ELEMENTARY CONCEPTS IN STRUCTURAL OTIMIZATION

Pierre DUYSINX
LTAS – Automotive Engineering

Academic year 2020-2021

Introduction

- Structural optimization applied to sizing (weight minimization) problem

$$\min_{x} \quad W(x) = \sum_{i=1}^{n} w_i \, x_i$$

$$\text{s.t.:} \quad g_j(x) \leq 0 \qquad (j = 1 \ldots m)$$

$$\underline{x}_i \leq x_i \leq \overline{x}_i \qquad (i = 1 \ldots n)$$

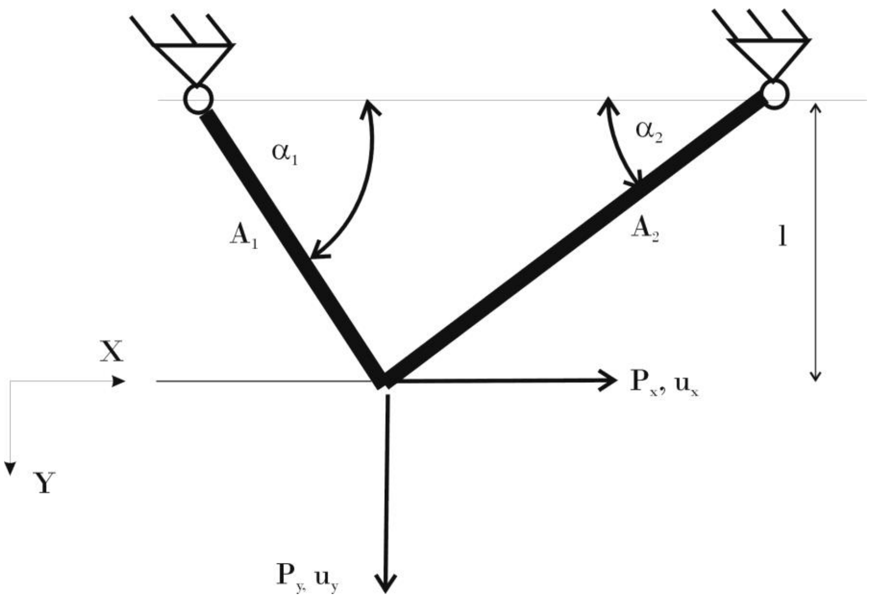
  - Finite element model
  - Design variables are the transverse sizes of the structural members (Fixed geometry and material properties)
  - Design restrictions

$$g_j(x) \equiv \begin{cases} u_j(x) - \overline{u}_j \leq 0 \\ \sigma_j(x) - \overline{\sigma}_j \leq 0 \\ \underline{\omega}_j^2 - \omega_j^2(x) \leq 0 \\ \underline{\lambda}_j^2 - \lambda_j^2(x) \leq 0 \end{cases}$$

TWO BAR TRUSS PROBLEM

TWO-BAR TRUSS

- Let's consider the example of the two-bar truss



- **Equilibrium** between the external loads P_x, P_y and the internal efforts N_1 and N_2:

$$\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & -\cos\alpha_2 \\ \sin\alpha_1 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

- For a general structure with n bars and m node, this matrix equation becomes

$$\mathbf{P} = \mathbf{B}^T\,\mathbf{N}$$

TWO-BAR TRUSS

- Internal forces can be found by solving the matrix equations to yield

$$\mathbf{N} = (\mathbf{B}^T)^{-1}\,\mathbf{P}$$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \frac{1}{\sin(\alpha_1 + \alpha_2)} \begin{bmatrix} \sin\alpha_2 & \cos\alpha_2 \\ -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

- For statically determinate structures, the number of equilibrium equations is equal to the number of unknown member internal forces and so the matrix **B** is square and full rank. However generally speaking for indeterminate structure, the matrix **B** is rectangular, and this does not hold in general as it will be seen for the three-bar-truss

TWO-BAR TRUSS

- Thus for statically determinate structures, the internal bar forces depends only on the applied loads and of the direction cosines of the individual bars.

- **Stresses**

$$\sigma_i = \frac{N_i}{x_i}$$

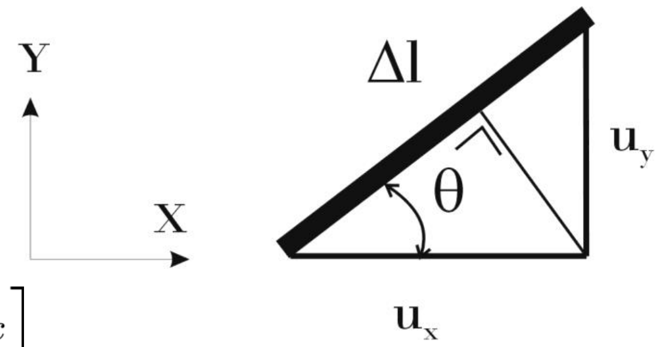
- They also depend on the applied load the geometry of the structure and the bar cross sectional areas **x***.

TWO-BAR TRUSS

- Let's write the compatibility conditions and relate nodal displacements to the applied loads.

- The <u>elongation of the bars</u> are related to the free node displacements

$$\Delta l_i = u_x \cos\theta + u_y \sin\theta$$



- It comes

$$\begin{bmatrix} \Delta l_1 \\ \Delta l_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\cos\alpha_2 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

- We recognize the strain matrix **B** connecting the strains to the nodal displacement.

$$\begin{bmatrix} \Delta l_1 \\ \Delta l_2 \end{bmatrix} = \mathbf{B} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

TWO-BAR TRUSS

- Bar strains

$$\epsilon_i = \frac{\Delta l_i}{l_i}$$

- Hook's law

$$\sigma_i = E\,\epsilon_i$$

- Bar forces

$$N_i = x_i\,\sigma_i = x_i\,E\,\frac{\Delta l_i}{l_i}$$

- It yields

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \frac{E_1 x_1}{l_1} & 0 \\ 0 & \frac{E_2 x_2}{l_2} \end{bmatrix} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \end{bmatrix}$$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \frac{E\,x_1}{l_1} & 0 \\ 0 & \frac{E\,x_2}{l_2} \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\cos\alpha_2 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

TWO-BAR TRUSS

- Write the applied loads in terms of the displacements.

$$\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & -\cos\alpha_2 \\ \sin\alpha_1 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} \frac{E\,x_1}{l_1} & 0 \\ 0 & \frac{E\,x_2}{l_2} \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\cos\alpha_2 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

- **Equation relating the applied loads to the displacements**

$$\mathbf{P} = \mathbf{B}^T \mathbf{D} \mathbf{B}\,\mathbf{u} = \mathbf{K}\,\mathbf{u}$$

- Generalized Hook matrix

$$\mathbf{D} = \begin{bmatrix} \frac{E\,x_1}{l_1} & 0 \\ 0 & \frac{E\,x_2}{l_2} \end{bmatrix}$$

- Stiffness matrix

$$\mathbf{K} = \mathbf{B}^T \mathbf{D} \mathbf{B}$$

TWO-BAR TRUSS

- Evaluation of the displacements

$$\begin{aligned}\mathbf{u} &= \mathbf{K}^{-1}\,\mathbf{P} \\ &= \mathbf{B}^{-1}\mathbf{D}^{-1}(\mathbf{B}^{T})^{-1}\,\mathbf{P} = \mathbf{B}^{-1}\mathbf{D}^{-1}\,\mathbf{N}\end{aligned}$$

- **u** gives all the displacement at all nodes and it is more usual in optimization to be interested in a specific displacement $\mathbf{u}_j$ corresponding to the jth degrees of freedom.
- In order to extract the required components, the vector **u** can be multiplied by a vector $\mathbf{e}_j$ which contains '0' elements everywhere except for the jth component which contains a '1' at this position.

$$\mathbf{e}_{(j)}^{T} = \begin{bmatrix} 0 & 0\ldots & 1 & 0\ldots & 0 \end{bmatrix}$$

$$u_j = \mathbf{e}_{(j)}^{T}\,\mathbf{u} = \mathbf{e}_{(j)}^{T}\,\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{N}$$

TWO-BAR TRUSS

- Remember that

$$\mathbf{N} = (\mathbf{B}^T)^{-1}\,\mathbf{P}$$

- It is interesting to remark that

$$(\mathbf{B}^T)^{-1}\mathbf{e}_{(j)} = \mathbf{n}_{(j)}$$

- It is interpreted as the set of internal forces in equilibrium with a unit load (1 N) applied on the degrees of freedom $\mathbf{u}_j$ and acting in the direction of displacement component.
- A unit load applied along degree of freedom j is called a dummy load.

