FINITE ELEMENT AND STRUCTURAL OPTIMIZATION

Pierre DUYSINX

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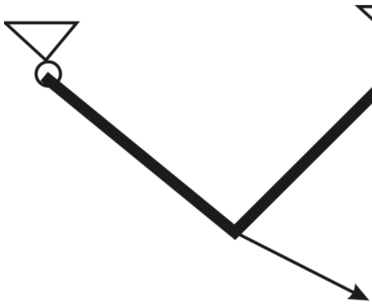
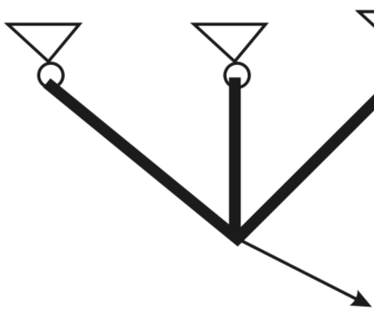
Equations of analysis

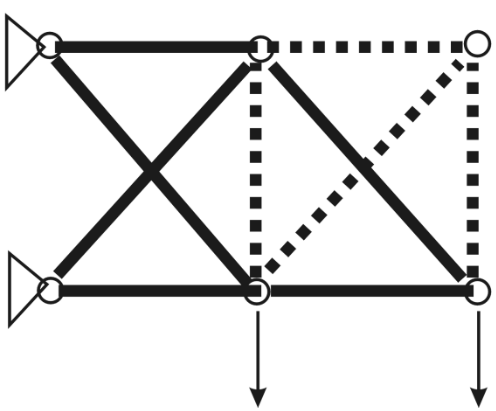
STATICALLY DETERMINATE AND INDETERMINATE STRUCTURES



Determinate
2 dof - 2 bars



Indeterminate
2 dof - 3 bars

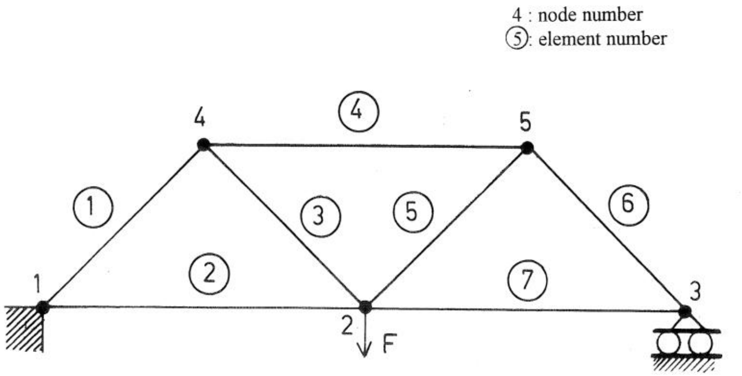


Determinate
6 dof - 6 bars
Indeterminate
8 dof - 10 bars

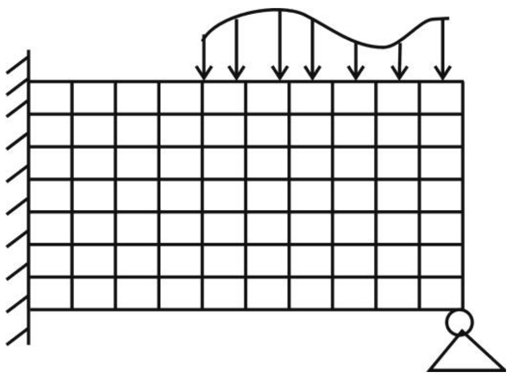
- Statically determinate structures
  - #Unknowns = #Equilibrium equations
  - Equilibrium determines completely the problem
- Indeterminate structures
  - #Unknowns > #Equilibrium equations
  - Elastic redistribution of internal loads with the stiffness
  - Principle of minimum energy to determine unknowns

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Let's suppose that the body is discrete in nature, for instance truss structure, or it is discretized into finite elements, i.e. continuum structure.



