}{V} - \beta$$
$$\alpha_r \simeq \frac{cr}{V} - \beta$$



Behavioural equations of tyres

- Cornering forces and side slip angles

$$F_{yf} = C_{\alpha f}\alpha_f \qquad F_{yr} = C_{\alpha r}\alpha_r$$



Source: Gillespie (fig 6.2)

Vehicle dynamic model

- Dynamic equilibrium

$$\begin{aligned} mV(\dot{\beta}+r) &= F_{yr}+F_{yf} \\ J_{zz}\dot{r} &= -F_{yr}\,c+F_{yf}\,b \end{aligned}$$

- Let's introduce the behaviour law of tyres

$$\begin{aligned} mV(\dot{\beta}+r) &= C_{\alpha r}\alpha_r+C_{\alpha f}\alpha_f \\ J_{zz}\dot{r} &= -C_{\alpha r}\alpha_r\,c+C_{\alpha f}\alpha_f\,b \end{aligned}$$

- And then the compatibility equations

$$\begin{aligned} mV(\dot{\beta}+r) &= C_{\alpha r}(\frac{cr}{V}-\beta)+C_{\alpha f}(\delta-\frac{br}{V}-\beta) \\ J_{zz}\dot{r} &= -C_{\alpha r}(\frac{cr}{V}-\beta)\,c+C_{\alpha f}(\delta-\frac{br}{V}-\beta)\,b \end{aligned}$$

Vehicle dynamic model

- Reshuffling the terms in β, r, and δ, one gets the equations related to the lateral forces and the moments about vertical axis

$$
\begin{aligned}
mV(\dot{\beta}+r) &= -(C_{\alpha f}+C_{\alpha r})\beta-(b\,C_{\alpha f}-c\,C_{\alpha r})\frac{1}{V}\,r+C_{\alpha f}\,\delta \\
J_{zz}\dot{r} &= -(b\,C_{\alpha f}-c\,C_{\alpha r})\beta-(b^2\,C_{\alpha f}+c^2\,C_{\alpha r})\frac{1}{V}\,r+b\,C_{\alpha f}\,\delta
\end{aligned}
$$

- And so

$$
\begin{aligned}
mV(\dot{\beta}+r)+(C_{\alpha f}+C_{\alpha r})\,\beta+(b\,C_{\alpha f}-c\,C_{\alpha r})\frac{1}{V}\,r &= C_{\alpha f}\,\delta \\
\underbrace{J_{zz}\dot{r}}_{\text{Differential terms in } r \text{ and } \beta}+\underbrace{(b\,C_{\alpha f}-c\,C_{\alpha r})\,\beta}_{\text{Terms in } r \text{ and } \beta}+\underbrace{(b^2\,C_{\alpha f}+c^2\,C_{\alpha r})\frac{1}{V}\,r}_{\text{Terms in } r \text{ and } \beta} &= \underbrace{b\,C_{\alpha f}\,\delta}_{\text{Control terms in } \delta}
\end{aligned}
$$