- The expression of the displacement $\mathbf{u}_j$ becomes

$$u_j = \mathbf{n}_{(j)}^T\,\mathbf{D}^{-1}\mathbf{N}$$

TWO-BAR TRUSS

- The expression of the displacement $\mathbf{u}_j$

$$u_j = \mathbf{n}_{(j)}^T \, \mathbf{D}^{-1} \mathbf{N}$$

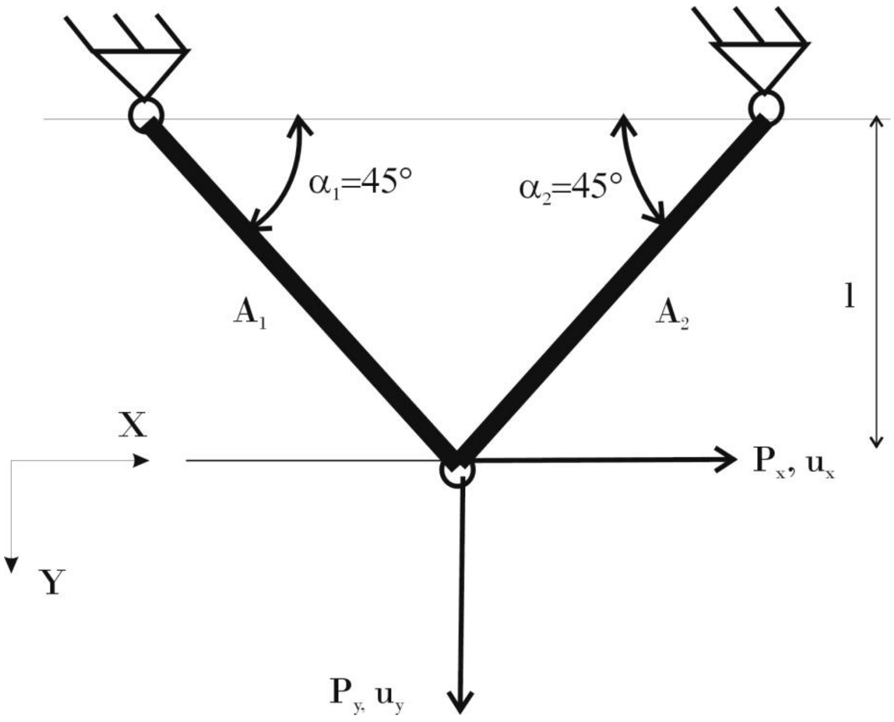
- Expanding this matrix product, we recover the familiar expression for calculating the magnitude of a specific nodal displacement in truss structures

$$u_j = \sum_{i=1}^{n} \frac{N_i \, l_i \, n_i^{(j)}}{E \, x_i}$$

- where $n_i^{(j)}$ represented components of the vector $\mathbf{n}^{(j)}$ and l_i and x_i are again the bar length and its cross-sectional areas respectively.

TWO-BAR TRUSS

- Let's consider the particular case the two-bar truss with 45° angles



$$\alpha_1 = 45^\circ \qquad \alpha_2 = 45^\circ$$

- **Equilibrium** between the external loads P_x, P_y and the internal efforts N_1 and N_2:

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \frac{1}{2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

$$\begin{cases} N_1 = \frac{\sqrt{2}}{2} P_x + \frac{\sqrt{2}}{2} P_y \\ N_2 = -\frac{\sqrt{2}}{2} P_x + \frac{\sqrt{2}}{2} P_y \end{cases}$$

TWO –BAR TRUSS

- We shall consider the following particular cases:

$$P_x = P, \qquad P_y = 0$$

- It comes

$$N_1 = \frac{\sqrt{2}}{2} P$$

$$N_2 = -\frac{\sqrt{2}}{2} P$$

- And the stresses

$$\sigma_1 = \frac{\sqrt{2}\, P}{2\, x_1}$$

$$\sigma_2 = -\frac{\sqrt{2}\, P}{2\, x_2}$$

TWO-BAR TRUSS

- We want now to evaluate the **displacement at the free node**.

- We use the dummy load case approach. Let's compute first the dummy load cases in both x and y directions at the free node.

  - $P_x=1$, $P_y=0$

$$n_1 = \frac{\sqrt{2}}{2} \qquad n_2 = -\frac{\sqrt{2}}{2}$$

  - $P_x=0$, $P_y=1$

$$n_1 = \frac{\sqrt{2}}{2} \qquad n_2 = \frac{\sqrt{2}}{2}$$

TWO-BAR TRUSS

- Insert these results into the expression

$$u_j = \sum_{i=1}^{n} \frac{N_i\, l_i\, n_i^{(j)}}{E\, x_i}$$

- For a horizontal displacement:

$$u_x = \frac{l_1}{E\, x_1}\left(\frac{\sqrt{2}}{2}P\right)\left(\frac{\sqrt{2}}{2}\right) + \frac{l_2}{E\, x_2}\left(-\frac{\sqrt{2}}{2}P\right)\left(-\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}\, P\, l}{2\, E}\left(\frac{1}{x_1} + \frac{1}{x_2}\right)$$

- For a horizontal displacement:

$$u_y = \frac{l_1}{E\, x_1}\left(\frac{\sqrt{2}}{2}P\right)\left(\frac{\sqrt{2}}{2}\right) + \frac{l_2}{E\, x_2}\left(-\frac{\sqrt{2}}{2}P\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}\, P\, l}{2\, E}\left(\frac{1}{x_1} - \frac{1}{x_2}\right)$$

TWO-BAR TRUSS

- The most elementary optimum design problem for this class of structure consists in finding a set of bar cross sectional areas which minimizes structural weight subject to limits on the allowable stresses in individual members.

$$\min \quad W = \sum_{i=1}^{2} \rho_i l_i x_i$$

$$\textbf{s.t.}: \quad -\bar{\sigma} \leq \frac{N_i}{x_i} \leq \bar{\sigma}_i \quad i = 1, 2$$

$$0 \leq x_i \quad i = 1, 2$$

- Although the problem is in many aspects trivial, it nevertheless forms a useful model for illustrating some of the concepts which play important roles when more complex problems are considered.