- The continuous displacement field **u**(x) in the elements can be approximated using local shape functions **N**(x) while the unknowns are the nodal displacements, which can be collected in the unknown vector **q**.



$$\mathbf{u}(\mathbf{X}) = \mathbf{N}(\mathbf{X})\,\mathbf{q}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

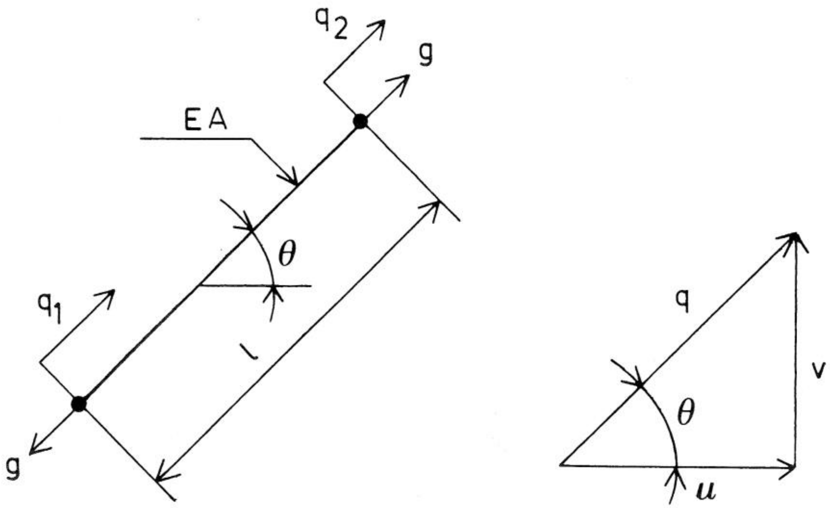
- For truss elements, one assumes a linear displacement field:

$$u_x(X) = N_1(X)\, q_1 \; + \; N_2(X)\, q_2$$

- With the shape functions (interpolation functions)

$$N_1(X) = 1 - \frac{X}{L}$$

$$N_2(X) = \frac{X}{L}$$



ANALYSIS OF FE DISCRETIZED STRUCTURES

- The compatibility equations relates the displacements u to the strain components ε.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j}\right)$$

- For one dimensional truss element, only the axial strain is non-trivial

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial X}$$

- Coming back to the Finite Element framework, it comes that one can apply the differentiation operator to shape function matrix

$$\partial = \left[\frac{\partial}{\partial X}\right] \qquad \varepsilon = \partial \mathbf{N}\,\mathbf{q} = \mathbf{B}\,\mathbf{q}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- For a bar, the strain matrix writes:

$$\mathbf{B} = \left[\frac{\partial}{\partial X}\right] [N_1(X) \; N_2(X)]$$

$$= \left[\frac{\partial N_1(X)}{\partial X} \; \frac{\partial N_2(X)}{\partial X}\right] = \left[\frac{-1}{L} \; \frac{+1}{L}\right]$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- The constitutive equations describe the relation between the stresses and the strain. For a linear elastic behaviour, the stress-strain relation is linear and writes in terms of the Hook coefficients:

$$\sigma = \mathbf{D}\,\varepsilon$$

- In the particular case of truss structure, the Hook matrix degenerates into a single scalar value

$$\mathbf{D} = [E]$$

$$\sigma_{xx} = E\,\varepsilon_{xx}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Let's express the element strain energy U_e.

$$U_e = \frac{1}{2} \int_{V_e} \sigma^T \, \varepsilon \, dV_e$$

- Introducing the stress-strain relationships, the compatibility equations, and the discretization scheme, one finds:

$$\begin{aligned} U_e &= \frac{1}{2} \int_{V_e} \sigma^T \, \varepsilon \, dV_e \\ &= \frac{1}{2} \int_{V_e} \varepsilon^T \, \mathbf{D} \, \varepsilon \, dV_e \\ &= \frac{1}{2} \int_{V_e} \mathbf{q}_e^T \, \mathbf{B}^T \, \mathbf{D} \, \mathbf{B} \, \mathbf{q}_e \, dV_e \\ &= \frac{1}{2} \, \mathbf{q}_e^T \left[\int_{V_e} \mathbf{B}^T \, \mathbf{D} \, \mathbf{B} \, dV_e \right] \mathbf{q}_e \end{aligned}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- This expression puts forward the expression of the element stiffness matrix:

$$\mathbf{K}_e = \int_{V_e} \mathbf{B}^T \, \mathbf{D} \, \mathbf{B} \, dV_e$$

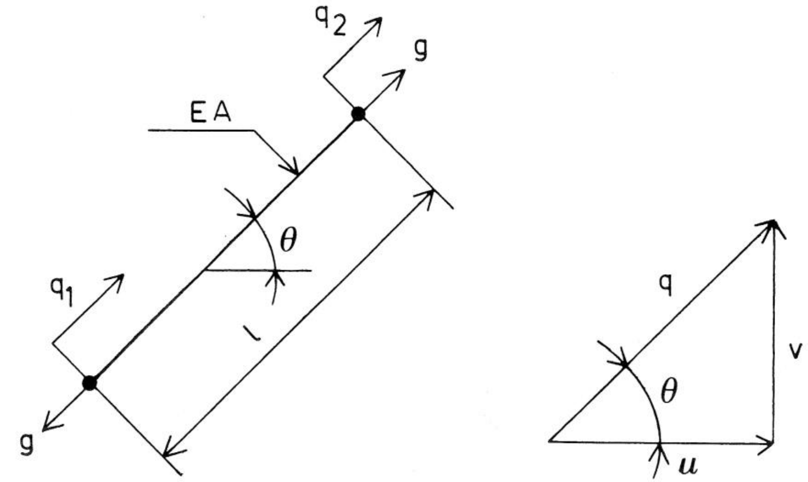
- In the particular case of bar truss element, it comes

$$\mathbf{K}_e \; = \; \int_0^L \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{bmatrix} \, E \, \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \, A \, dx \; = \; \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- For truss structures, local axial displacements must be expressed in terms of their components in the structural frame.

$$q_i = u_i \cos\theta + v_i \sin\theta$$



- In matrix form, one can write

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

$$= \mathbf{R}_e(\theta)\, \mathbf{q}_e$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Finally the element stiffness matrix in structural frame is given by:

$$
{}^{S}\mathbf{K}_e = \frac{E\,A}{L}\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}
$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Strain energy of a structure

$$U = \frac{1}{2} \int_V \sigma_{ij} \, \varepsilon_{ij} \, dV \; = \; \frac{1}{2} \int_V \sigma^T \varepsilon \, dV$$

- The strain energy can be calculated as the sum of element strain energies.

$$U = \frac{1}{2} \sum_{e=1}^{NE} \int_{V_e} \varepsilon^T \, \mathbf{D} \, \varepsilon \, dV_e$$

- Using the expression of the element strain energy and the element displacements and stiffness matrices, one can write

$$U = \frac{1}{2} \sum_{e=1}^{NE} \mathbf{q}_e^T \, \mathbf{K}_e \, \mathbf{q}_e$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- We have now to express the element degrees of freedom in terms of the degrees of freedom of the whole structure. Formally, the element displacement vector can be extracted from la the structural displacement vector by using a localization matrix $\mathbf{L}_e$ made of a few identity terms placed at the terms to be extracted.

$$\mathbf{q}_e = \mathbf{L}_e \, \mathbf{q}$$

- with

$$\mathbf{L}_e = \begin{bmatrix} 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 \end{bmatrix}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- The structural strain energy takes the form

$$U = \frac{1}{2} \sum_{e=1}^{NE} \mathbf{q}^T \mathbf{L}_e^T \mathbf{K}_e \mathbf{L}_e \mathbf{q} = \mathbf{q}^T \left[\sum_{e=1}^{NE} \mathbf{L}_e^T \mathbf{K}_e \mathbf{L}_e \right] \mathbf{q}$$
$$= \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q}$$

- With the structural stiffness matrix:

$$\mathbf{K} = \sum_{e=1}^{NE} \mathbf{L}_e^T \mathbf{K}_e \mathbf{L}_e$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- A similar development can be performed to express the generalized load vector:

$$
\begin{aligned}
P &= \int_V \mathbf{f}^T \, \mathbf{u} \, dV \; + \int_{\Gamma_\sigma} \mathbf{t}^T \, \mathbf{u} \, d\Gamma \\
&= \sum_{e=1}^{NE} \{ \int_{V_e} \mathbf{f}^T \, \mathbf{N} \mathbf{q}_e \, dV_e \; + \int_{\Gamma_{\sigma e}} \mathbf{t}^T \, \mathbf{N} \mathbf{q}_e \, d\Gamma \} \\
&= \sum_{e=1}^{NE} \mathbf{g}_e^T \, \mathbf{q}_e
\end{aligned}
$$