Differential terms in r and β

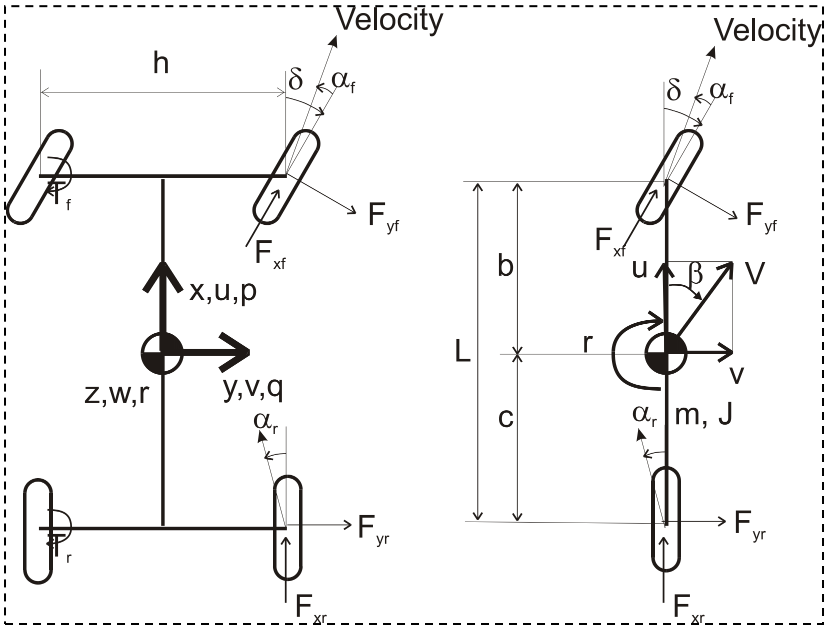
Terms in r and β

Control terms in δ

Vehicle dynamic model

- Dynamic equations ruling the motion of the single track vehicle

$$
\begin{aligned}
mV(\dot{\beta}+r)+(C_{\alpha f}+C_{\alpha r})\,\beta+(b\,C_{\alpha f}-c\,C_{\alpha r})\frac{1}{V}\,r &= C_{\alpha f}\,\delta \\
J_{zz}\dot{r}+(b\,C_{\alpha f}-c\,C_{\alpha r})\,\beta+(b^2\,C_{\alpha f}+c^2\,C_{\alpha r})\frac{1}{V}\,r &= b\,C_{\alpha f}\,\delta
\end{aligned}
$$



Stability derivatives

- Alternatively, it is the equivalent to perform a linear Taylor expansion of the forces and moments around the current configuration (that is reference configuration)

$$
\begin{aligned}
F_y &= \frac{\partial F_y}{\partial \beta}\beta + \frac{\partial F_y}{\partial r}r + \frac{\partial F_y}{\partial \delta}\delta \dots \\
N &= \frac{\partial N}{\partial \beta}\beta + \frac{\partial N}{\partial r}r + \frac{\partial N}{\partial \delta}\delta \dots
\end{aligned}
$$

- It is usual to denote them as *stability derivatives*

$$
Y_\beta = \frac{\partial F_y}{\partial \beta} \qquad Y_r = \frac{\partial F_y}{\partial r} \qquad Y_\delta = \frac{\partial F_y}{\partial \delta}
$$

$$
N_\beta = \frac{\partial N}{\partial \beta} \qquad N_r = \frac{\partial N}{\partial r} \qquad N_\delta = \frac{\partial N}{\partial \delta}
$$

Stability derivatives

- Comparing with the initial developments,

$$
\begin{aligned}
mV(\dot{\beta} + r) &= -(C_{\alpha f} + C_{\alpha r})\beta - (b\,C_{\alpha f} - c\,C_{\alpha r})\frac{1}{V}\,r + C_{\alpha f}\,\delta \\
J_{zz}\dot{r} &= -(b\,C_{\alpha f} - c\,C_{\alpha r})\beta - (b^2\,C_{\alpha f} + c^2\,C_{\alpha r})\frac{1}{V}\,r + b\,C_{\alpha f}\,\delta
\end{aligned}
$$

- one finds the expression of the stability derivatives

$$
\begin{aligned}
Y_\beta &= -(C_{\alpha f} + C_{\alpha r}) \quad (< 0) & N_\beta &= -(b\,C_{\alpha f} - c\,C_{\alpha r}) \\
Y_r &= -(b\,C_{\alpha f} - c\,C_{\alpha r})\frac{1}{V} & N_r &= -(b^2\,C_{\alpha f} + c^2\,C_{\alpha r})\frac{1}{V} \quad (< 0) \\
Y_\delta &= C_{\alpha f} \quad (> 0) & N_\delta &= b\,C_{\alpha f} \quad (> 0)
\end{aligned}
$$

Stability derivatives

- The equilibrium equations writes

$$\begin{array}{rcl} mV(\dot{\beta}+r) & = & Y_\beta \beta + Y_r r + Y_\delta \delta \\ J_{zz}\dot{r} & = & N_\beta \beta + N_r r + N_\delta \delta \end{array}$$

$$\begin{array}{rcl} Y_\beta & = & -(C_{\alpha f} + C_{\alpha r}) \quad (< 0) \\ Y_r & = & -(b\,C_{\alpha f} - c\,C_{\alpha r})\dfrac{1}{V} \\ Y_\delta & = & C_{\alpha f} \quad (> 0) \end{array}$$

$$\begin{array}{rcl} N_\beta & = & -(b\,C_{\alpha f} - c\,C_{\alpha r}) \\ N_r & = & -(b^2\,C_{\alpha f} + c^2\,C_{\alpha r})\dfrac{1}{V} \quad (< 0) \\ N_\delta & = & b\,C_{\alpha f} \quad (> 0) \end{array}$$