TWO-BAR TRUSS

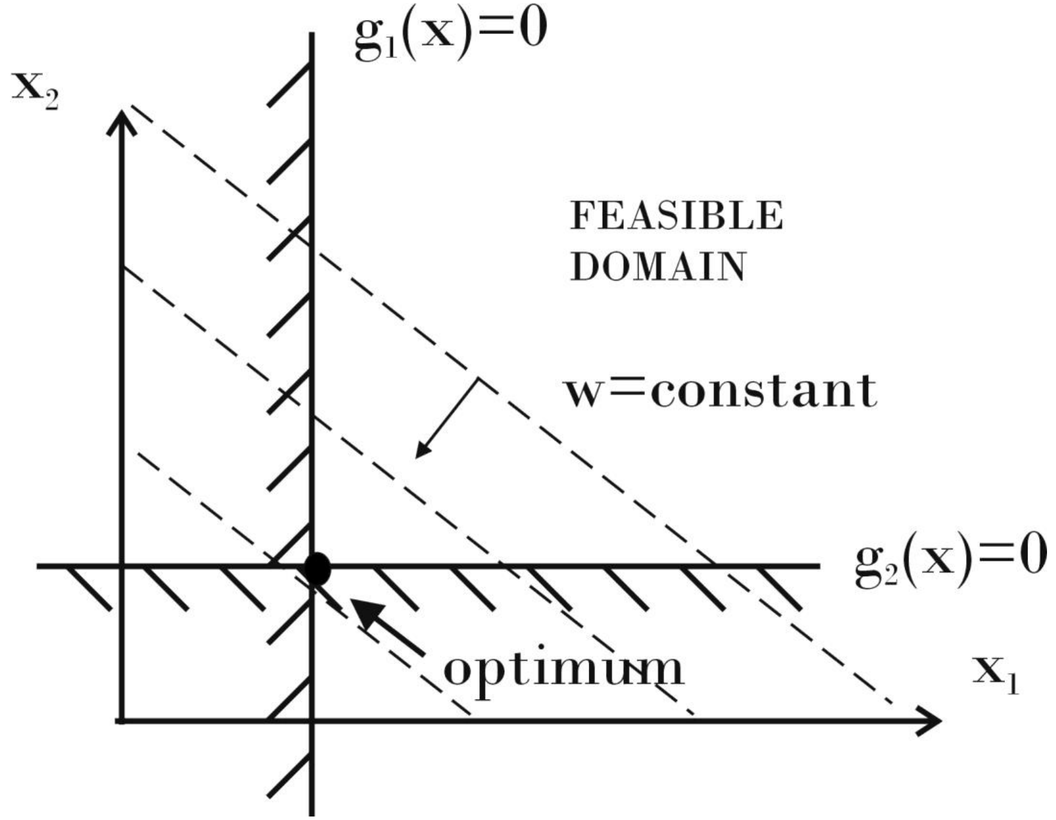
$$W = \rho l \sqrt{2}(x_1 + x_2)$$

$$\sigma_1 = \frac{\sqrt{2}\,P}{2\,x_1} \leq \bar{\sigma}$$

$$\sigma_2 = -\frac{\sqrt{2}\,P}{2\,x_2} \leq \bar{\sigma}$$

$$g_1(x_1) \equiv \frac{\sqrt{2}\,P}{2\,\bar{\sigma}} - x_1 \leq 0$$

$$g_2(x_2) \equiv \frac{\sqrt{2}\,P}{2\,\bar{\sigma}} - x_2 \leq 0$$



TWO-BAR TRUSS

- The design problems requires that we find the vector **x*** for a structure minimizing the weight subject to stress constraints σ.
- Because the structure is determinate each bar can be sized separately at the minimum value of the cross section to carry the applied loads.
- They optimized cross sectional areas are given by

$$x_i^\star = \frac{N_i}{\bar{\sigma}_i}$$

TWO-BAR TRUSS

- If we take as starting point the set of bar cross sections $\mathbf{x}^{(0)}$, and that we compute the related internal bar forces $N_i^{(0)}$ and stresses $\sigma_i^{(0)}$, the optimized cross sections that lead to reach the maximum allowable stress σ are given by the above formula:

$$x_i^\star = \frac{\sigma_i^{(0)}}{\bar{\sigma}} \, x_i^{(0)} \qquad i = 1, 2 \ldots n$$

- which we can immediately recognize as this stress-ratioing resuming reserving formula of the fully stressed design concepts, familiar in many practical application of machine design.

TWO BAR TRUSS

- Returning to the simple two bar truss problem with 45 degrees angle
- If a minimum weight design is now sought subject to limitation on the bar stresses, then the constraints imposed on the design problem becomes

$$\sigma_1 = \frac{\sqrt{2}\,P}{2\,x_1} \leq \bar{\sigma} \qquad\qquad g_1(x_1) \equiv \frac{\sqrt{2}\,P}{2\,\bar{\sigma}} - x_1 \leq 0$$

$$\sigma_2 = -\frac{\sqrt{2}\,P}{2\,x_2} \geq -\bar{\sigma} \qquad\qquad g_2(x_2) \equiv \frac{\sqrt{2}\,P}{2\,\bar{\sigma}} - x_2 \leq 0$$

- The design restrictions are linear

TWO-BAR TRUSS

- The problem is linear, and the constraints are parallel to the axis defined by the design variables x_1, and x_2.
- It is clearly seen that each of these variable is associated with one and only one constraint and then the optimum design occurs at a vertex in design space.
- The optimum can therefore be fought by seeking to simultaneously satisfy the design constraints rather than seeking to actually minimize the objective function.

TWO-BAR TRUSS

- In later developments, we will show it is convenient to linearize the design constraints by using design variable defined as the reciprocal of the bar cross sectional areas.

$$z_i = \frac{1}{x_i} \qquad i = 1, 2 \ldots n$$

- The weight now becomes a nonlinear function

$$W = \sum_{i=1}^{n} \frac{\rho_i \, l_i}{z_i}$$

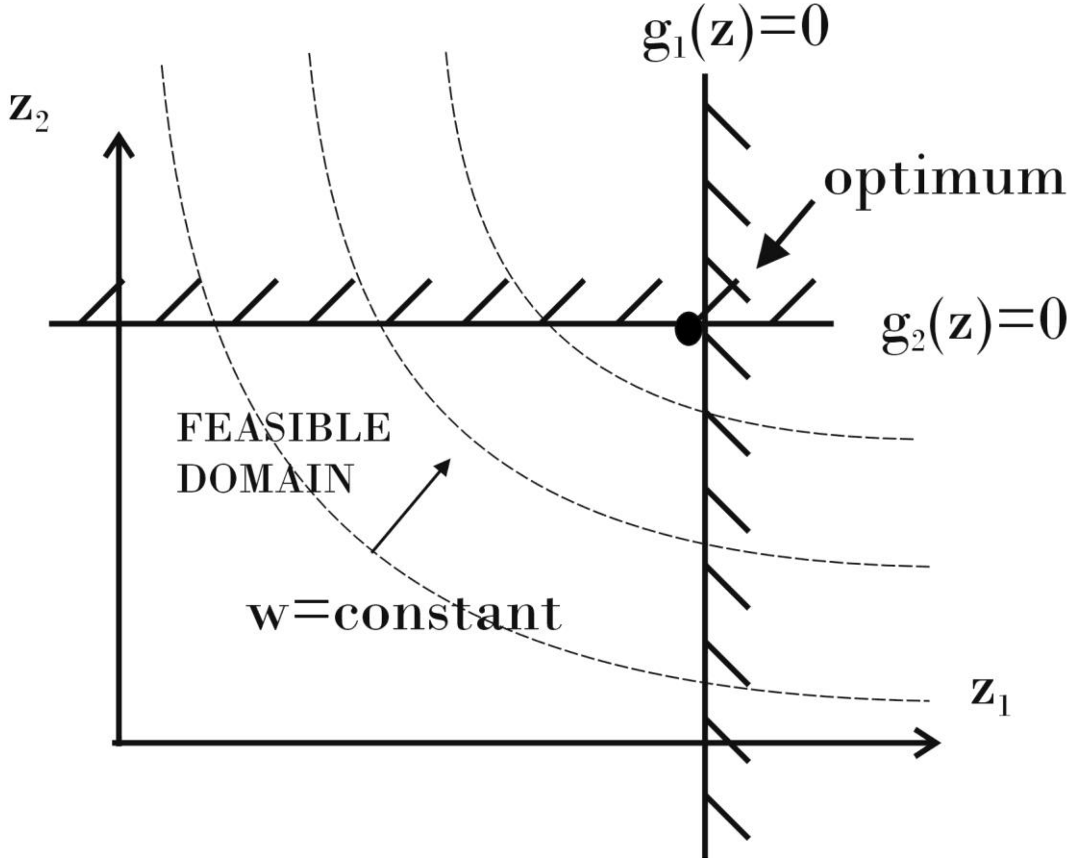
- The stress constraints remain linear function of the reciprocal variables