- With the element and structural load vectors

$$
\mathbf{g}_e^T = \int_V \mathbf{f}^T \, \mathbf{N} \, dV_e \; + \int_{\Gamma_{\sigma e}} \mathbf{t}^T \, \mathbf{N} \, d\Gamma \qquad \mathbf{g} = \sum_{e=1}^{NE} \mathbf{L}_e \, \mathbf{g}_e
$$

- The external work of the applied loads

$$
P = \mathbf{g}^T \, \mathbf{q}
$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- The total potential energy of the structure is:

$$\Pi = U(\mathbf{q}) - P(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T\,\mathbf{K}\,\mathbf{q} \;-\; \mathbf{g}^T\,\mathbf{q}$$

- The principle of the minimum total potential energy yields the equilibrium equation

$$\boxed{\mathbf{K}\,\mathbf{q} \;=\; \mathbf{g}}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Let's define two unrelated states for the body:
  - The σ-state : This shows external surface forces T, body forces f, and internal stresses σ in equilibrium.
  - The ε-state : This shows continuous displacements u* and consistent strains ε*.
  - The superscript * emphasizes that the two states are unrelated.

- The principle of virtual work then states: *External virtual work is equal to internal virtual work when equilibrated forces and stresses undergo unrelated but consistent displacements and strains*.

$$\int_V \epsilon^{*\,T}\,\sigma^T\,dV = \int_V \mathbf{u}^{*\,T}\,\mathbf{f}\,dV + \int_S \mathbf{u}^{*\,T}\,\mathbf{t}\,dS$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

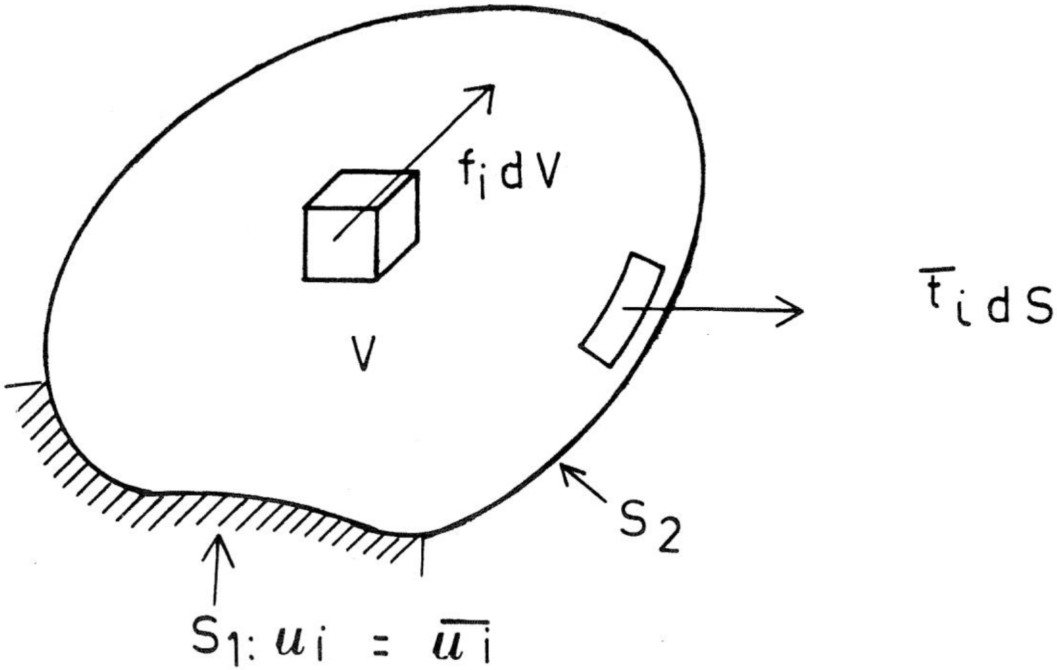
- We may specialize the virtual work equation and derive the principle of virtual displacements in variational notations :
  - Virtual displacements and strains as variations of the real displacements and strains using variational notation such as $\delta u = u^*$ and $\delta\varepsilon = \varepsilon^*$;
  - Virtual displacements be zero on the part of the surface that has prescribed displacements, and thus the work done by the reactions is zero. There remains only external surface forces on the part S_t that do work.

