INTRODUCTION
TO
THE FINITE ELEMENT METHOD

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HISTORICAL PERSPECTIVE

ANALYTICAL SOLUTIONS USING HYPOTHESES

- For a time, structural analysis has made use of some hypotheses concerning the diffusion of forces or kinematics of the deformation of structural parts to find analytical or numerical solutions to complex problems.

- For instance, the classical NAVIER beam theory is based on the following assumptions:
  - The stress state is unidimensional,
  - Cross sections remain plane during deformation,
  - Cross sections remain orthogonal to the neutral axis.

ANALYTICAL SOLUTIONS USING HYPOTHESES

- Assumptions of the classical NAVIER beam theory:
  - The stress state is unidimensional,

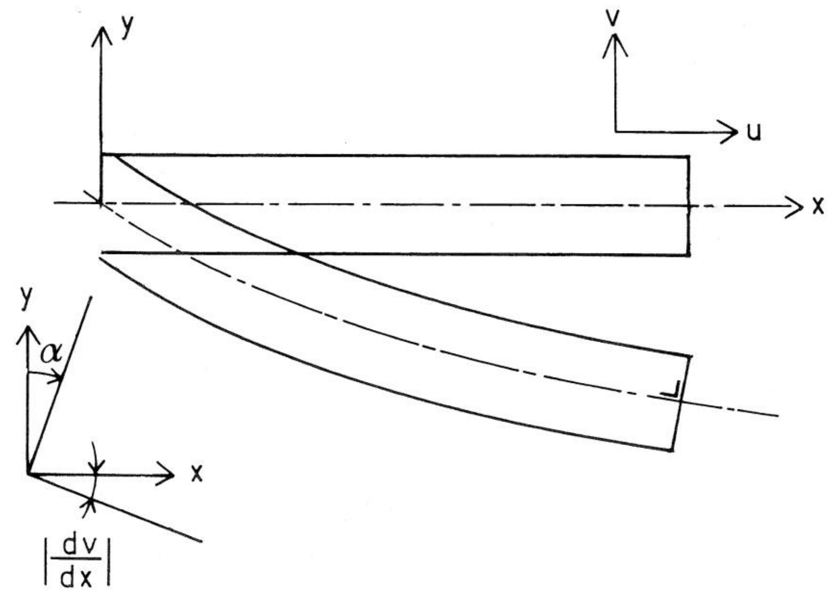
$$\sigma = E\epsilon$$

  - Cross sections remain plane during deformation,

$$u(x, y) = \alpha(x)\, y$$

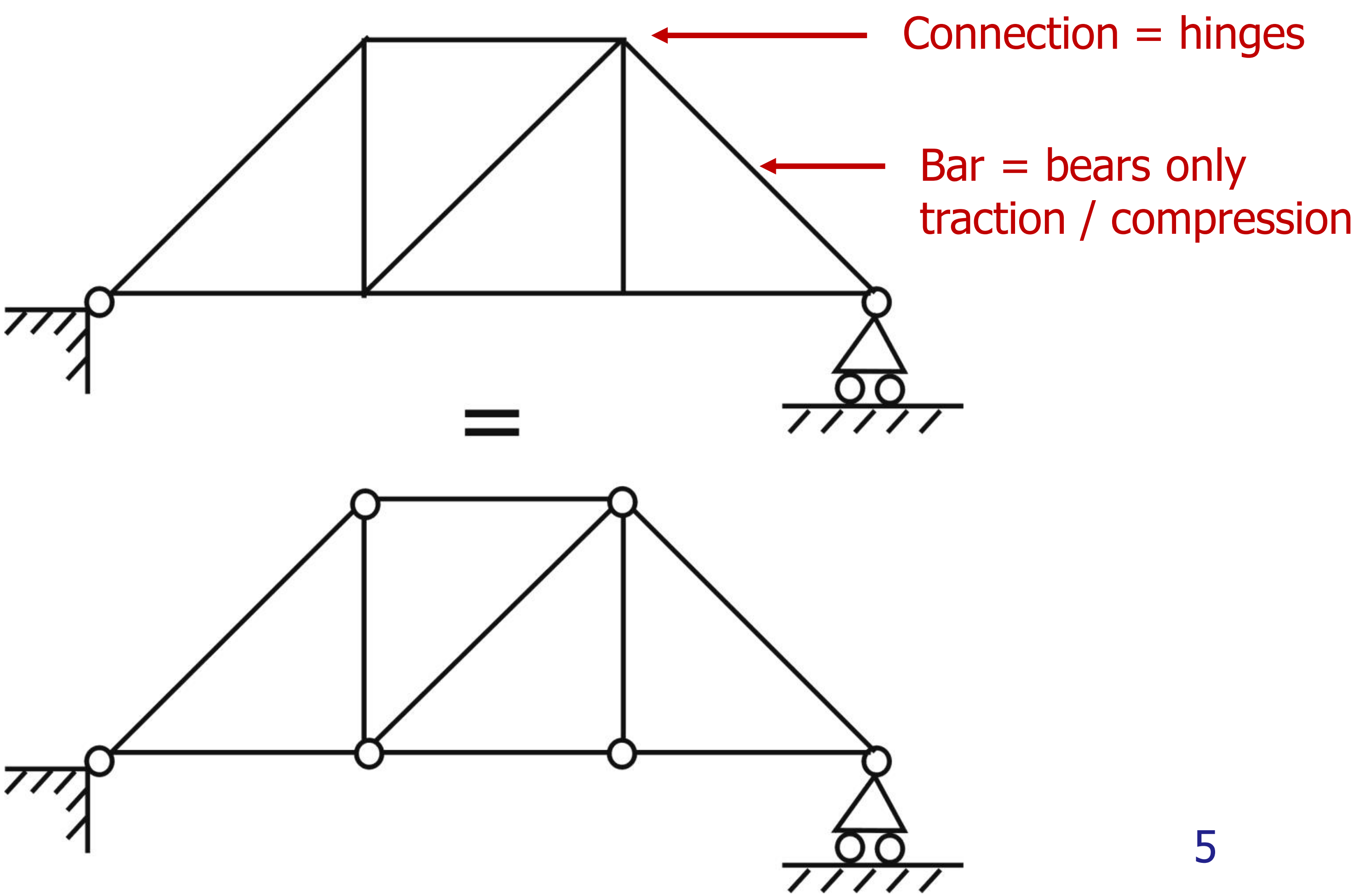
  - Cross sections remain orthogonal to the neutral

$$\alpha(x) + \frac{\partial v}{\partial x} = 0$$



BAR AND TRUSS MODEL

- Another famous case of approximation is the articulated truss.



BAR AND TRUSS MODEL

- Another famous case of approximation is the articulated truss.
  - An idealization consisting to consider that each bar only works in stretching while carrying negligible bending moments
  - The end connections, called nodes, are able to transmit poorly any moment and it thus supposed that they are close to pin join connection transmitting only traction and compressive forces

BAR AND TRUSS MODEL

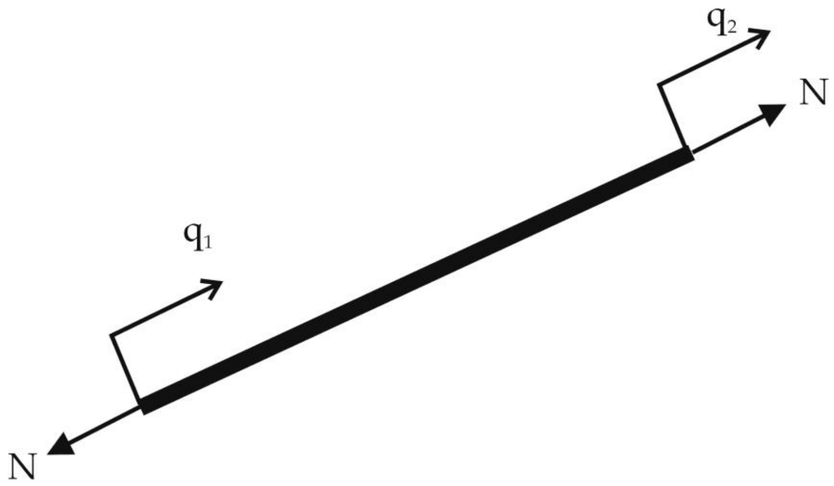
- Bar model:
  - Stress state and effort

$$N = A\sigma$$

  - Strain

$$\varepsilon = \frac{q_1 - q_2}{L}$$

  - Behavior: Hooke law



$$\sigma = E\varepsilon$$

  - Bar stiffness

$$N = k(q_1 - q_2) \quad k = \frac{EA}{L}$$

BAR AND TRUSS MODEL

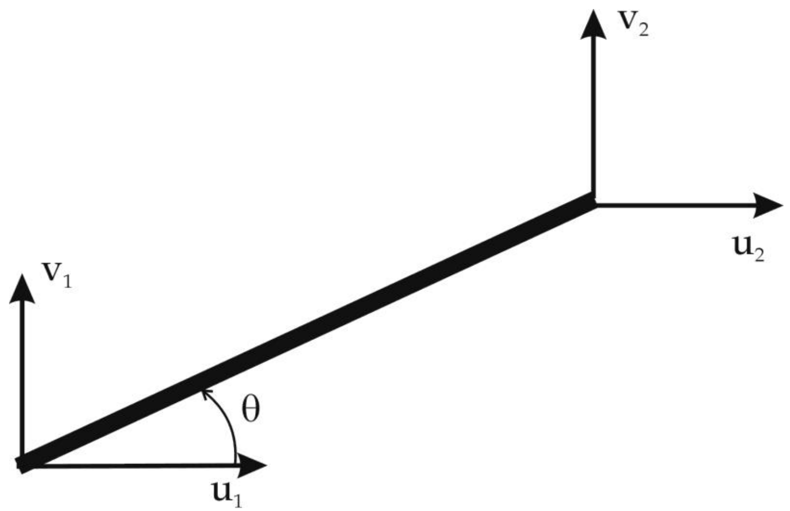
- Assembling the bars into the structure