$$-\bar{\sigma} \leq N_i \, z_i \leq \bar{\sigma}_i$$

$$\sigma_1 = \frac{\sqrt{2} \, P \, z_1}{2} \leq \bar{\sigma}$$

$$\sigma_2 = -\frac{\sqrt{2} \, P \, z_2}{2} \geq -\bar{\sigma}$$

TWO-BAR TRUSS



$$g_1(x_1) \equiv \frac{\sqrt{2}\,P}{2}\,z_1 - \bar{\sigma} \leq 0$$

$$g_2(x_2) \equiv \frac{\sqrt{2}\,P}{2}\,z_2 - \bar{\sigma} \leq 0$$

TWO-BAR TRUSS

- We can continue our study of structural optimality theory by considering a statically determinate truss structure subject to constraints on specified nodal displacements.

- We seek for the minimum of the objective function, that is the structural weight, while satisfying to restriction over the two components of the nodal displacement.

$$\min \quad W = \sum_{i=1}^{2} \rho_i l_i x_i$$

$$\textbf{s.t.}: \quad u_x \leq \bar{u}_x$$

$$u_y \leq \bar{u}_y$$

$$0 \leq x_i \quad i = 1, 2$$

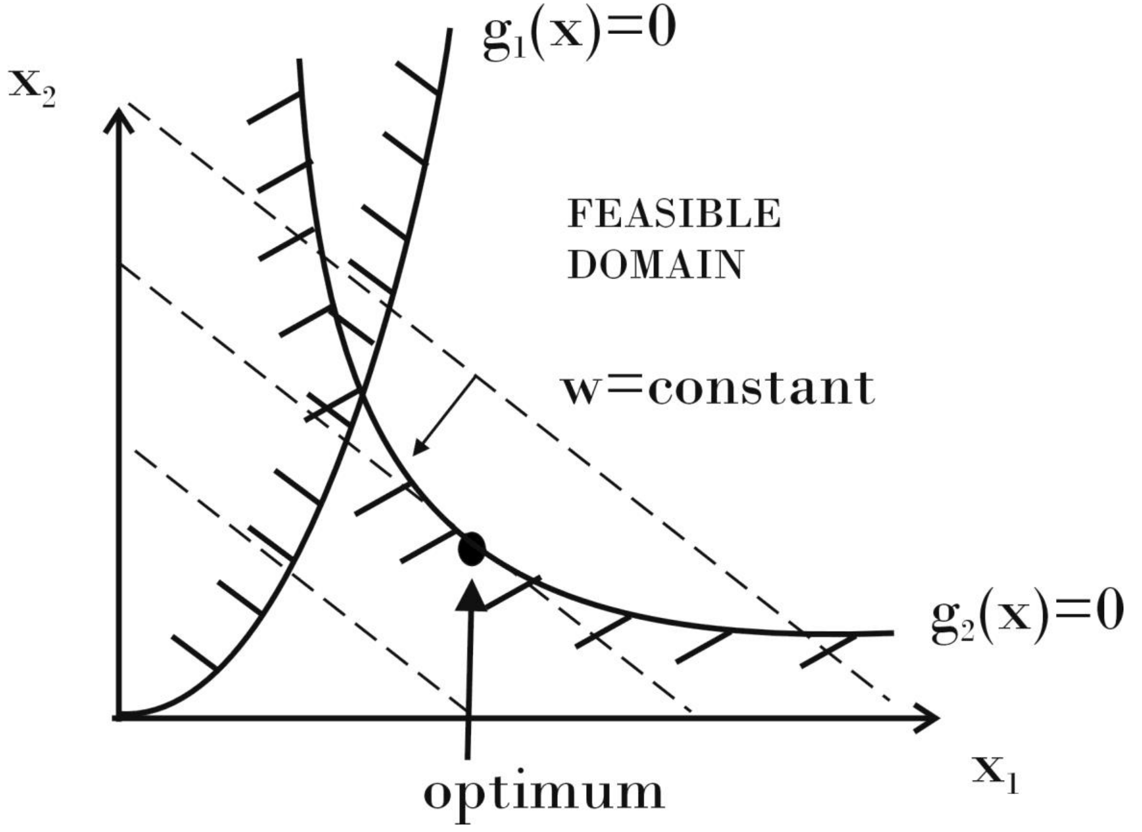
TWO-BAR TRUSS

- If we assume that the same material is used in each bar, the problem statement reads in the case of the two-bar truss with 45 degree:

$$
\begin{aligned}
\min \quad & W = \sqrt{2}\, l\, \rho\, (x_1 + x_2) \\
\textbf{s.t.}: \quad & g_1(x_1, x_2) \equiv \frac{\sqrt{2}\, P\, l}{2\, E} \left(\frac{1}{x_1} + \frac{1}{x_2} \right) - \bar{u}_x \leq 0 \\
& g_2(x_1, x_2) \equiv \frac{\sqrt{2}\, P\, l}{2\, E} \left(\frac{1}{x_1} - \frac{1}{x_2} \right) - \bar{u}_y \leq 0 \\
& 0 \leq x_i \quad i = 1, 2
\end{aligned}
$$

TWO-BAR TRUSS

- If the cross-sectional areas are taken as design variables, the problem may not be convex as it is usually illustrated by returning to the two-bar example.