- The virtual work equation then becomes the principle of virtual displacements:

$$\int_V \delta\epsilon^T \sigma \, dV = \int_V \delta\mathbf{u}^T \mathbf{f} \, dV + \int_S \delta\mathbf{u}^T \mathbf{t} \, dS$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- This relation is equivalent to the set of equilibrium equations written for a differential element in the deformable body as well as of the stress boundary conditions on the part S_t of the surface.



ANALYSIS OF FE DISCRETIZED STRUCTURES

- Let's consider a system with known actual deformations ε, which are supposedly consistent, giving rise to displacements u throughout the system.
- For example, a point P has moved to P', and one wants to compute the displacement u_P of P in a considered direction **n**.
- For this particular purpose, we choose the following virtual unit force system:
  - The unit force $F^{(1)}$ is located at P and acts in the direction of **n** so that the external virtual work done by $F^{(1)}$ is, noting that the displacement in P along direction **n**.

$$F^{(1)} \times u_P = 1 \times u_P$$

  - The internal virtual work done by the virtual stresses is

$$\int_V \sigma^{(1)\,T} \epsilon \, dV$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Equating the two work expressions gives the desired displacement:

$$1 \times u_P = \int_V \sigma^{(1)\,T} \epsilon \, dV$$

- Let's consider the Principle of Virtual Work under discretized finite element form. Let's consider a variation of the displacement field δu. It is consistent with the strain field $\delta\varepsilon$.

$$\delta\epsilon = \mathbf{B}\,\delta\mathbf{q}$$

- The Principle of Virtual Work states the internal work of the stress under the variation of the strains is equal to the external work of the applied loads against the variation of the displacement field.

$$\delta\mathbf{q}^T\{\mathbf{K}\,\mathbf{q}\} = \delta\mathbf{q}^T\mathbf{g}$$

Berke’s approximation

VIRTUAL WORK

- The virtual work theorem states that the equality holds for any kinematically admissible virtual fields

$$\delta\mathbf{q}^T\{\mathbf{K}\,\mathbf{q}\} \;=\; \delta\mathbf{q}^T\mathbf{g}$$

- Let's consider the compatible displacement in equilibrium with any virtual load vector:

$$\mathbf{K}\,\tilde{\mathbf{q}} \;=\; \tilde{\mathbf{g}}$$

- The theorem of virtual work leads to:

$$\tilde{\mathbf{q}}^T\,\mathbf{g} = \mathbf{q}^T\,\tilde{\mathbf{g}} = \mathbf{q}^T\,\mathbf{K}\,\tilde{\mathbf{q}}$$

VIRTUAL WORK

- Use the virtual work,

$$\tilde{\mathbf{q}}^T\,\mathbf{g} = \mathbf{q}^T\,\tilde{\mathbf{g}} = \mathbf{q}^T\,\mathbf{K}\,\tilde{\mathbf{q}}$$

- By choosing a smart virtual displacement / vector field for instance, if the virtual load vector is chosen as a unit load vector under the displacement u that one wants to determine,

$$\tilde{\mathbf{g}} = (0\,0\,0\ldots1\ldots0\,0)^T$$

- One gets

$$\mathbf{q}^T\,\tilde{\mathbf{g}}\;=\;u\;\times\;1\;=\;\mathbf{q}^T\,\mathbf{K}\,\tilde{\mathbf{q}}$$

- With

$$\mathbf{K}\;=\;\sum_{i=1}^{NE}\mathbf{L}_i^T\mathbf{K}_i\,\mathbf{L}_i$$

VIRTUAL WORK

- For many design variables, the stiffness matrix takes the interesting form:

$$\mathbf{K}_i = x_i \, \bar{\mathbf{K}}_i$$

- For instance:
  - Truss structures $x_i = A_i$
  - Plate structures $x_i = t_i$
  - Beam structures $x_i = h_i^3$
  - Shell structures $x_i = t_i^3$

VIRTUAL WORK

- One can decompose the contribution of each element:

$$u = \mathbf{q}^T \mathbf{K} \tilde{\mathbf{q}} = \sum_{i=1}^{NE} \mathbf{q}^T \mathbf{L}_i^T \mathbf{K}_i \mathbf{L}_i \tilde{\mathbf{q}}$$

$$= \sum_{i=1}^{NE} \mathbf{q}_i^T \mathbf{K}_i \tilde{\mathbf{q}}_i = \sum_{i=1}^{NE} x_i \mathbf{q}_i^T \bar{\mathbf{K}}_i \tilde{\mathbf{q}}$$