- Reorganizing the terms, one has

$$\begin{array}{rcl} mV\dot{\beta} & = & Y_\beta \beta + (Y_r - mV)r + Y_\delta \delta \\ J_{zz}\dot{r} & = & N_\beta \beta + N_r r + N_\delta \delta \end{array}$$

Canonical form of the equations

- It is also valuable to notice that the single track model lag to a linear time invariant (LTI) model. It is usual to cast this model under the standard form

$$\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} \;+\; \mathbf{B}\,\mathbf{u}$$

- The system state variables and the command vector are:

$$\mathbf{z} = \begin{pmatrix} \beta \\ r \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} \delta \end{pmatrix}$$

- The system matrices A and B are easily identified and write

$$\mathbf{A} = \begin{bmatrix} \frac{Y_\beta}{mV} & \frac{Y_r}{mV} - 1 \\ \frac{N_\beta}{J_{zz}} & \frac{N_r}{J_{zz}} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{Y_\delta}{mV} \\ \frac{Y_\beta}{J_{zz}} \end{bmatrix}$$

Stability analysis

Study of system stability

- Use Laplace transform

$$\beta(t) \to \beta(s) \quad r(t) \to r(s) \quad \delta(t) \to \delta(s)$$
$$\dot{\beta}(t) \to s\,\beta(s) \quad \dot{r}(t) \to s\,r(s)$$

- The system becomes

$$\begin{aligned} (s\,mV - Y_\beta)\ \beta(s) + (mV - Y_r)\ r(s) &= Y_\delta\ \delta(s) \\ -N_\beta\ \beta(s) + (s\,J_{zz} - N_r)\ r(s) &= N_\delta\ \delta(s) \end{aligned}$$

- The stability of the free response stems from the study of the roots of the characteristic equation

$$\Delta = (smV - Y_\beta)\,(sJ_{zz} - N_r) + (mV - Y_r)\,N_\beta = 0$$

$$mV J_{zz}\ s^2 - (Y_\beta J_{zz} + mV N_r)\,s + (Y_\beta N_r - Y_r N_\beta + N_\beta mV) = 0$$

Study of system stability

- Characteristic equation

$$\boxed{mVJ_{zz}\;s^2-(Y_\beta J_{zz}+mVN_r)\;s+(Y_\beta N_r-Y_rN_\beta+N_\beta mV)=0}$$

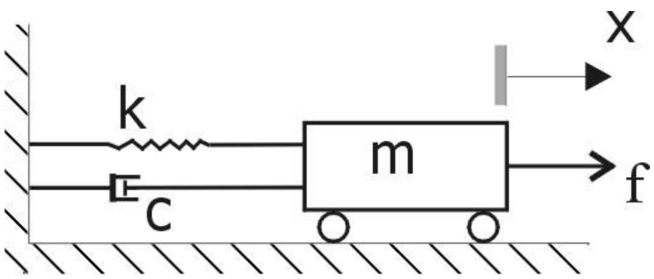
$$s^2-(\frac{Y_\beta}{mV}+\frac{N_r}{J_{zz}})\;s+(\frac{Y_\beta}{mV}\frac{N_r}{J_{zz}}-\frac{Y_r}{mV}\frac{N_\beta}{J_{zz}}+\frac{N_\beta}{J_{zz}})=0$$