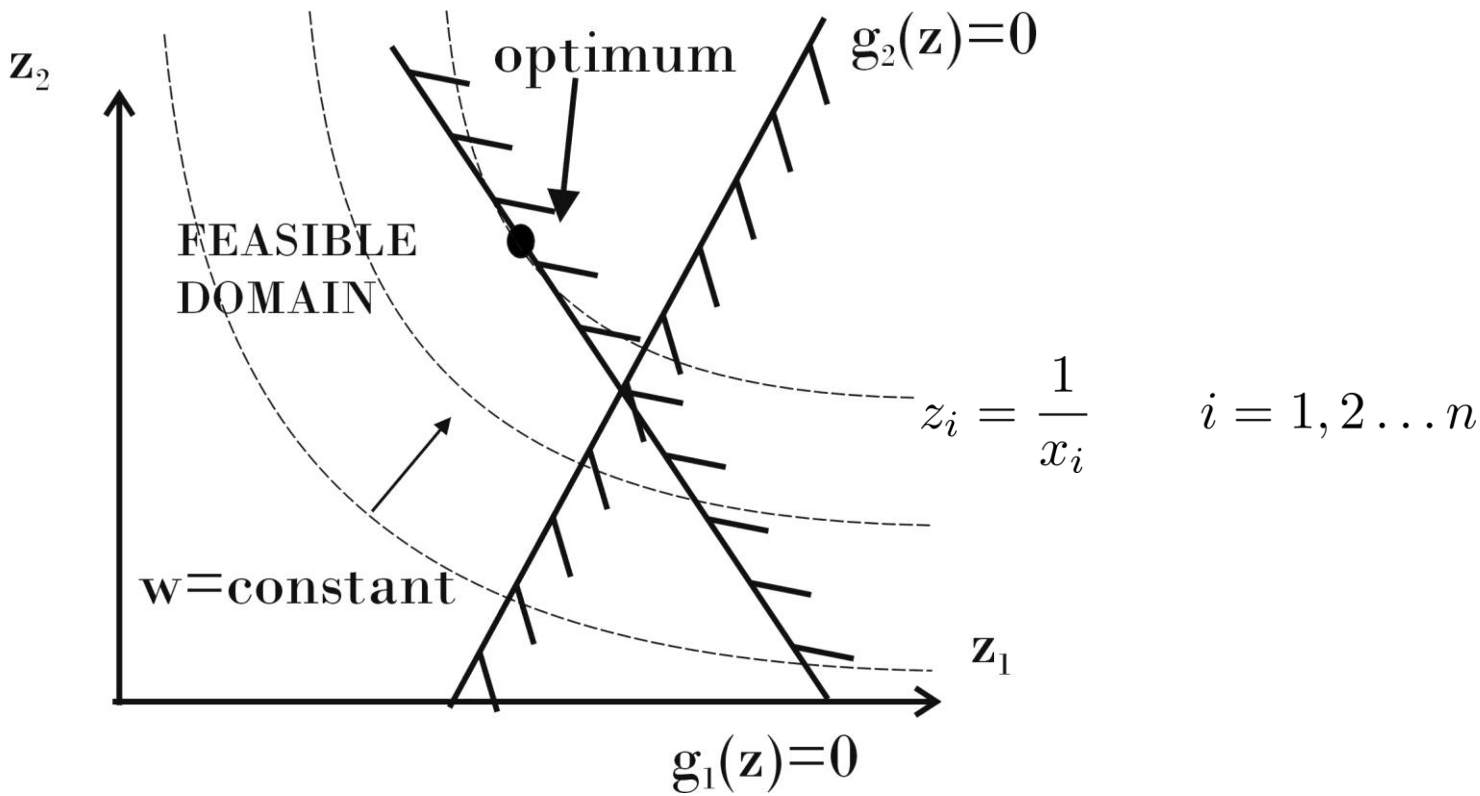
TWO-BAR TRUSS

- To circumvent this difficulty, we can take the hint given in the previous section and use the reciprocal of the cross-sectional areas as design variables.

- The two-bar truss displacement constraint problem now becomes

$$
\begin{aligned}
\min \quad & W = \sqrt{2}\, l\, \rho\, \left(\frac{1}{z_1} + \frac{1}{z_2}\right) \\
\textbf{s.t.}: \quad & g_1(z_1, z_2) \equiv \frac{\sqrt{2}\, P\, l}{2\, E}(z_1 + z_2) - \bar{u}_x \leq 0 \\
& g_2(z_1, z_2) \equiv \frac{\sqrt{2}\, P\, l}{2\, E}(z_1 - z_2) - \bar{u}_y \leq 0 \\
& 0 \leq z_i \quad i = 1, 2
\end{aligned}
$$

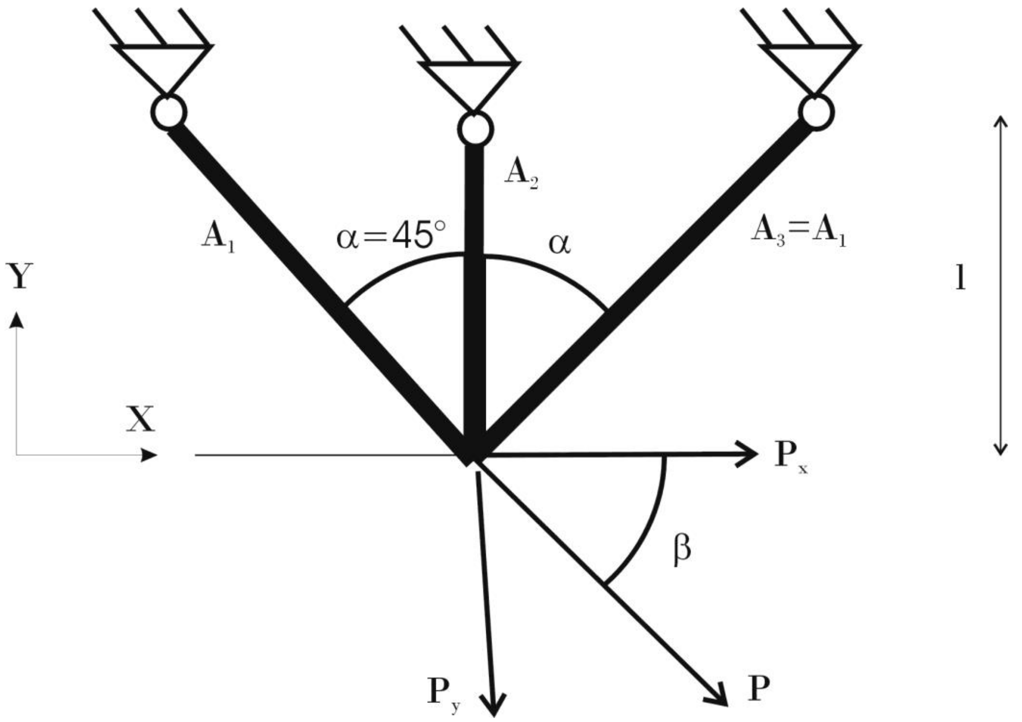
TWO-BAR TRUSS



THREE BAR TRUSS PROBLEM

THREE BAR TRUSS

- Famous example



$$E = 70.000\ MPa \qquad l = 500\ mm$$
$$\nu = 0.3 \qquad \alpha = 45^\circ$$
$$\rho = 2800\ kg/m^3 \qquad \beta = 45^\circ$$
$$\bar{\sigma} = 500MPa \qquad \underline{\sigma} = -250MPa$$
$$P = 10.000\ N$$
$$P_x = P\cos\beta$$
$$P_y = P\sin\beta$$

- The load case is given by a given force P applied with an orientation β with respect to the horizontal line.

THREE BAR TRUSS

- The design variables: the bar cross sections

$$x_i = A_i \quad i = 1, 2, 3$$

  - Geometrical symmetry conditions

$$x_1 = A_1 = A_3$$
$$x_2 = A_2$$

- Problem statement: mass minimization subject to stress constraints

$$\begin{aligned} \min \quad & W(A_1, A_2) \\ \textbf{s.t.}: \quad & \underline{\sigma} \leq \sigma_1(A_1, A_2) \leq \bar{\sigma} \\ & \underline{\sigma} \leq \sigma_2(A_1, A_2) \leq \bar{\sigma} \\ & \underline{\sigma} \leq \sigma_3(A_1, A_2) \leq \bar{\sigma} \\ & 0 \leq A_1, A_2 \end{aligned}$$

THREE BAR TRUSS

- The mass of the truss can be easily expressed as the volume of the bar times the density of the material:

$$W(A_1, A_2) = \rho\, l_1\, A_1 + \rho\, l_2\, A_2 + \rho\, l_3\, A_3$$

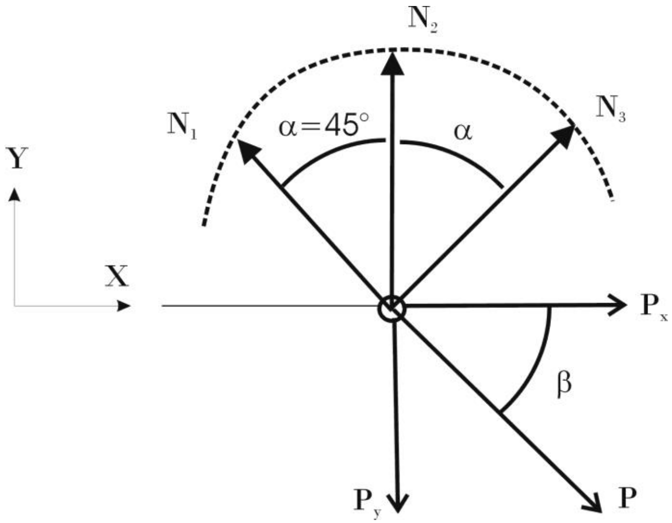
  - In particular case

$$W(A_1, A_2) = \rho\, l\, (2\sqrt{2}\, A_1 + A_2)$$

THREE BAR TRUSS

- Equilibrium:
  - Can be obtained by looking at the free body diagram of the free node

$$
\begin{aligned}
-N_1 \sin\alpha + N_3 \sin\alpha + P_x &= 0 \\
N_1 \cos\alpha + N_2 + N_3 \cos\alpha - P_y &= 0
\end{aligned}
$$



  - The problem is hyper static which means that the equilibrium equations are not sufficient to determine all unknown. We have only two equilibrium equations and three internal forces. To determine the bar loads, it is necessary to express the compatibility of the displacements at the free node.