  - Displacement of the nodes in the structural frame

$$q_i = u_i \cos\theta \; + \; v_i \sin\theta$$

  - Forces components

$$X = N\cos\theta \qquad \text{and} \qquad Y = N\sin\theta$$



BAR AND TRUSS MODEL

- Now assembling the different bars may be performed by two ways:
  - Displacement method: Express that the displacement of each node is uniquely determined and then find the value of the nodal displacements which leads to the equilibrium.
  - Force method: Express the sum of the forces at each node is equal to zero or to the applied nodes on the considered node. The solution of these equations is undetermined, depending on the arbitrary self-stresses whose number is equal to the hyperstaticity index of the structure. The self-stresses are adjusted to ensure the uniqueness of the displacement at the nodes.
- Both methods lead to a matrix system to be solved.

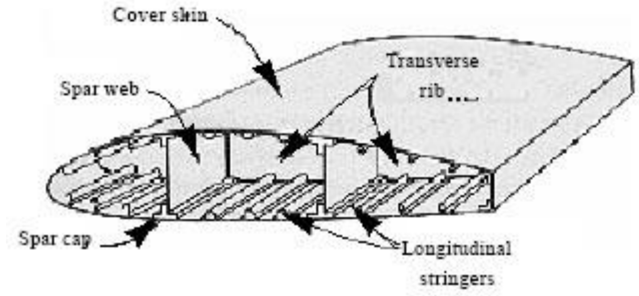
BAR AND SHEAR PANELS

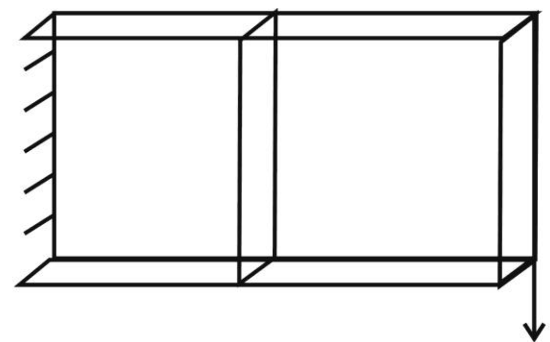
- Aeronautical structures use stiffened panels.

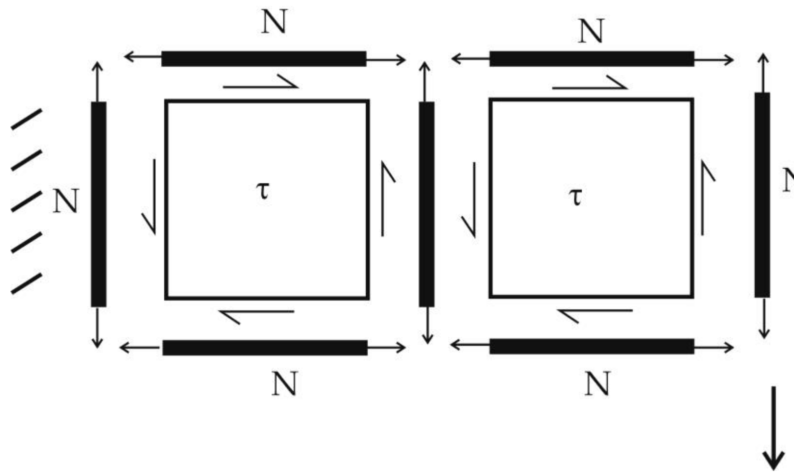
- A classical approximation consists in considering that a panel can only resist to shear loads.

- Stretching is only supported by stiffeners (stringers and frames) which are modelled as bars.

- Associating displacements to the shear forces, one comes to a matrix system to solve.





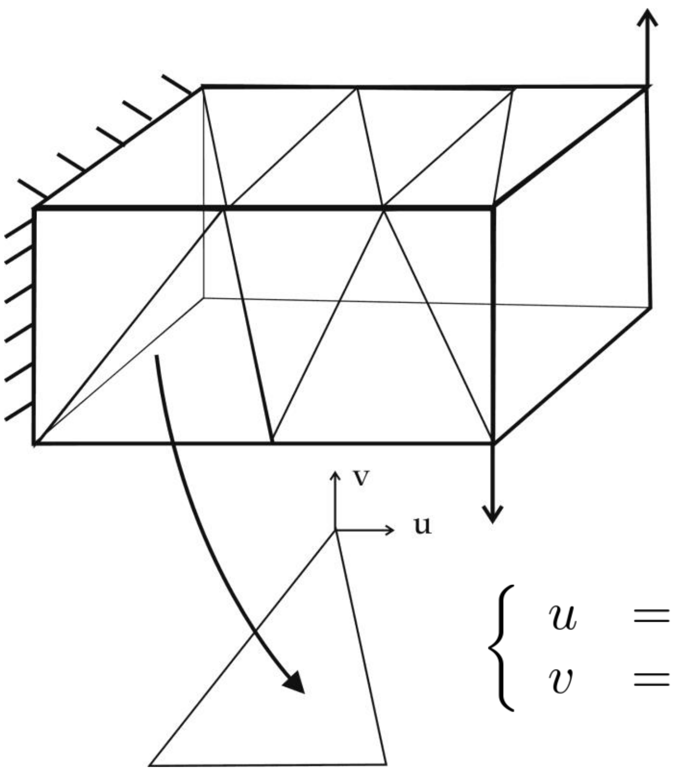


MATRIX STRUCTURAL ANALYSIS

- These idealizations belong to the so-called matrix structural analysis, which was defined by Argyris (1954)
- Exact solution of a structure idealized on a physical basis.
- A structure is really composed of frames, bars, panels, etc.
- In each element, use is made of classical hypotheses that enable to evaluate their stiffness.
- Matrix structural analysis is thus only a systematic way of solving a structure.

TURNER, CLOUGH, MARTIN and TOPP (1956)

- In 1956, Turner, Clough, Martin, Topp, working on thin walled structures, imagined a new procedure. Subdividing arbitrarily each wall in triangular elements, they supposed that in each triangle, the displacement is linear:



$$\begin{cases} u = a_1 + a_2 x + a_3 y \\ v = a_4 + a_5 x + a_6 y \end{cases}$$

TURNER, CLOUGH, MARTIN and TOPP (1956)

- They show that it is possible to express the coefficients a_i in terms of the displacements of the corners

$$\begin{cases} u = a_1 + a_2\,x + a_3\,y \\ v = a_4 + a_5\,x + a_6\,y \end{cases}$$

- In each element, the stresses and strains are constant in such a way that it is easy to reckon the force corresponding to any nodal displacement.

- Finally connecting the nodal displacements ensures the compatibility at each interface.

- This was the starting point of the finite element method!

RAYLEIGH RITZ METHOD

- A further step was the recognition of the fact that the finite element method is nothing than a particular form of the well-known Rayleigh-Ritz procedure.

- As is well-known, the Rayleigh-Ritz procedure consists to define a basis of functions and to seek the coefficients of these functions which minimize a given functional.

- In fact, finite elements lead to a particular appropriate basis.

MATHEMATICAL FOUNDATIONS

- From this time, large progress have been made in finite element techniques and in their analysis.
- Mathematical foundations of FEM proceed from the functional analysis using for instance Sobolev spaces.