- This equation is similar to the one of single dof oscillating mass

$$m\,s^2+c\,s+k=0$$

$$s^2+\xi\,s+\Omega^2=0$$

$$\xi=c/m \qquad \Omega^2=k/m$$



Study of system stability

- Stability equation

$$s^2 + \xi\, s + \Omega^2 = 0$$

- Roots of the characteristic equations

$$s_{1,2} = -\xi/2 \pm 1/2\sqrt{\xi^2 - 4\Omega^2}$$

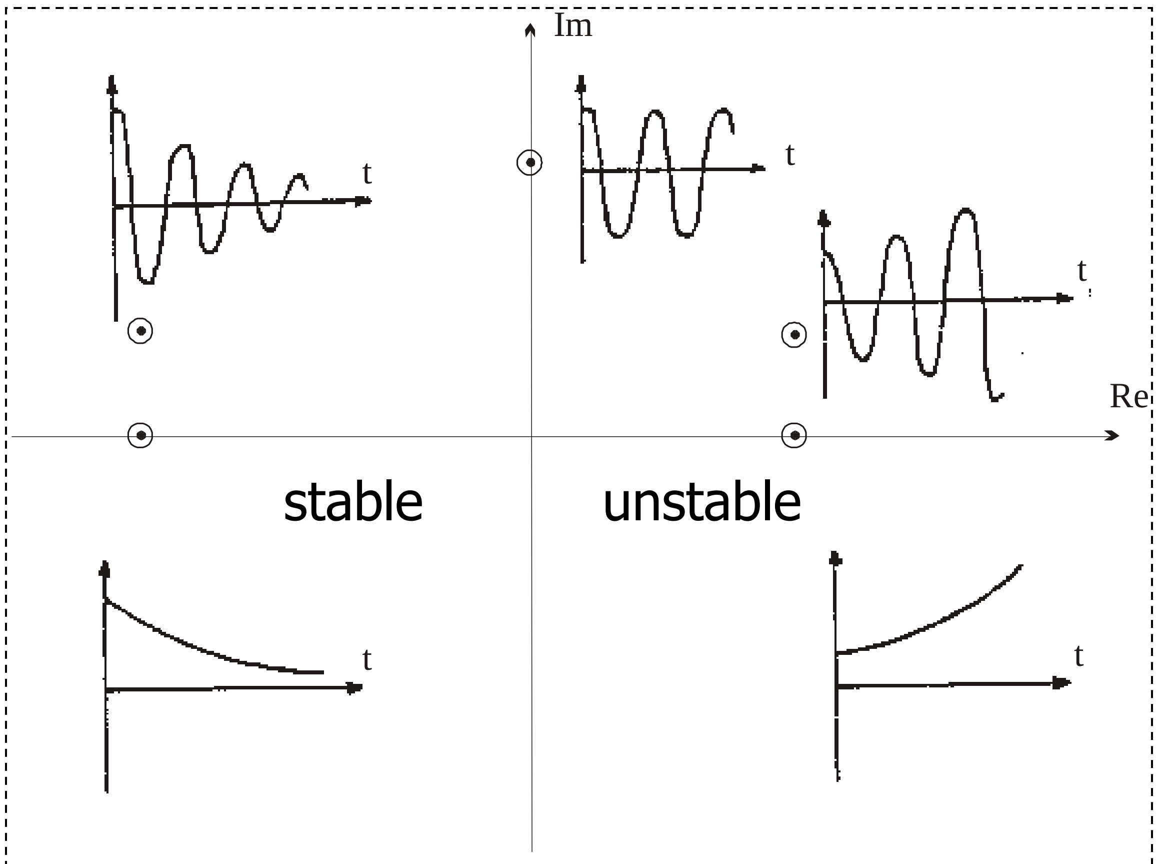
- Stability criterion: The real parts of all roots must be negative
  - In case of conjugate roots, their sum must be negative
  - In case of real roots, their sum must be negative and their product must be positive

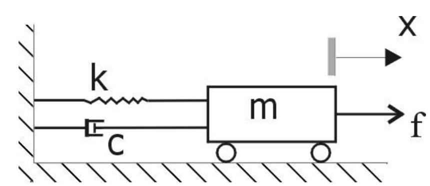
  That is:

$$s_1 + s_2 = -b/a < 0 \qquad s_1.s_2 = c/a > 0$$

- This criterion is equivalent to Routh Hurwitz.

Study of system stability





$$m\,s^2 + c\,s + k = 0$$

$$s^2 + \xi\,s + \Omega^2 = 0$$

Study of system stability

- Characteristic equations (reminder)

$$mVJ_{zz}\ s^2 - (Y_\beta J_{zz} + mVN_r)\ s + (Y_\beta N_r - Y_r N_\beta + N_\beta mV) = 0$$

a b c

- To be checked

$$s_1 + s_2 = -b/a < 0 \quad \Leftrightarrow \quad mVN_r + J_{zz}Y_\beta < 0$$

$$s_1.s_2 = c/a > 0 \quad \Leftrightarrow \quad mVN_\beta + Y_\beta N_r - Y_r N_\beta > 0$$