THREE BAR TRUSS

- Compatibility
  - The elongation in terms of the longitudinal strains:

$$\Delta l_1 = \epsilon_1 \, l_1 = \epsilon_1 \, l \, \sqrt{2}$$
$$\Delta l_2 = \epsilon_2 \, l_2 = \epsilon_2 \, l$$
$$\Delta l_3 = \epsilon_3 \, l_3 = \epsilon_3 \, l \, \sqrt{2}$$

  - Using the material behaviour, i.e. Hook's law relating stresses and strains, in the bars:

$$\sigma_i = E \, \epsilon_i$$
$$N_i = \sigma_i \, A_i$$

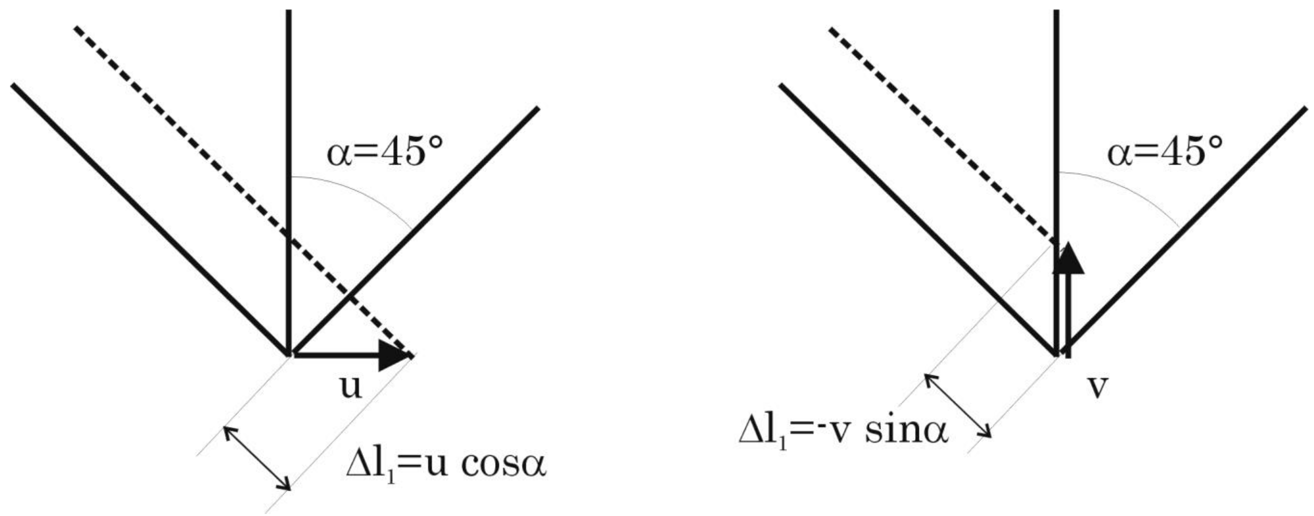
$$\Delta l_1 = N_1 \, \frac{\sqrt{2}\, l}{E A_1}$$
$$\Delta l_2 = N_2 \, \frac{l}{E A_2}$$
$$\Delta l_3 = N_3 \, \frac{\sqrt{2}\, l}{E A_3}$$

THREE BAR TRUSS

- Express the compatibility of the displacements with the elongations. Let's denote by (u,v) the displacement of the lower node in the positive x and y directions.

  - Elongation in bar 1 is related to the node displacements (u,v).

$$\Delta l_1 = u \cos 45^\circ - v \sin 45^\circ = u\,\frac{\sqrt{2}}{2} - v\frac{\sqrt{2}}{2}$$



THREE BAR TRUSS

- More generally if the bar makes an angle θ with the horizontal direction, the displacement (u,v) along the local axis of the bar (x′,y′) can be obtained using the formula of the rotation with respect to the structural reference frame (x,y).

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u'_i = \Delta l_i \\ v'_i \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\Delta l_i = u\cos\theta + v\sin\theta$$

- Apply to bar 1: θ=-45°

$$\Delta l_1 = u\frac{\sqrt{2}}{2} - v\frac{\sqrt{2}}{2}$$

- Apply to bar 2: θ=-90°

$$\Delta l_2 = -v$$

- Apply to bar 3: θ=-135°

$$\Delta l_3 = -u\frac{\sqrt{2}}{2} - v\frac{\sqrt{2}}{2}$$

THREE BAR TRUSS

- Relations between bar elongations and node displacement

$$\Delta l_1 = u\frac{\sqrt{2}}{2} - v\frac{\sqrt{2}}{2}$$
$$\Delta l_2 = -v$$
$$\Delta l_3 = -u\frac{\sqrt{2}}{2} - v\frac{\sqrt{2}}{2}$$

- Combining with relations (1) and (3) and inserting (2) , it comes

$$\Delta l_1 \ + \ \Delta l_3 = \sqrt{2}\Delta l_2$$

THREE BAR TRUSS

- Combining compatibility and behaviour relations yields

$$\left.\begin{aligned}\Delta l_1 &= N_1\,\frac{\sqrt{2}\,l}{EA_1}\\ \Delta l_2 &= N_2\,\frac{l}{EA_2}\\ \Delta l_3 &= N_3\,\frac{\sqrt{2}\,l}{EA_3}\\ \Delta l_1 + \Delta l_3 &= \sqrt{2}\Delta l_2\end{aligned}\right\} \Rightarrow \frac{N_1}{A_1}+\frac{N_3}{A_3}=\frac{N_2}{A_2}$$

THREE BAR TRUSS

- To determine the three member forces N_1, N_2 and N_3, one has to solve the set of three equations:

$$-N_1 \sin\alpha + N_3 \sin\alpha + P_x = 0$$

$$N_1 \cos\alpha + N_2 + N_3 \cos\alpha - P_y = 0$$

$$\frac{N_1}{A_1} - \frac{N_2}{A_2} + \frac{N_3}{A_3} = 0$$

THREE BAR TRUSS

- Solution in the particular case β=45°

$$P_x = P_y = \frac{\sqrt{2}}{2}\, P$$

- That is

$$-N_1 + N_3 + P = 0$$

$$N_1 + \sqrt{2}\, N_2 + N_3 - P = 0$$

$$\frac{N_1}{A_1} - \frac{N_2}{A_2} + \frac{N_3}{A_3} = 0$$

- Summing the two first equations yields:

$$N_1 = N_3 + P$$

- Inserting into the second equation provides

$$N_2 = -\sqrt{2}\, N_3$$

- Substitute these results into the third equation

$$N_3 = -\frac{P\, A_2}{\sqrt{2}(A_1 + \sqrt{2}\, A_2)}$$

THREE BAR TRUSS

- From the value of N_3,

$$N_3 = -\frac{P\,A_2}{\sqrt{2}(A_1 + \sqrt{2}\,A_2)}$$

- one obtains the value of N_1 and N_2:

$$N_2 = -\sqrt{2}\,N_3 = \frac{P\,A_2}{A_1 + \sqrt{2}\,A_2}$$

$$N_1 = N_3 + P = \frac{P\,(\sqrt{2}A_1 + A_2)}{\sqrt{2}\,(A_1 + \sqrt{2}\,A_2)}$$