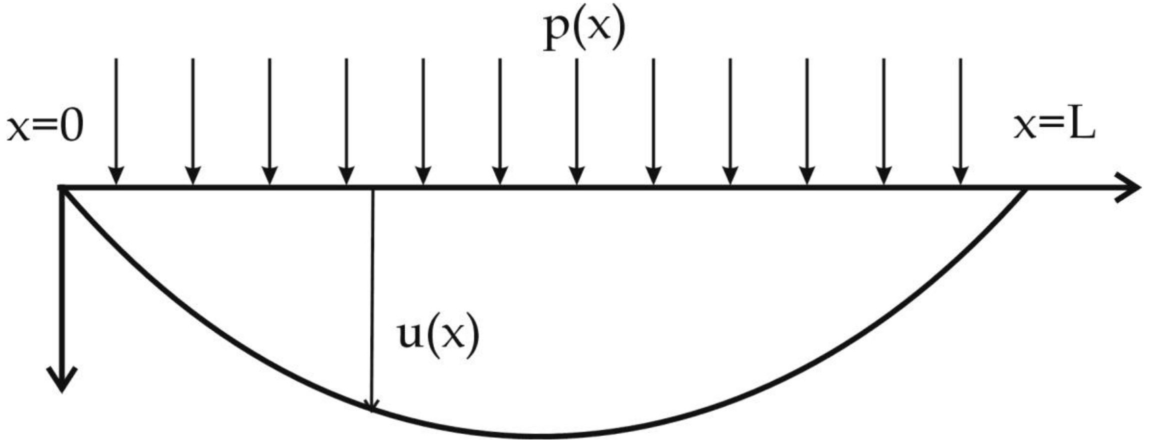
INTRODUCTORY PROBLEM

- Illustrate the FEM approach on a simple string problem taut by a force N

- Differential equations

$$-N\,\frac{d^2u}{dx^2} \;=\; p(x)$$

- With boundary conditions



ANALYTICAL SOLUTION p(x)=cste

- Integration of the differential equation

$$\frac{d^2u}{dx^2} = -\frac{p}{N}$$

$$\frac{du}{dx} = -\frac{p}{N}\, x + C \qquad u(x) = -\frac{p}{2N}\, x^2 + C\, x + D$$

- Applying the boundary conditions $u(0) = D = 0$

$$u(L) = \frac{p}{2N}\, L^2 + C\, L = 0$$

- It comes

$$u(x) = \frac{p}{2N}\,(-x^2 + L\, x)$$

- Displacement at mid-span x=L/2

$$u(L/2) = \frac{pL^3}{8N} = \frac{4pL^2}{8\pi^3 N}\,\frac{\pi^3}{32} = 0,9689\,\frac{4pL^2}{8\pi^3 N}$$

INTRODUCTORY PROBLEM

- The first step to find approximation solutions is to find a variational principle which is equivalent to the differential equation and the boundary conditions

- Every candidate for the solution u(x) is equal to zero at both ends:

$$u = 0 \quad x = 0 \text{ and } x = L$$

- Such a displacement filed will be called kinematically admissible displacement.

- A kinematically admissible displacement variation δu is then defined as an arbitrary difference between two admissible displacements. It is therefore equal to zero at both ends.

$$\delta u = 0 \quad x = 0 \text{ and } x = L$$

INTRODUCTORY PROBLEM

- Let's multiply the differential equation by a kinematically admissible variation and integrate on the domain

$$\int_0^L \left(-N \frac{d^2u}{dx^2} - p(x) \right) \delta u \, dx = 0$$

- Use integration by part

$$-\int_0^L N \frac{d^2u}{dx^2} \delta u \, dx = -\left[N \frac{du}{dx} \delta u \right]_0^L + \int_0^L N \frac{du}{dx} \frac{d\delta u}{dx} \, dx$$

- Since we have kinematically admissible variation of displacement

$$\delta u(0) = 0 \qquad \text{and} \qquad \delta u(L) = 0$$

INTRODUCTORY PROBLEM

- Noticing that

$$\frac{d\delta u}{dx} = \delta \frac{du}{dx}$$

- We get

$$-\int_0^L N \frac{d^2u}{dx^2} \delta u \, dx = \int_0^L N \frac{du}{dx} \delta(\frac{du}{dx}) \, dx = \delta \int_0^L N \frac{1}{2} \left(\frac{du}{dx}\right)^2 dx$$

- And finally, the variational principle writes

$$\delta \left[\int_0^L \frac{1}{2} N \left(\frac{du}{dx}\right)^2 - p \, u \right] dx = 0$$

INTRODUCTORY PROBLEM

- The solution of the differential equation renders the functional F(u) stationary:

$$\delta \, \mathcal{F}(u) = 0$$

- with

$$\mathcal{F}(u) \, = \int_0^L \frac{1}{2} \, N \, \left(\frac{du}{dx} \right)^2 \, dx \, - \int_0^L p \, u \, dx$$

RAYLEIGH RITZ SOLUTION

- The Rayleigh-Ritz method consists in determining among all superpositions of given test functions u, the combination that renders the functional F(u) stationary.

- Since the approximation is built from a combination of a set of basis (independent) functions, Rayleigh-Ritz method aims at determining the coefficients that are a minimizer of the functional.

- Assume a solution of the form

$$u = \sum_{k=1}^{m} A_k \sin \frac{k\pi x}{L}$$

RAYLEIGH RITZ SOLUTION

- For the sake of simplicity, we assume

$$p(x) = p = cst$$

- The functional writes

$$\mathcal{F}(u) = \frac{1}{2} \sum_{k=1}^{m} N\, A_k^2 \,\frac{k^2\pi^2}{L^2} \;-\; p \sum_{k=1}^{m} \frac{L}{k\pi}[1-(-1)^k]\, A_k \;=\; \mathcal{F}(A_1,\ldots,A_m)$$

- To determine the value of the coefficients A_k, solve stationary conditions

$$\frac{\partial \mathcal{F}(u)}{\partial A_k} = \; N\, A_k \,\frac{k^2\pi^2}{L^2} \;-\; p\, \frac{L}{k\pi}[1-(-1)^k] \;=\; 0$$

- It comes

$$A_k \;=\; \begin{cases} \frac{4L^2}{k^3\pi^3 N}\, p & \text{if k is odd} \\ 0 & \text{if k is even} \end{cases}$$

RAYLEIGH RITZ SOLUTION

- The displacement at mid-span x=L/2

$$u(\frac{L}{2}) = \sum_{k=1}^{m} A_k = \frac{4pL^2}{\pi^3 N} \sum_{k=1,3,\ldots}^{m} \frac{(-1)^k}{k^3}$$

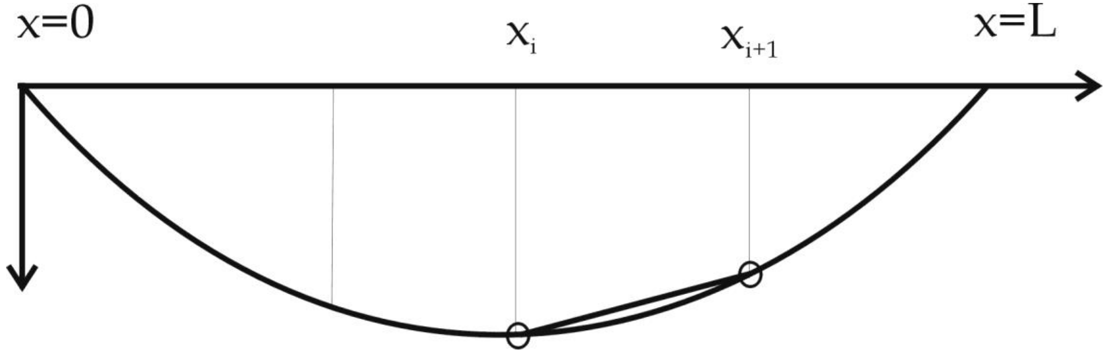
- Increasing the number of test functions, increases the precision

m	1	3	5	7	9	11	13	15
$\frac{N\pi^3 u(L/2)}{4pL^2}$	1,0000	0,9630	9710	0,9680	0,9694	0,9687	0,9691	0,9688

FINITE ELEMENT SOLUTION

- The finite element method consists to split the domain into n intervals $]x_i, x_{i+1}[$ and to interpolate linearly the displacement between the values at points

$$u(x) = u_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + u_{i+1} \frac{x - x_i}{x_{i+1} - x_i}$$



FINITE ELEMENT SOLUTION

- Let's calculate the variational principle to determine the local values x_i:

$$u(x) = u_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + u_{i+1} \frac{x - x_i}{x_{i+1} - x_i}$$

$$\frac{du(x)}{dx} = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$$

- Integrate on each interval

$$\frac{1}{2} \int_{x_i}^{x_{i+1}} N \left(\frac{du}{dx} \right)^2 dx = \frac{1}{2} \int_{x_i}^{x_{i+1}} N \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} \right)^2 dx$$

$$\int_{x_i}^{x_{i+1}} p(x)\, u\, dx = u_i \int_{x_i}^{x_{i+1}} p(x) \frac{x - x_{i+1}}{x_i - x_{i+1}} dx + u_{i+1} \int_{x_i}^{x_{i+1}} p(x) \frac{x - x_i}{x_{i+1} - x_i} dx$$