- First condition: always satisfied

$$mVN_r = -m(b^2\, C_{\alpha f} + c^2\, C_{\alpha r}) < 0$$

$$J_{zz}Y_\beta = -J_{zz}(C_{\alpha f} + C_{\alpha r}) < 0$$

Study of system stability

- Second condition:

$$s_1.s_2 = c/a > 0 \quad \Leftrightarrow \quad mVN_\beta + Y_\beta N_r - Y_r N_\beta > 0$$

- So

$$mVN_\beta = -mV(b\,C_{\alpha f} - c\,C_{\alpha r})$$

$$N_r Y_\beta = (C_{\alpha f} + C_{\alpha r})\frac{b^2\,C_{\alpha f} + c^2\,C_{\alpha r}}{V}$$

$$N_\beta Y_r = (b\,C_{\alpha f} - c\,C_{\alpha r})\frac{b\,C_{\alpha f} - c\,C_{\alpha r}}{V}$$

$$N_r Y_\beta - N_\beta Y_r = \frac{C_{\alpha f} C_{\alpha r} L^2}{V}$$

- It comes the condition

$$-mV(b\,C_{\alpha f} - c\,C_{\alpha r}) + \frac{C_{\alpha f} C_{\alpha r} L^2}{V} > 0$$

Study of system stability

- The second condition is satisfied if

$$-mV\,(b\,C_{\alpha f} - c\,C_{\alpha r}) + \frac{C_{\alpha f} C_{\alpha r} L^2}{V} > 0$$

- For an understeer vehicle: $N_\beta = -(b\,C_{\alpha f} - c\,C_{\alpha r}) > 0$
  the dynamic behaviour is always stable

- For an oversteer vehicle $N_\beta = -(b\,C_{\alpha f} - c\,C_{\alpha r}) < 0$
  the dynamic behaviour is unstable above the critical speed

$$\boxed{V_{crit}^2 = \frac{C_{\alpha f} C_{\alpha r} L^2}{m(b\,C_{\alpha f} - c\,C_{\alpha r})} = -\frac{C_{\alpha f} C_{\alpha r} L^2}{m N_\beta}}$$

Reminder:
Neutral steer, understeer and oversteer vehicles

- Understeer gradient in Steady State Cornering

$$K = \frac{m\,c}{C_{\alpha f}\,L} - \frac{m\,b}{C_{\alpha r}\,L} = m\,\frac{c\,C_{\alpha r} - bC_{\alpha f}}{C_{\alpha f}C_{\alpha r}\,L}$$

- If K=0, the vehicle is said to be *neutralsteer*:

$$K = 0 \quad \Leftrightarrow \quad c\,C_{\alpha r} = b\,C_{\alpha f} \qquad N_\beta = -(b\,C_{\alpha f} - c\,C_{\alpha r}) = 0$$

- If K>0, the vehicle is *understeer* :

$$K > 0 \quad \Leftrightarrow \quad c\,C_{\alpha r} > b\,C_{\alpha f} \qquad N_\beta = -(b\,C_{\alpha f} - c\,C_{\alpha r}) > 0$$

- If K<0, the vehicle is *oversteer*:

$$K < 0 \quad \Leftrightarrow \quad c\,C_{\alpha r} < b\,C_{\alpha f} \qquad N_\beta = -(b\,C_{\alpha f} - c\,C_{\alpha r}) < 0$$

Investigation of the motion nature

- Investigation of the discriminant of the stability equation

$$\ddot{x} + \xi\,\dot{x} + \Omega^2\,x = 0$$

$$\begin{aligned} \rho &= \xi^2 - 4\Omega^2 \\ &= (\frac{Y_\beta}{mV} + \frac{N_r}{J_{zz}})^2 - 4\,(\frac{Y_\beta}{mV}\frac{N_r}{J_{zz}} - \frac{Y_r}{mV}\frac{N_\beta}{J_{zz}} + \frac{N_\beta}{J_{zz}}) \end{aligned}$$

$$\xi = -(\frac{Y_\beta}{mV} + \frac{N_r}{J_{zz}}) \qquad \Omega^2 = (\frac{Y_\beta}{mV}\frac{N_r}{J_{zz}} - \frac{Y_r}{mV}\frac{N_\beta}{J_{zz}} + \frac{N_\beta}{J_{zz}})$$

  - If $\rho>0$: 2 real roots, damping is greater than the critical damping and one experiences an aperiodic motion.
  - If $\rho<0$: 2 complex conjugate roots, damping is below the critical damping. One experiences an oscillation motion.
  - If $\rho=0$, 2 identical roots, critical damping of the system.