THREE BAR TRUSS

- The optimization problem writes :

$$\min \quad W(A_1, A_2) = \rho\, l_1\, A_1 + \rho\, l_2\, A_2 + \rho\, l_3\, A_1$$

$$\textbf{s.t.} : \quad \sigma_1 = \frac{N_1}{A_1} = \frac{P\,(\sqrt{2}A_1 + A_2)}{\sqrt{2}\, A_1\, (A_1 + \sqrt{2}\, A_2)} \leq 500$$

$$\sigma_2 = \frac{N_2}{A_2} = \frac{P}{(A_1 + \sqrt{2}\, A_2)} \leq 500$$

$$\sigma_3 = \frac{N_3}{A_3} = -\frac{P\, A_2}{\sqrt{2}\, A_1\, (A_1 + \sqrt{2}\, A_2)} \geq -250$$

THREE BAR TRUSS

- In the case of particular values of the parameters l=500 mm, α=45°, and P=10.000 N, the optimization problem statement becomes:

$$\min \quad W(A_1, A_2) = \rho\, l\, (2\sqrt{2}\, A_1 + A_2)$$

$$\mathbf{s.t.}: \quad \sigma_1 = \frac{(\sqrt{2} A_1 + A_2)}{\sqrt{2}\, A_1\, (A_1 + \sqrt{2}\, A_2)} \leq 1/20$$

$$\sigma_2 = \frac{1}{(A_1 + \sqrt{2}\, A_2)} \leq 1/20$$

$$\sigma_3 = \frac{A_2}{\sqrt{2}\, A_1\, (A_1 + \sqrt{2}\, A_2)} \leq 1/40$$

THREE BAR TRUSS

- Three bar truss structure
  - Objective function: lines with decreasing mass towards point (0,0)

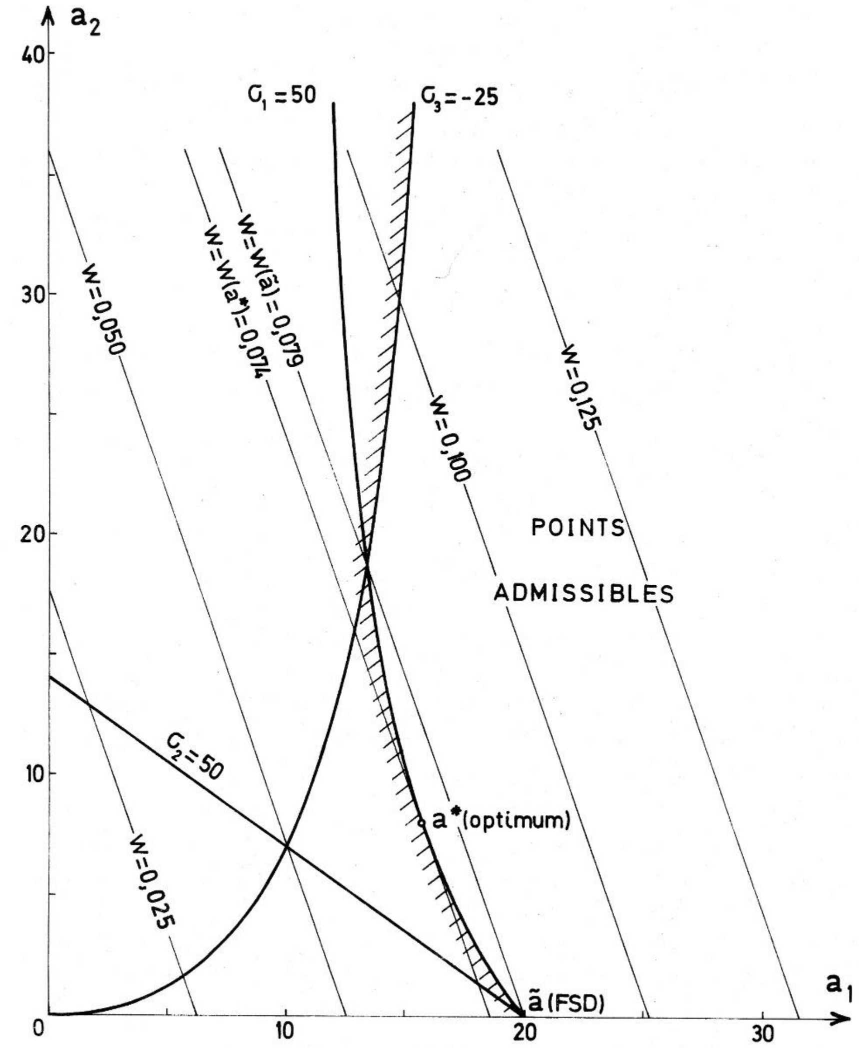
$$2\sqrt{2}\,A_1 \;+\; A_2$$

  - Stress constraint 2 is a linear

$$\sigma_2 = \frac{1}{(A_1 + \sqrt{2}\,A_2)} \leq 1/20$$

$$\Longleftrightarrow (A_1 + \sqrt{2}\,A_2) \geq 20.$$

  - Stress constraint 1 is clearly active at optimum



THREE BAR TRUSS

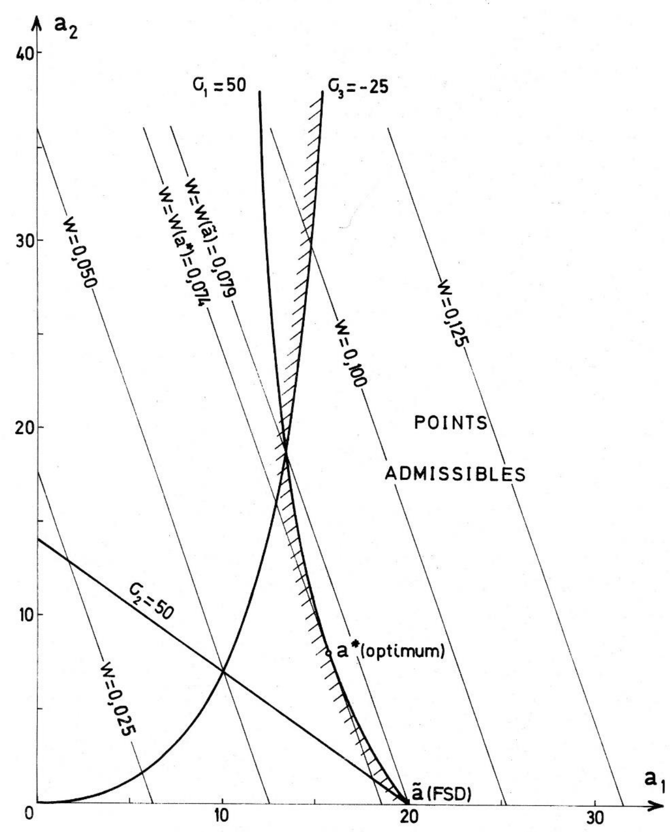
- Analytical solution of the optimization problem

$$\min \quad W(A_1, A_2) = \rho\, l\, (2\sqrt{2}\, A_1 + A_2)$$

$$\textbf{s.t.}: \quad \sigma_1 = P\,\frac{(\sqrt{2}A_1 + A_2)}{\sqrt{2}\,A_1\,(A_1 + \sqrt{2}\,A_2)} \leq 500$$

$$\sigma_2 = P\,\frac{1}{(A_1 + \sqrt{2}\,A_2)} \leq 500$$

$$\sigma_3 = P\,\frac{A_2}{\sqrt{2}\,A_1\,(A_1 + \sqrt{2}\,A_2)} \leq 250$$



- Only restriction 1 is active

$$\min \quad W(A_1, A_2) = \rho\, l\, (2\sqrt{2}\, A_1 + A_2)$$

$$\textbf{s.t.}: \quad \sigma_1 = P\,\frac{(\sqrt{2}A_1 + A_2)}{\sqrt{2}\,A_1\,(A_1 + \sqrt{2}\,A_2)} \leq 500$$