FINITE ELEMENT SOLUTION

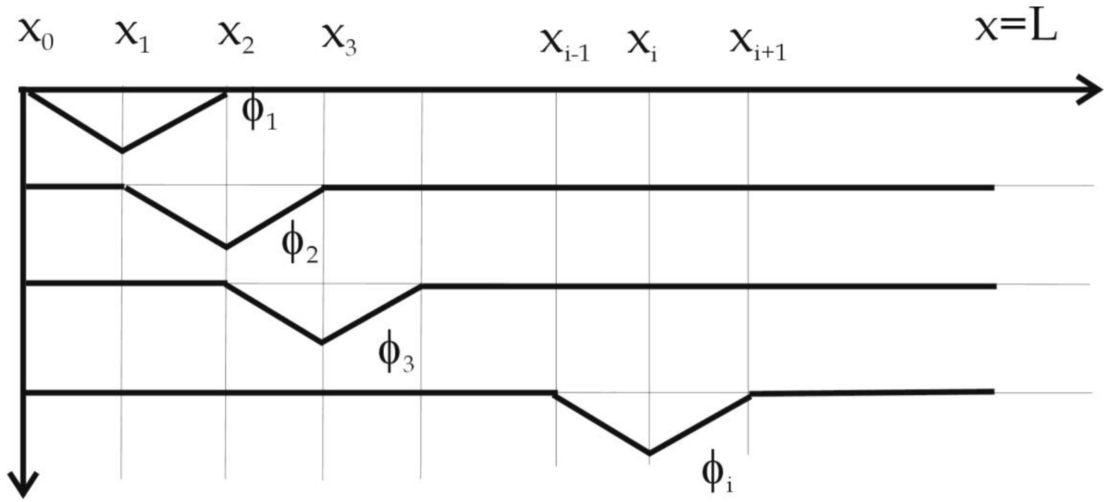
- Summing on all elements

$$\mathcal{F}(u) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left[\frac{1}{2} N \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} \right)^2 - p \left(u_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + u_{i+1} \frac{x - x_i}{x_{i+1} - x_i} \right) \right] dx$$

- Values of u_i coefficients, the local values of the displacements are obtained by minimizing the functional

FINITE ELEMENT SOLUTION

- It is important to note that the finite element discretization is equivalent to a Rayleigh-Ritz method where the basis functions are the roof-type functions:



$$u(x) = \sum_{i=0}^{n-1} M_i(x)\, u_i$$

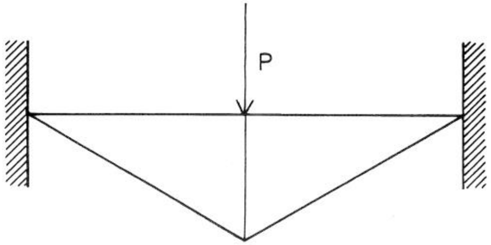
$$M(x) = \begin{cases} 0 & \text{if } x \leq x_{i-1} \\ \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x - x_{i+1}}{x_i - x_{i+1}} & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{if } x \geq x_{i+1} \end{cases}$$

FINITE ELEMENT SOLUTION

- Note that finally for piecewise linear approximations, which are used, the original differential equation has no sense.

- However, the weak form based on the functional is meaningful. The variational principal, however, is well defined.

- This is a specificity of finite elements: the approximate solutions are just able to ensure the existence of the variational principle, not of the local differential equation. In this direction note that regular functions are not unusual in engineering problems in practice. As an example if the string is submitted to a concentrated load the solution is precisely of roof-type.



FINITE ELEMENTS OF BARS AND TRUSS STRUCTURES

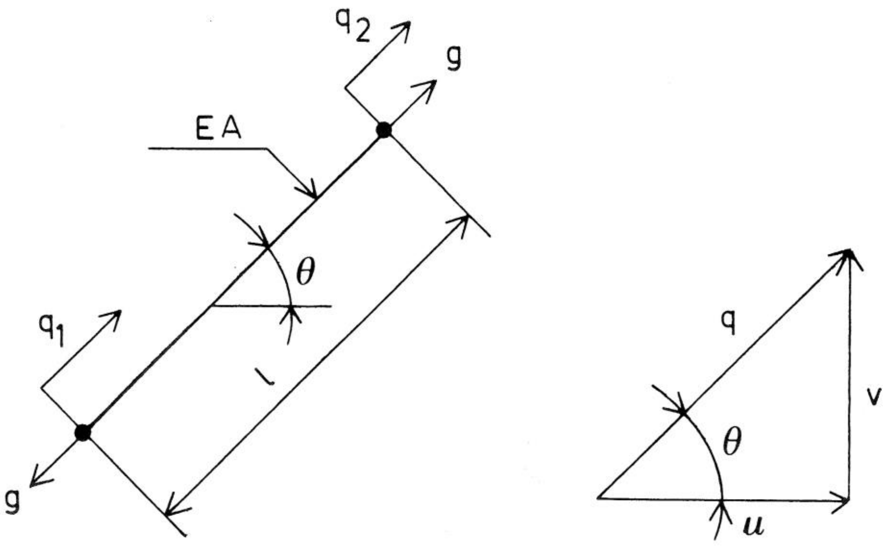
FINITE ELEMENT OF BARS

- We illustrate the development of displacement based finite elements with bar truss structures.

- Simple case will be used to illustrate the different steps of matrix structural analysis, avoiding complications related to a more complex problem.

- Consider a bar of section A,
- Young's modulus E,
- Length L

- The end displacements of the bar along its axes are q_1 and q_2.



FINITE ELEMENT OF BARS

- Let's assume that the displacement along the bar is a linear piece wise function.

$$q(x) = q_1 \frac{L-x}{L} + q_2 \frac{x}{L}$$

- Collect the displacements into a column vector $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$$q(x) = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{N}(x)\,\mathbf{q}$$

- The deformation along the bar is obtained by differentiating the displacement with respect to x variable:

$$\epsilon = \frac{\partial q}{\partial x}(x) = q_1 \frac{\partial}{\partial x}\left(\frac{L-x}{L}\right) + q_2 \frac{\partial}{\partial x}\left(\frac{x}{L}\right)$$

FINITE ELEMENT OF BARS

- The deformation along the bar is obtained by differentiating the displacement with respect to x variable:

$$\epsilon = \frac{\partial q}{\partial x}(x) = q_1 \frac{\partial}{\partial x}\left(\frac{L-x}{L}\right) + q_2 \frac{\partial}{\partial x}\left(\frac{x}{L}\right)$$

- That is

$$\epsilon = q_1 \frac{-1}{L} + q_2 \frac{1}{L} = \frac{q_2 - q_1}{L}$$

- In matrix form:

$$\epsilon = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{B}\,\mathbf{q}$$

- With the strain matrix

$$\mathbf{B} = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} N_1(x) & \frac{\partial}{\partial x} N_2(x) \end{bmatrix} = \partial \mathbf{N}(x)$$

FINITE ELEMENT OF BARS

- The axial stress is given by the uni directional Hooke's law:

$$\sigma = E\,\epsilon = E\,\frac{q_2 - q_1}{L}$$

- And

$$\sigma = E\,\begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{T}\,\mathbf{q}$$

- The tension matrix T

$$\mathbf{T} = E\,\begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}$$

FINITE ELEMENT OF BARS

- To solve the problem using approximation, it is better to use a variational principle instead of local differential equations. In displacement based finite elements, the variational principle is given by the minimum of the total potential energy:

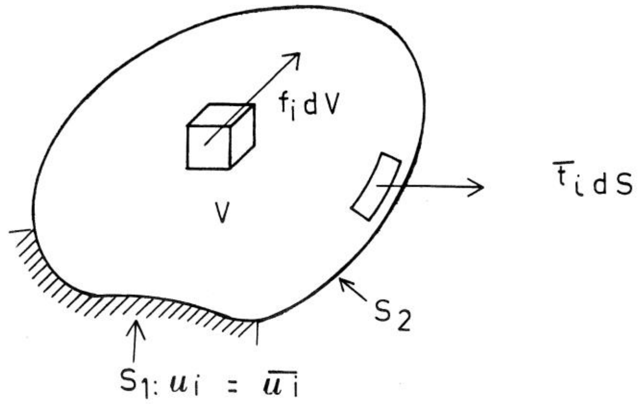
$$\min_{u} \; \mathcal{U} + \mathcal{P}$$

- The elastic energy

$$\mathcal{U} = \int_V \frac{1}{2}\sigma_{ij}\,\epsilon_{ij}\,dV = \int_V \frac{1}{2}E_{ijkl}\,\epsilon_{ij}\epsilon_{kl}\,dV$$

- The work of the load

$$\mathcal{P} = \int_V f_i\,u_i\,dV + \int_{S_\sigma} \bar{t}_i\,u_i\,dS$$



FINITE ELEMENT OF BARS

- For a bar, these expressions take a simplified form

$$\mathcal{U}_e = \int_0^L \frac{1}{2} E \epsilon^2 A \, dx$$

- If we consider the FE approximation

$$\mathcal{U}_e = \frac{1}{2} \frac{EA}{L} (q_2 - q_1)^2$$

- In structural matrix analysis, this expression finds the matrix form:

$$\mathcal{U}_e = \frac{1}{2} \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{EA}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad k = \frac{EA}{L}$$

FINITE ELEMENT OF BARS

- The matrix

$$\mathbf{K}_e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

- is called stiffness matrix in element local axes. The three last words refer to the fact that the displacements are expressed in the local axis of the bar.