Investigation of the motion nature

- Value of the discriminant

$$\begin{aligned}\rho &= \left(\frac{Y_\beta}{mV} + \frac{N_r}{J_{zz}}\right)^2 - 4\left(\frac{Y_\beta}{mV}\frac{N_r}{J_{zz}} - \frac{Y_r}{mV}\frac{N_\beta}{J_{zz}} + \frac{N_\beta}{J_{zz}}\right) \\ &= \left(\frac{Y_\beta}{mV} - \frac{N_r}{J_{zz}}\right)^2 + 4\frac{Y_r}{mV}\frac{N_\beta}{J_{zz}} - 4\frac{N_\beta}{J_{zz}} \\ &= \left(\frac{C_{\alpha f} + C_{\alpha r}}{mV} - \frac{b^2\, C_{\alpha f} + c^2\, C_{\alpha r}}{J_{zz}V}\right)^2 + 4\frac{N_\beta^2}{mV^2 J_{zz}} - 4\frac{N_\beta}{J_{zz}}\end{aligned}$$

- One finally finds

$$\rho = \left[\left(\frac{C_{\alpha f} + C_{\alpha r}}{m} - \frac{b^2\, C_{\alpha f} + c^2\, C_{\alpha r}}{J_{zz}}\right)^2 + 4\frac{(b\, C_{\alpha f} - c\, C_{\alpha r})^2}{mJ_{zz}}\right]\frac{1}{V^2} - 4\frac{N_\beta}{J_{zz}}$$

Investigation of the motion nature

- Discriminant (reminder)

$$\rho = \left[\left(\frac{C_{\alpha f} + C_{\alpha r}}{m} - \frac{b^2\, C_{\alpha f} + c^2\, C_{\alpha r}}{J_{zz}}\right)^2 + 4\frac{(b\, C_{\alpha f} - c\, C_{\alpha r})^2}{mJ_{zz}}\right]\frac{1}{V^2} - 4\frac{N_\beta}{J_{zz}}$$

- If $N_\beta<0$ (oversteer machine), $\rho>0$.
  - The dynamic response is aperiodic
  - Stable as long as $V < V_{crit}$.
- If $N_\beta>0$ (understeer machine), $\rho<0$.
  - Positive term decreases as $1/V^2$
  - The response becomes oscillation (damped) over the speed

$$V_{osc.}^2 = \frac{J}{N_\beta}\left[\left(\frac{C_{\alpha f} + C_{\alpha r}}{m} - \frac{b^2\, C_{\alpha f} + c^2\, C_{\alpha r}}{J_{zz}}\right)^2 + 4\frac{(b\, C_{\alpha f} - c\, C_{\alpha r})^2}{mJ_{zz}}\right]$$

Steady state circular motion

Particular case: steady state turn

- The circular motion is characterized by:

$$\dot{\beta} = \dot{r} = 0 \qquad a_y = \frac{V^2}{R} \qquad r = \dot{\theta} = \frac{V}{R}$$

- It comes

$$\begin{cases} -Y_\beta\,\beta + & (mV - Y_r)\,r & = & Y_\delta\,\delta \\ -N_\beta\,\beta - & N_r\,r & = & N_\delta\,\delta \end{cases}$$

- One extracts the value of the slip angle

$$\beta = -\frac{N_\delta\,\delta + N_r\,r}{N_\beta}$$

- The value of the yaw angle writes

$$(-mV\;N_\beta + N_\beta Y_r - N_r Y_\beta)\,r = (N_\delta Y_\beta - N_\beta Y_\delta)\,\delta$$

Particular case: steady state turn

- It yields the gain between the yaw speed and the steering angle:

$$\frac{r}{\delta} = \frac{N_\beta Y_\delta - N_\delta Y_\beta}{N_r Y_\beta - N_\beta Y_r + mV\ N_\beta}$$