- It is usual to define the *flexibility coefficients*:

$$c_i = x_i^2 \mathbf{q}_i^T \bar{\mathbf{K}}_i \tilde{\mathbf{q}}_i = x_i \mathbf{q}_i^T \mathbf{K}_i \tilde{\mathbf{q}}_i$$

- So that the expression of displacement writes

$$u = \sum_{i=1}^{NE} x_i^2 \mathbf{q}_i^T \bar{\mathbf{K}}_i \tilde{\mathbf{q}}_i \frac{1}{x_i} = \sum_{i=1}^{NE} \frac{c_i}{x_i}$$

VIRTUAL WORK

$$c_i = x_i^2 \, \mathbf{q}_i^T \bar{\mathbf{K}}_i \, \tilde{\mathbf{q}}_i \; = \; x_i \, \mathbf{q}_i^T \mathbf{K}_i \, \tilde{\mathbf{q}}_i$$

- For isostatic structures, we will show that these flexibility coefficient c_i are constant.

- One can intuitively understand the result. If the internal load remains constant, increasing the sizing variables will reduce the element displacements as the inverse of variables. In the proposed flexibility coefficient, the denominators and the numerators both evolves as x_i^2 and cancels each other. Of course in case of indeterminate structures, there is a redistribution of the load and the flexibility coefficient do not remain strictly constant so the assuming that the c_i coefficients are constant is only a local approximation around the current design point.

VIRTUAL WORK

- Let's now investigate the physical interpretation of the Berke expression using truss structures.

- Indeed in this particular case, it is easy to express the formula in terms of the forces. It comes:

$$
\begin{aligned}
u &= \mathbf{q}^T\,\mathbf{K}\,\tilde{\mathbf{q}} = \mathbf{g}^T\,\mathbf{K}^{-1}\,\tilde{\mathbf{g}} \\
&= \sum_i \mathbf{g}^T \mathbf{L}_i^T \mathbf{K}_i^{-1}\,\mathbf{L}_i\,\tilde{\mathbf{g}} \\
&= \sum_i \mathbf{g_i}^T \mathbf{K}_i^{-1}\,\tilde{\mathbf{g}}_i
\end{aligned}
$$

VIRTUAL WORK

- For truss structures, the compliance matrix (inverse of stiffness matrix) and the element load vectors have simple expressions since they are simple scalars:

$$\mathbf{g_i} = Q_i$$
$$\mathbf{K}_i^{-1} = \frac{l_i}{E_i\, x_i}$$

- Applying a unit dummy load case generates a system of internal loads $\tilde{Q}_i$ which are in equilibrium

- It comes

$$u \times 1 = \sum_i \frac{Q_i\, \tilde{Q}_i\, l_i}{E_i\, x_i}$$

VIRTUAL WORK

- Therefore the flexibility coefficients writes

$$c_i = x_i \, \mathbf{q}_i^T \mathbf{K}_i \, \tilde{\mathbf{q}}_i = \sum_i \mathbf{g_i}^T \mathbf{K}_i^{-1} \, \tilde{\mathbf{g}}_i = \frac{Q_i \, \tilde{Q}_i \, l_i}{E_i \, x_i} \, x_i$$

- For isostatic trusses, c_i is obviously constant since the element loads Q_i and $\tilde{Q}_i$ remain independent of the sizing variables!

- In the next chapter, it will be proved that Berke's explicit expression are in fact first order approximations of the real displacement. This approximation is equivalent to a first order Taylor expansion using a change of design variables, i.e. after using intermediate reciprocal variables.

VIRTUAL WORK

- For indeterminate structures, the load redistribution is generally weak and the c_i are nearly constant:

$$c_i \;=\; x_i\,\mathbf{q}_i^T\mathbf{K}_i\,\tilde{\mathbf{q}}_i \;=\; \sum_i x_i^2 \mathbf{q_i}^T\bar{\mathbf{K}}_i\,\tilde{\mathbf{q}}_i \;=\; \frac{Q_i\,\tilde{Q}_i\,l_i}{E_i\,x_i}\,x_i$$

- And the following expression is generally a very good expression of the displacement u:

$$\boxed{u \;=\; \sum_{i=1}^{NE} \frac{c_i}{x_i}}$$