- Note that this matrix is singular. Indeed one can observe that applying a uniform displacement corresponding to a rigid body motion yields no strain energy:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

FINITE ELEMENT OF BARS

- One should also notice that the expression can be obtained by inserting the finite element approximation into the variational principle. The strain energy can be written:

$$
\begin{aligned}
\mathcal{U}_e &= \int_0^L \frac{1}{2}\, E\, \epsilon^2\, A\, dx \\
&= \int_0^L \frac{1}{2}\, \mathbf{q}^T \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{bmatrix} E \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \mathbf{q}\, A\, dx \\
&= \int_0^L \frac{1}{2}\, \mathbf{q}^T \frac{EA}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{q}\, dx
\end{aligned}
$$

- One notices that the integrant is constant and it comes

$$
\mathcal{U}_e = \frac{1}{2}\, \mathbf{q}^T \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{q}
$$

FINITE ELEMENT OF BARS

- And one identifies the expression of the stiffness matrix

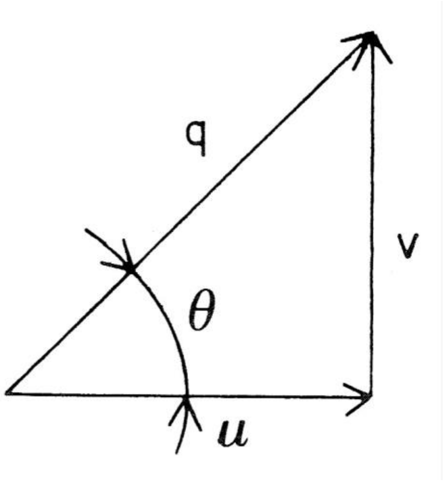
$$\mathbf{K}_e = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

BAR STIFFNESS IN GLOBAL AXES

- Assembling the different bars of a truss implies that we have to use a unique axes system at each nodes ➔ cartesian system attached to the structure

- Displacement of each node is represented by its components u and v along respectively axis x and y

$$q_i = u_i \cos\theta + v_i \sin\theta$$



BAR STIFFNESS IN GLOBAL AXES

- Let's denote by 'e' the bar index and by 'S' the structural coordinates.
- The local displacements of the two nodes of the bar are expressed in terms of the node cartesian components of the displacements measured in the structural frame S.

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

$$^{loc}\mathbf{q}_e = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \mathbf{R}_e = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \qquad {}^{S}\mathbf{q}e = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

BAR STIFFNESS IN GLOBAL AXES

- The element strain energy is thus given either in local or in structural frames by

$$\mathcal{U}_e = \frac{1}{2}\, {}^{loc}\mathbf{q}_e^T\, {}^{loc}\mathbf{K}_e\, {}^{loc}\mathbf{q}_e \;=\; \frac{1}{2}\, {}^{S}\mathbf{q}_e^T\, {}^{S}\mathbf{K}_e\, {}^{S}\mathbf{q}_e$$

- We can deduce that the element stiffness in structural axes is

$${}^{S}\mathbf{K}_e \;=\; \mathbf{R}_e^T\, {}^{loc}\mathbf{K}_e\, \mathbf{R}_e$$

$${}^{S}\mathbf{K}_e \;=\; \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

BAR STIFFNESS IN GLOBAL AXES

- The rank of the element stiffness matrix in structural axes is the same as the stiffness matrix in local axes.

- The stiffness being of dimension 4 and the rank being of 1, one can see that there are three singularity modes that can be interpreted as three rigid body modes:
  - $u_1=1$, $v_1=0$, $u_2=1$, $v_2=0$, i.e. translation along the axis x;
  - $u_1=0$, $v_1=1$, $u_2=0$, $v_2=1$, i.e. translation along the axis y;
  - $u_1=0$, $v_1=0$, $u_2=-\sin\theta$, $v_2=\cos\theta$, i.e. rotation about the point 1.

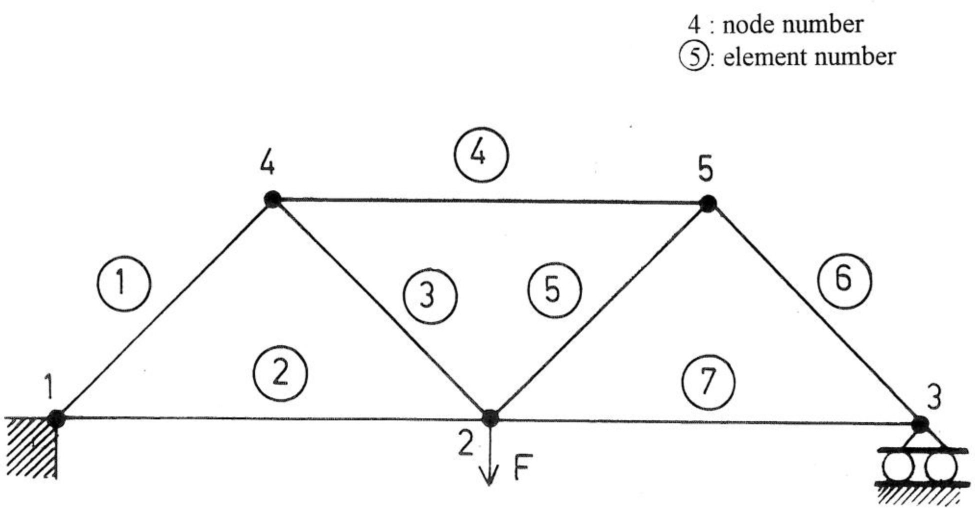
ASSEMBLING THE BARS

- The global displacement vector

$$ {}^{S}\mathbf{q}^{T} = \left[\underbrace{u_1\ v_1}_{\text{node 1}}\ u_2\ v_2\ \ldots \underbrace{u_e\ v_e}_{\text{node e}}\ \ldots\ u_n\ v_n \right] $$

- Considering the truss structure, the global displacement vector

$$ {}^{S}\mathbf{q}^{T} = [u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ u_4\ v_4\ u_5\ v_5] $$



ASSEMBLING THE BARS

- Each element has 4 displacements components which have to be indexed in the global displacement vector by an element localization matrix $\mathbf{L}_e$.

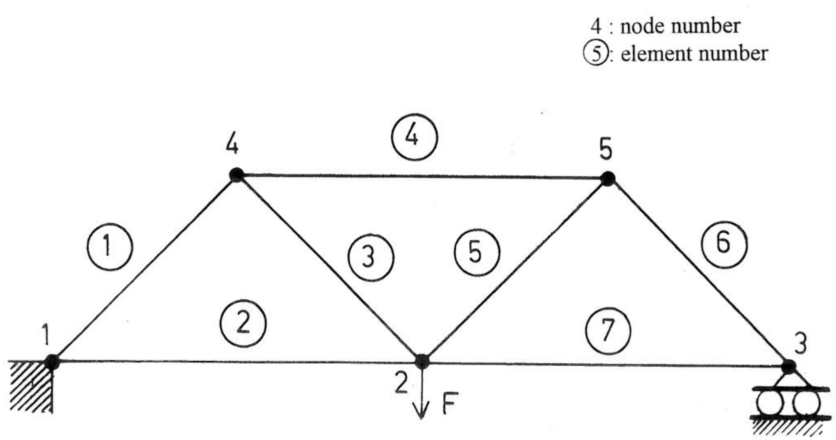
- From a formal point of view the link between the element displacement vector ${}^S\mathbf{q}_e$ and the structural displacement vector ${}^S\mathbf{q}$ can be realized by applying a matrix with appropriate 1 in the right positions and 0 everywhere else.

$$ {}^S\mathbf{q}_e = \mathbf{L}_e \, {}^S\mathbf{q} $$

ASSEMBLING THE BARS

- Example: the bar 1 connecting node 1 and 4 takes the form:

$$
\underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \end{bmatrix}}_{{}^S\mathbf{q}_1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{L}_1} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{bmatrix}}_{{}^S\mathbf{q}}
$$



ASSEMBLING THE BARS

- From a computational point of view, this approach would be a complete nonsense because it would involve a large number of trivial operations.