THREE BAR TRUSS

- The problem can be rewritten to be easier to solve

$$\begin{aligned} \min \quad & W(A_1, A_2) = \rho\, l\, (2\sqrt{2}\, A_1 + A_2) \\ \textbf{s.t.}: \quad & \sigma_1 = P\, \frac{(\sqrt{2}A_1 + A_2)}{\sqrt{2}\, A_1\, (A_1 + \sqrt{2}\, A_2)} \leq 500 \end{aligned}$$

$$\begin{aligned} \min \quad & \rho\, l\, (2\sqrt{2}\, A_1 + A_2) \\ \textbf{s.t.}: \quad & \frac{(\sqrt{2}A_1 + A_2)}{A_1\, (A_1 + \sqrt{2}\, A_2)} \leq \frac{\sqrt{2}\, 500}{P} \end{aligned}$$

$$\begin{aligned} \min \quad & \rho\, l\, (2\sqrt{2}\, A_1 + A_2) \\ \textbf{s.t.}: \quad & (\sqrt{2}A_1 + A_2) \leq \frac{\sqrt{2}\, 500}{P}\, A_1\, (A_1 + \sqrt{2}\, A_2) \end{aligned}$$

THREE BAR TRUSS

- The problem can be rewritten to be easier to solve

$$\begin{aligned} \min \quad & (2\sqrt{2}\,x_1 + x_2) \\ \textbf{s.t.}: \quad & (\sqrt{2}x_1 + x_2) \leq \frac{\sqrt{2}\,500}{P}\,(x_1^2 + \sqrt{2}\,x_1\,x_2) \end{aligned}$$

- Let's denote $C = \dfrac{\sqrt{2}\,500}{P}$

$$\begin{aligned} \min \quad & (2\sqrt{2}\,x_1 + x_2) \\ \textbf{s.t.}: \quad & (\sqrt{2}x_1 + x_2) - C\,x_1\,(x_1 + \sqrt{2}\,x_2) \leq 0 \end{aligned}$$

- Lagrangian function

$$L(x_1, x_2, \lambda) = (2\sqrt{2}\,x_1 + x_2) + \lambda\,\{(\sqrt{2}x_1 + x_2) - C\,(x_1^2 + \sqrt{2}\,x_1\,x_2)\}$$

THREE BAR TRUSS

- Lagrangian function

$$L(x_1, x_2, \lambda) = (2\sqrt{2}\,x_1 + x_2) + \lambda\,\{(\sqrt{2}x_1 + x_2) - C\,x_1\,(x_1 + \sqrt{2}\,x_2)\}$$

- KKT conditions

$$\frac{\partial L}{\partial x_1} = 2\sqrt{2} + \lambda(\sqrt{2} - 2\,C\,x_1 - \sqrt{2}\,C\,x_2) = 0$$

$$\frac{\partial L}{\partial x_2} = 1 + \lambda(1 - \sqrt{2}\,C\,x_1) = 0$$

- Solving KKT conditions for given λ:

$$1 + \lambda(1 - \sqrt{2}\,C\,x_1) = 0$$

$$\Longleftrightarrow 1 + \lambda = \sqrt{2}\,\lambda\,C\,x_1$$

$$\Longleftrightarrow x_1 = \frac{1+\lambda}{\sqrt{2}\,\lambda\,C}$$

THREE BAR TRUSS

- Solving KKT conditions for given λ:

$$2\sqrt{2} + \lambda(\sqrt{2} - 2\,C\,x_1 - \sqrt{2}\,C\,x_2) = 0$$

$$\iff 2\sqrt{2} + \sqrt{2}\,\lambda - 2\,C\,\lambda\,\frac{1+\lambda}{\sqrt{2}C\lambda} - \lambda\,\sqrt{2}\,C\,x_2) = 0$$

$$\iff 2\sqrt{2} + \sqrt{2}\,\lambda - \sqrt{2}\,(1+\lambda) - \lambda\,\sqrt{2}\,C\,x_2) = 0$$

$$\iff 2 + \lambda - (1+\lambda) - \lambda\,C\,x_2) = 0$$

$$\iff 1 - \lambda\,C\,x_2 = 0$$

$$\iff x_2 = \frac{1}{\lambda\,C}$$

- For given λ, we get the optimal values of the design variables

$$x_1 = \frac{\lambda+1}{2\,C\,\lambda} \qquad x_2 = \frac{1}{C\,\lambda}$$

THREE BAR TRUSS

- Determine Lagrange multiplier using the constraint

$$\frac{\partial L}{\partial \lambda} = (\sqrt{2} x_1 + x_2) - C\,(x_1^2 + \sqrt{2}\, x_1\, x_2) = 0$$

- Inserting the value of the design variables x_1 and x_2, it comes:

$$\begin{aligned}
& (\sqrt{2} x_1 + x_2) - C\,(x_1^2 + \sqrt{2}\, x_1\, x_2) = 0 \\
\Longleftrightarrow\ & (\sqrt{2}\frac{1+\lambda}{\sqrt{2} C \lambda} + \frac{1}{C\lambda}) - C\,\frac{(1+\lambda)^2}{2C^2\lambda^2} - C\,\sqrt{2}\,\frac{1+\lambda}{\sqrt{2} C\lambda}\,\frac{1}{C\lambda} = 0 \\
\Longleftrightarrow\ & \frac{1+\lambda}{C\lambda} + \frac{1}{C\lambda} - \frac{(1+\lambda)^2}{2C\lambda^2} - \frac{1+\lambda}{C\lambda^2} = 0 \\
\Longleftrightarrow\ & \frac{2+\lambda}{2C\lambda^2}\,2\lambda - \frac{(1+2\,\lambda+\lambda^2+2+2\lambda)}{2C\lambda^2} = 0 \\
\Longleftrightarrow\ & \frac{(4\lambda + 2\lambda^2 - 3 - 4\lambda - \lambda^2)}{2C\lambda^2} = 0 \\
\Longleftrightarrow\ & \frac{\lambda^2 - 3}{2C\lambda^2} = 0
\end{aligned}$$

THREE BAR TRUSS

- Determine Lagrange multiplier using the constraint

$$(\lambda^\star)^2 = 3$$

$$\lambda^\star = \sqrt{3} > 0\,!$$

- Come back to the primal variable design optimal values

$$C = \frac{500\,\sqrt{2}}{P} = \frac{\sqrt{2}\,500}{10.000} = \frac{\sqrt{2}}{20}$$

$$x_1^\star = \frac{20}{\sqrt{2}}\,\frac{\sqrt{3}+1}{\sqrt{2}\,\sqrt{3}} = 15,7735$$

$$x_2^\star = \frac{20}{\sqrt{2}\,\sqrt{3}} = 8,1650$$

THREE BAR TRUSS

- Solution

$$\lambda^{\star} = \sqrt{3}$$

$$x_1^{\star} = 15,7735$$

$$x_2^{\star} = 8,1650$$