- Given that:

$$N_r Y_\beta - N_\beta Y_r = \frac{C_{\alpha f} C_{\alpha r} L^2}{V}$$

$$\begin{array}{rcl} N_\beta Y_\delta & = & C_{\alpha f}(cC_{\alpha r} - bC_{\alpha f}) \\ -N_\delta Y_\beta & = & bC_{\alpha f}(C_{\alpha r} + C_{\alpha f}) \\ \hline \end{array}$$

$$N_\beta Y_\delta - N_\delta Y_\beta = C_{\alpha f}\ C_{\alpha r}\ L$$

- It comes

$$\frac{\delta}{r} = \frac{L}{V} + N_\beta\ \frac{mV}{C_{\alpha f}\ C_{\alpha r}\ L}$$

Particular case: steady state turn

- Now taking into consideration the circular motion nature $r = \frac{V}{R}$

$$\delta = \frac{L\,r}{V} + N_\beta \,\frac{mVr}{C_{\alpha f}\,C_{\alpha r}\,L} \qquad \delta = \frac{L}{R} + \frac{m\,N_\beta}{C_{\alpha f}\,C_{\alpha r}\,L}\,\frac{V^2}{R}$$

- And by introducing the value

$$N_\beta = -(b\,C_{\alpha f} - c\,C_{\alpha r})$$

- One recovers the classical expression for steady state turn

$$\boxed{\delta = \frac{L}{R} + (m\frac{c}{L}\frac{1}{C_{\alpha f}} - m\frac{b}{L}\frac{1}{C_{\alpha r}})\,\frac{V^2}{R}}$$

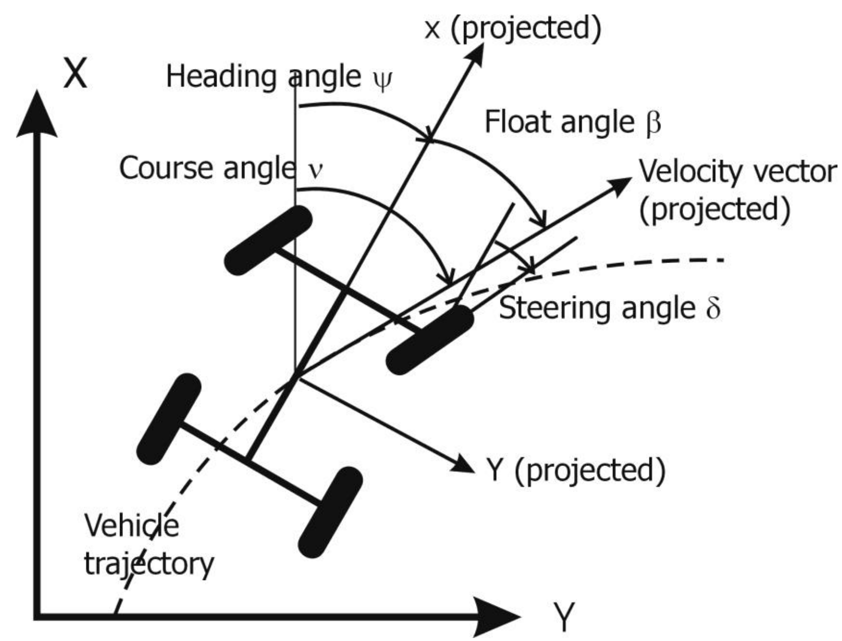
Trajectory description

Trajectory description

- The vehicle trajectory can be described using a parametric equation relating the space coordinates and the time

$$t \mapsto (\bar{X}(t), \bar{Y}(t))$$

- One defines:
  - θ the course angle between the trajectory and the frames axis X
  - ψ the heading angle between the X of the reference frame and the x axes of the car
  - β the side slide angle of the vehicle, the angle between the vehicle axis x and the velocity vector tangent to the trajectory



Trajectory description

- We can write the following relations between the course angle θ, the heading angle ψ and the side slip angle β:

$$\vartheta = \psi + \beta \qquad\qquad \dot{\vartheta} = r + \dot{\beta}$$

- The linear velocities are obtained by using frame transformation between the vehicle body axes and the inertial reference frame

$$\begin{aligned} \frac{dX}{dt} &= V\cos\vartheta \\ \frac{dY}{dt} &= V\sin\vartheta \end{aligned} \qquad\qquad \begin{aligned} \frac{dX}{dt} &= u\cos\psi - v\sin\psi \\ \frac{dY}{dt} &= u\sin\psi + v\cos\psi \end{aligned}$$