- Thus in practice it is preferred to resort to a localization procedure involving a pointer type approach. For each element, the element localization table

$$l_e(i)$$

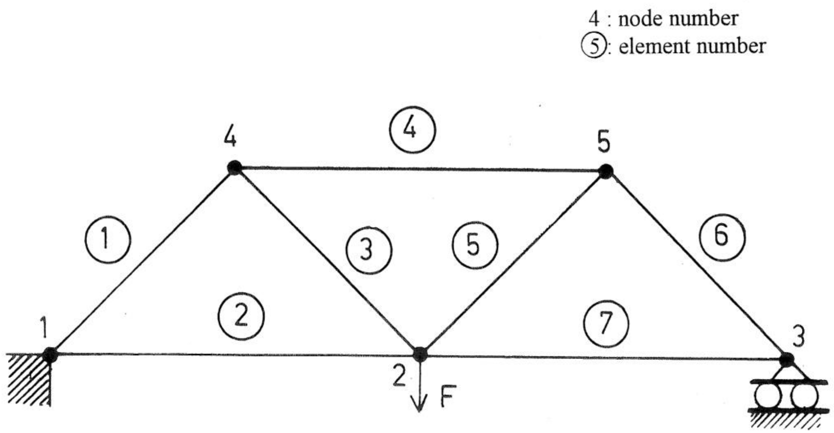
- returns the list of the degrees of freedom in the structural vector that corresponds to the local displacements. It comes

$$^S\mathbf{q}_e(i) = {}^S\mathbf{q}(l_e(i)) \quad \Leftrightarrow \quad {}^S\mathbf{q}_e = \mathbf{L}_e\, {}^S\mathbf{q}$$

ASSEMBLING THE BARS

- Coming back to the example of the 7-bar truss, the localization element table

Element	node 1	node 2	$l_e(1)$	$l_e(2)$	$l_e(3)$	$l_e(4)$
1	1	4	1	2	7	8
2	1	2	1	2	3	4
3	2	4	3	4	7	8
4	4	5	7	8	9	10
5	2	5	3	4	9	10
6	3	5	5	6	9	10
7	2	3	3	4	5	6



ASSEMBLING THE BARS

- The energy being a local quantity, everything can be carried out element by element. Thus the strain energy is evaluated as follows

$$\begin{aligned}\mathcal{U} &= \frac{1}{2}\,{}^S\mathbf{q}^T\,\mathbf{K}\,{}^S\mathbf{q} \\ &= \frac{1}{2}\sum_{e=1}^{n}{}^S\mathbf{q}_e^T\,\mathbf{K}_e\,{}^S\mathbf{q}_e \\ &= \frac{1}{2}\sum_{e=1}^{n}{}^S\mathbf{q}^T\,\mathbf{L}_e^T\,\mathbf{K}_e\,\mathbf{L}_e\,{}^S\mathbf{q}_e\end{aligned}$$

- So the stiffness matrix of the structure is clearly

$$ {}^S\mathbf{K} = \sum_{e=1}^{n}\mathbf{L}_e^T\,\mathbf{K}_e\,\mathbf{L}_e $$

ASSEMBLING THE BARS

- Again, using the localization matrices would be not computationally efficient so one uses the localization table:

$$\mathcal{U} = \frac{1}{2} \sum_{e=1}^{n} {}^S\mathbf{q}_e^T \, \mathbf{K}_e \, {}^S\mathbf{q}_e = \frac{1}{2} \sum_{e=1}^{n} \sum_{i=1}^{4} \sum_{j=1}^{4} K_{e,ij} \, {}^S q_{e,i} \, {}^S q_{e,j}$$

$$= \frac{1}{2} \sum_{e=1}^{n} \sum_{i=1}^{4} \sum_{j=1}^{4} {}^S K_{e,ij} \, {}^S q(l_e(i)) \, {}^S q(l_e(j))$$

- In practice the structural matrix is assembled by assigning the contribution of each stiffness term at the corresponding position determined by the localization table.

ASSEMBLING THE BARS

Algorithm 1 Assemble $\mathbf{K}$

for $i = 1, N$ and $j = 1, N$ **do**
  $^{S}\mathbf{K}(i, j) \leftarrow 0$
end for
for $e = 1, NEL$ **do**
  for $i = 1, 4$ and $j = 1, 4$ **do**
    $^{S}K(l_e(i), l_e(j)) \leftarrow {}^{S} K(l_e(i), l_e(j)) + K_e(i, j)$
  end for
end for

ASSEMBLING THE BARS

- In the case study, the truss is such that we have the following contributions of the elements to the different degrees of freedom of the structural stiffness matrix

	1	2	3	4	5	6	7	8	9	10
1	①②	①②	②	②			①	①		
2	①②	①②	②	②			①	①		
3	②	②	②③⑤⑦	②③⑤⑦	⑦	⑦	③	③	⑤	⑤
4	②	②	②③⑤⑦	②③⑤⑦	⑦	⑦	③	③	⑤	⑤
5			⑦	⑦	⑥⑦	⑥⑦			⑥	⑥
6			⑦	⑦	⑥⑦	⑥⑦			⑥	⑥
7	①	①	③	③			①③④	①③④	④	④
8	①	①	③	③			①③④	①③④	④	④
9			⑤	⑤	⑥	⑥			⑤⑥	⑤⑥
10			⑤	⑤	⑥	⑥			⑤⑥	⑤⑥

SOLUTION OF THE ELASTIC PROBLEM

- Considering the application example, let us assume that there is a load F along -y direction at node 2. The load vector in structural axes is given by setting the applied force component in the global load vector. In the particular case study, one has:

$$\mathbf{g}^T = [0\,0\,0\,-F\,0\,0\,0\,0\,0\,0]$$

- The variational principle reads:

$$\min_{\mathbf{q}} = \mathcal{U} + \mathcal{P}$$

$$\min_{\mathbf{q}} = \frac{1}{2}\,{}^S\mathbf{q}\,{}^S\mathbf{K}\,{}^S\mathbf{q} - {}^S\mathbf{g}^T\,{}^S\mathbf{q}$$

SOLUTION OF THE ELASTIC PROBLEM

- The stationary conditions provide the linear system that is the equilibrium equation of the system.

$$ {}^{S}\mathbf{K}\,{}^{S}\mathbf{q} = {}^{S}\mathbf{g} $$

- Taking into account the boundary conditions, the stiffness matrix of the system becomes invertible.

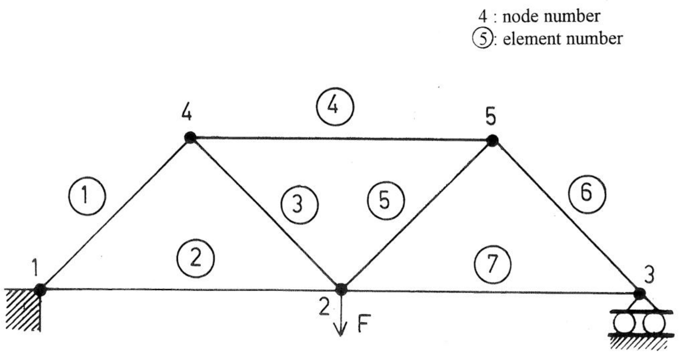
$$ u_1 = 0,\ v_1 = 0,\ v_3 = 0 \quad \Leftrightarrow \quad {}^{S}q_1 = 0\ {}^{S}q_2 = 0\ {}^{S}q_6 = 0 $$

- The boundary conditions can be taken into account by suppressing the corresponding lines and columns to these degrees of freedom.

SOLUTION OF THE ELASTIC PROBLEM

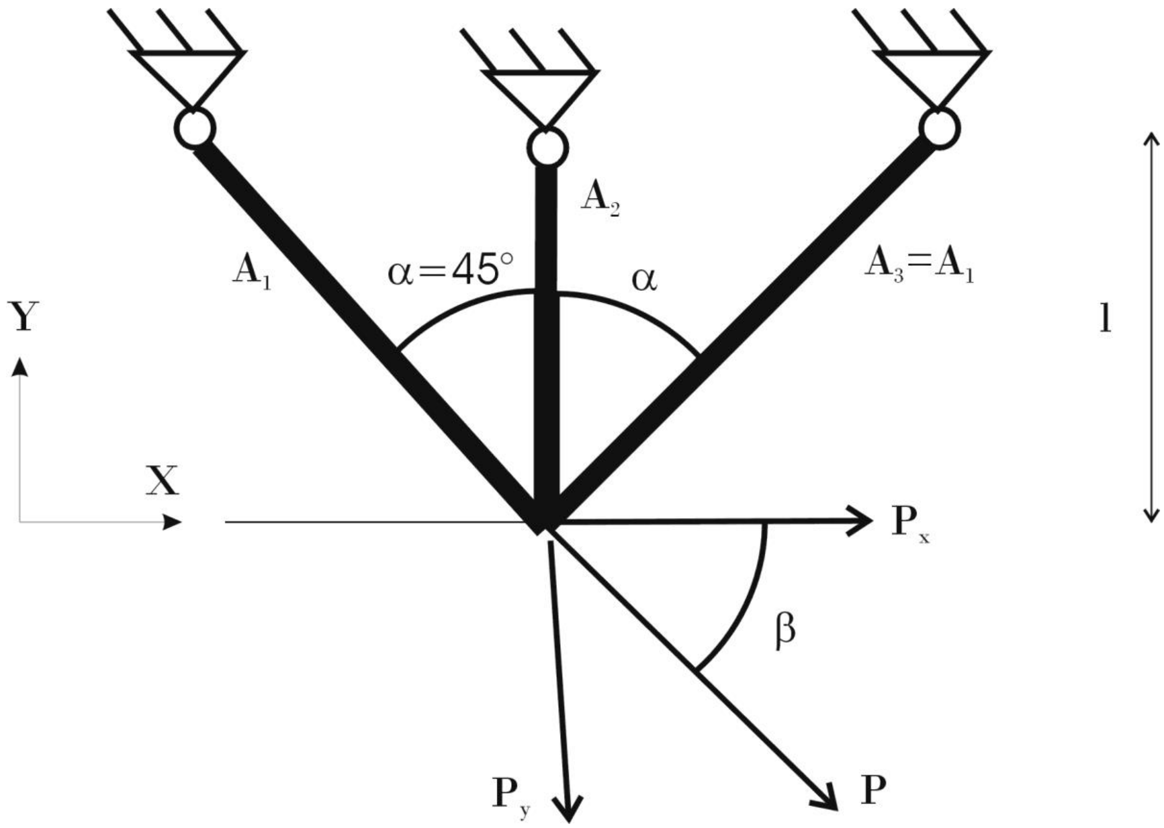
- The boundary conditions can be taken into account by suppressing the corresponding lines and columns to these degrees of freedom.

$$\begin{bmatrix} K_{3,3} & K_{3,4} & K_{3,5} & K_{3,7} & K_{3,8} & K_{3,9} & K_{3,10} \\ K_{4,3} & K_{4,4} & K_{4,5} & K_{4,7} & K_{4,8} & K_{4,9} & K_{4,10} \\ K_{5,3} & K_{5,4} & K_{5,5} & K_{5,7} & K_{5,8} & K_{5,9} & K_{5,10} \\ K_{7,3} & K_{7,4} & K_{7,5} & K_{7,7} & K_{7,8} & K_{7,9} & K_{7,10} \\ K_{8,3} & K_{8,4} & K_{8,5} & K_{3,7} & K_{8,8} & K_{8,9} & K_{8,10} \\ K_{9,3} & K_{9,4} & K_{9,5} & K_{9,7} & K_{9,8} & K_{9,9} & K_{9,10} \\ K_{10,3} & K_{10,4} & K_{10,5} & K_{10,7} & K_{10,8} & K_{10,9} & K_{10,10} \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \\ q_5 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -F \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



EXAMPLE: THREE BAR TRUSS

- Let's consider the three-bar truss problem



EXAMPLE: THREE BAR TRUSS

- Stiffness matrices of bars are derived from the general expression

$$\mathbf{K}_j = \frac{EA_j}{l_j} \begin{bmatrix} c^2 & s\,c & -c^2 & -s\,c \\ s\,c & s^2 & -s\,c & -s^2 \\ -c^2 & -s\,c & c^2 & s\,c \\ -s\,c & -s^2 & s\,c & s^2 \end{bmatrix}$$

$$c = \cos\theta \quad s = \sin\theta$$

- Bar 1

$$\theta = -\pi/4 \quad c = \cos\theta = \sqrt{2}/2 \quad s = \sin\theta = -\sqrt{2}/2$$

$$c^2 = 1/2 \quad s^2 = 1/2 \quad s\,c = -1/2$$

$$\mathbf{K}_1 = \frac{EA_1}{2\sqrt{2}\,l} \begin{bmatrix} +1 & -1 & -1 & +1 \\ -1 & +1 & +1 & -1 \\ -1 & +1 & +1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix}$$

EXAMPLE: THREE BAR TRUSS

- Bar 2 $\theta = -\pi/2 \quad c = \cos\theta = 0 \quad s = \sin\theta = -1$

$$c^2 = 0 \quad s^2 = 1 \quad s\,c = 0$$

$$\mathbf{K}_2 = \frac{EA_2}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

- Bar 3 $\theta = -3\pi/4 \quad c = \cos\theta = -\sqrt{2}/2 \quad s = \sin\theta = -\sqrt{2}/2$

$$c^2 = 1/2 \quad s^2 = 1/2 \quad s\,c = 1/2$$

$$\mathbf{K}_3 = \frac{EA_3}{2\sqrt{2}\,l} \begin{bmatrix} +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & -1 \\ -1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 \end{bmatrix}$$

EXAMPLE: THREE BAR TRUSS

- Let's assemble the element stiffness matrices.
- Because of boundary conditions, displacements of nodes 1, 2, and 3 are eliminated
- Let's consider the symmetric geometrical configuration $A_1 = A_3$

$$\mathbf{K}_1 = \frac{EA_1}{2\sqrt{2}\,l}\begin{bmatrix} +1 & -1 & -1 & +1 \\ -1 & +1 & +1 & -1 \\ -1 & +1 & +1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix} \qquad \mathbf{K}_3 = \frac{EA_3}{2\sqrt{2}\,l}\begin{bmatrix} +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & -1 \\ -1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 \end{bmatrix}$$

$$\mathbf{K}_2 = \frac{EA_2}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

EXAMPLE: THREE BAR TRUSS

- In structural axis, the full system equations write:

$$\mathbf{L}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^S\mathbf{K}_1 = \mathbf{L}_1^T \mathbf{K}_1 \mathbf{L}_1$$

$${}^S\mathbf{K}_1 = \frac{EA_1}{2\sqrt{2}\,l} \begin{bmatrix} +1 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \\ -1 & +1 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 & +1 & -1 \\ +1 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \end{bmatrix}$$

EXAMPLE: THREE BAR TRUSS

- In structural axis, the full system equations write:

$$\mathbf{L}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{S}\mathbf{K}_2 = \mathbf{L}_2^T \mathbf{K}_2 \mathbf{L}_2$$

$$\mathbf{K}_2 = \frac{EA_2}{l} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE: THREE BAR TRUSS

- In structural axis, the full system equations write:

$$\mathbf{L}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{S}\mathbf{K}_1 = \mathbf{L}_3^T \mathbf{K}_3 \mathbf{L}_3$$

$$\mathbf{K}_3 = \frac{EA_3}{2\sqrt{2}\,l} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & +1 & +1 \\ 0 & 0 & 0 & 0 & -1 & -1 & +1 & +1 \end{bmatrix}$$

EXAMPLE: THREE BAR TRUSS

- Assembling the element stiffness matrices

$$
{}^S\mathbf{K} = {}^S\mathbf{K}_1 + {}^S\mathbf{K}_2 + {}^S\mathbf{K}_3
$$

$$
= \begin{bmatrix}
C_1 & -C_1 & 0 & 0 & 0 & 0 & -C_1 & +C_1 \\
-C_1 & +C_1 & 0 & 0 & 0 & 0 & C_1 & -C_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_2 & 0 & 0 & 0 & -C_2 \\
0 & 0 & 0 & 0 & C_3 & C_3 & -C_3 & -C_3 \\
0 & 0 & 0 & 0 & C_3 & C_3 & -C_3 & -C_3 \\
-C_1 & +C_1 & 0 & 0 & -C_3 & -C_3 & +C_1 + C_3 & -C_1 + C_3 \\
+C_1 & -C_1 & 0 & -C_2 & -C_3 & -C_3 & -C_1 + C_3 & C_1 + C_2 + C_3
\end{bmatrix}
$$

EXAMPLE: THREE BAR TRUSS

- Let's apply the boundary conditions
  - Nodes 1, 2 and 3 are fixed

    $$u_1 = v_1 = 0 \qquad u_2 = v_2 = 0 \qquad u_3 = v_3 = 0$$

  - Lines and columns corresponding to these dof are deleted

$$^S\mathbf{K} =^S \mathbf{K}_1 +^S \mathbf{K}_2 +^S \mathbf{K}_3$$

$$= \begin{bmatrix} C_1 & -C_1 & 0 & 0 & 0 & 0 & -C_1 & +C_1 \\ -C_1 & +C_1 & 0 & 0 & 0 & 0 & C_1 & -C_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & 0 & 0 & -C_2 \\ 0 & 0 & 0 & 0 & C_3 & C_3 & -C_3 & -C_3 \\ 0 & 0 & 0 & 0 & C_3 & C_3 & -C_3 & -C_3 \\ -C_1 & +C_1 & 0 & 0 & -C_3 & -C_3 & +C_1 + C_3 & -C_1 + C_3 \\ +C_1 & -C_1 & 0 & -C_2 & -C_3 & -C_3 & -C_1 + C_3 & C_1 + C_2 + C_3 \end{bmatrix}$$

EXAMPLE: THREE BAR TRUSS

- Let's consider the symmetric geometrical configuration $A_1=A_3$
- The reduced system writes

$$ {}^{S}\mathbf{K} = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} $$

$$ k_1 = \frac{EA_1}{2\sqrt{2}\,l} + \frac{EA_3}{2\sqrt{2}\,l} = \frac{2\,EA_1}{2\sqrt{2}\,l} = \frac{EA_1}{\sqrt{2}\,l} $$

$$ k_2 = -\frac{EA_1}{2\sqrt{2}\,l} + \frac{EA_3}{2\sqrt{2}\,l} = 0 $$

$$ k_3 = \frac{EA_1}{2\sqrt{2}\,l} + \frac{EA_2}{l} + \frac{EA_3}{2\sqrt{2}\,l} = \frac{EA_1}{\sqrt{2}\,l} + \frac{EA_2}{l} $$

EXAMPLE: THREE BAR TRUSS

- Let's calculate the displacements at the free node:

$$\begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

- The solution writes

$$u_4 = \frac{k_3 P_x - k_2 P_y}{k_1 k_3 - k_2^2}$$

$$v_4 = \frac{k_1 P_y - k_2 P_x}{k_1 k_3 - k_2^2}$$

- If k_2=0

$$u_4 = \frac{k_3 P_x}{k_1 k_3} = \frac{P_x}{k_1}$$

$$v_4 = \frac{k_1 P_y}{k_1 k_3} = \frac{P_y}{k_3}$$

EXAMPLE: THREE BAR TRUSS

- Let's calculate the displacements at the free node:

$$u_4 = \frac{P_x}{k_1} = \frac{P_x \sqrt{2}\, l}{E\, A_1}$$

$$v_4 = \frac{P_y}{k_3} = \frac{P_y \sqrt{2}\, l}{E\, (A_1 + \sqrt{2}\, A_2)}$$

- If β=45°

$$P_x = P\frac{\sqrt{2}}{2} \quad P_y = P\frac{\sqrt{2}}{2}$$

$$u_4 = \frac{P\, l}{E\, A_1}$$

$$v_4 = \frac{P\, l}{E\, (A_1 + \sqrt{2}\, A_2)}$$

FINITE ELEMENT IN ELASTICITY

FINITE ELEMENT DISCRETIZATION

- Strain energy of a structure

$$U = \frac{1}{2}\int_V \sigma_{ij}\,\varepsilon_{ij}\,dV \;=\; \frac{1}{2}\int_V \sigma^T\,\varepsilon\,dV$$

$$U = \frac{1}{2}\sum_{e=1}^{NE}\int_{V_e} \sigma^T\,\varepsilon\,dV_e$$

- Constitutive equations relating the stresses and the strains

$$\sigma \;=\; \mathbf{D}\,\varepsilon$$

- It comes

$$U = \frac{1}{2}\sum_{e=1}^{NE}\int_{V_e} \varepsilon^T\,\mathbf{D}\,\varepsilon\,dV_e$$

FINITE ELEMENT DISCRETIZATION

- The compatibility equations relate the strains to the displacements:

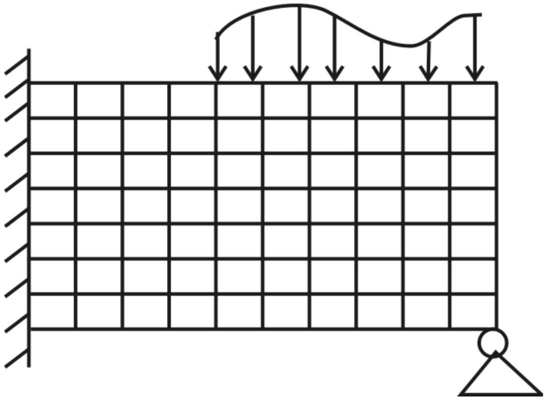
$$\varepsilon = \partial \mathbf{u}$$

- While the finite element approximation relies on the interpolation of the displacements using shape functions **N** and the nodal unknowns **q**.

$$\mathbf{u} = \mathbf{N}\,\mathbf{q}$$

- It comes

$$\varepsilon = \partial \mathbf{N}\,\mathbf{q} = \mathbf{B}\,\mathbf{q}$$

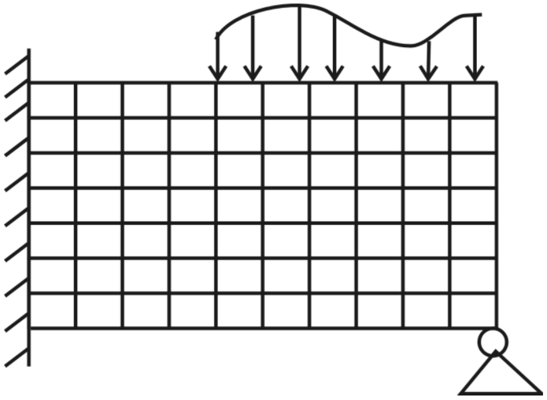


FINITE ELEMENT DISCRETIZATION

- The strain energy takes the form

$$U = \frac{1}{2} \sum_{e=1}^{NE} \int_{V_e} \mathbf{q}_e^T \, \mathbf{B}^T \, \mathbf{D} \, \mathbf{B} \, \mathbf{q}_e \, dV_e$$

$$= \frac{1}{2} \sum_{e=1}^{NE} \mathbf{q}_e^T \left(\int_{V_e} \mathbf{B}^T \, \mathbf{D} \, \mathbf{B} \, dV_e \right) \mathbf{q}_e$$



- The stiffness matrix of the element e is:

$$\mathbf{K}_e = \int_{V_e} \mathbf{B}^T \, \mathbf{D} \, \mathbf{B} \, dV_e$$

- The discretized strain energy

$$U = \frac{1}{2} \sum_{e=1}^{NE} \mathbf{q}_e^T \, \mathbf{K}_e \, \mathbf{q}_e$$

FINITE ELEMENT DISCRETIZATION

- The degrees of freedom of the element (node displacements) are related to the degrees of freedom of the whole structure using the localization matrix $\mathbf{L}_e$:

$$\mathbf{q}_e = \mathbf{L}_e \, \mathbf{q}$$

- The structural strain energy

$$\begin{aligned} U &= \frac{1}{2} \sum_{e=1}^{NE} \mathbf{q}^T \, \mathbf{L}_e^T \mathbf{K}_e \, \mathbf{L}_e \, \mathbf{q} \\ &= \frac{1}{2} \mathbf{q}^T \, \mathbf{K} \, \mathbf{q} \end{aligned}$$

- With the structural stiffness matrix

$$\mathbf{K} = \mathbf{L}_e^T \mathbf{K}_e \, \mathbf{L}_e$$

FINITE ELEMENT DISCRETIZATION

- A similar development can be performed to express the generalized load vector:

$$\begin{aligned} P &= \int_V \mathbf{f}^T \mathbf{u}\, dV + \int_{\Gamma_\sigma} \mathbf{t}^T \mathbf{u}\, d\Gamma \\ &= \sum_{e=1}^{NE} \{ \int_{V_e} \mathbf{f}^T \mathbf{N} \mathbf{q}_e\, dV_e + \int_{\Gamma_{\sigma e}} \mathbf{t}^T \mathbf{N} \mathbf{q}_e\, d\Gamma \} \\ &= \sum_{e=1}^{NE} \mathbf{g}_e^T \mathbf{q}_e \end{aligned}$$

- With the element and structural load vectors

$$\mathbf{g}_e^T = \int_V \mathbf{f}^T \mathbf{N}\, dV_e + \int_{\Gamma_{\sigma e}} \mathbf{t}^T \mathbf{N}\, d\Gamma$$

$$\mathbf{g} = \sum_{e=1}^{NE} \mathbf{L}_e\, \mathbf{g}_e$$

- The external work of the applied loads

$$P = \mathbf{g}^T\, \mathbf{q}$$

FINITE ELEMENT DISCRETIZATION

- The total potential energy of the structure is:

$$\Pi = U(\mathbf{q}) - P(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T\,\mathbf{K}\,\mathbf{q}\;-\;\mathbf{g}^T\,\mathbf{q}$$

- The principle of the minimum total potential energy yields the equilibrium equation

$$\boxed{\mathbf{K}\,\mathbf{q}\;=\;\mathbf{g}}$$

RECOMMENDED REFERENCES

- J.F. Debongnie. Fundamentals of Finite Elements. Les Editions de l'Université de Liège. 2003.
  - Download for free https://orbi.uliege.be/handle/2268/12679
- R.D. Cook, D.S. Malkus, M.E. Plesha. Concepts and Applications of Finite Element Analysis. 3rd Edition. John Wiley. 1989.
- J. Fish and T. Belytschko. A first Course in Finite Elements. John Wiley. 2007.