OPTIMALITY CRITERIA

Pierre DUYSINX
LTAS – Automotive Engineering
Academic year 2020-2021

LAYOUT OF THE LESSON

- Introduction & Motivation
- Analysis using Principle of Virtual Work and Finite Element Method
- Optimality criteria for fully stressed design
- Berke's approximation of displacement
- Optimality criteria for a single displacement constraint
- Optimality criteria for several displacement and stress constraints

INTRODUCTION & MOTIVATION

Introduction: Historical Context

- First developments: 1960 L. Schmit
  - Extension of the approach carried out in economy, in chemical engineering, etc.

- Structural analysis: first Finite Element Models.
  - Only simple elements: bars, shear panels, plates, beams, shells…
  - Conquest of air and space: lightweight thin-wall structures
  - Geometrical modelling, free mesh generation, sensitivity analysis, etc. are not yet developed
  - Design variables are sizing variables attached to the F.E.

- Development of computers and their application to engineering

Introduction: Historical Context

- Mathematical Programming methods are under construction
  - Unconstrained minimization → OK
  - Linear programming (linear objective function subject to linear constraints) → OK
  - Nonlinear programming: nonlinear objective function subject to nonlinear constraints → Under development in the 1960ies
    - Extension of optimization methods for linear constraints
    - Strategy based on the following the active constraints
    - Alternating minimization phases and restoring phases to come to feasible domain → projection methods
    - Feasible direction methods

    ➔ Costly and so not applicable to engineering problems because non explicit problems requires one FE analysis at each function evaluation

Introduction: Historical Context

- Focus on sizing problems of linear elastic structures with thin walls
  - Design variables are the cross-sectional areas and plate thicknesses
  - Typical problems: n-bar truss…

- Mass minimization

- Research for the development of fast convergence and reliable algorithms involving little novel concepts (apart from the sensitivity analysis)
  - No additional development at F.E. level
  - Design variables are properties of F.E. (fixed mesh, discrete variables)

Introduction: Historical Context

- Search for an alternative approach to Mathematical Programming methods, which are too costly.
- Optimization becomes a particular field concerned with aerospace and research.

Introduction: Mathematical Programming approach

- MP approach aims at solving structural optimization problems combining a general numerical optimization algorithm and a computer simulation code (e.g. FEM)

- Different approaches are possible

- <u>Direct methods</u>
  - Projection methods
  - Feasible direction (Zoutendijk)
  - Reduced gradient

  - Solution cost is proportional to the problem size

Introduction: Mathematical Programming approach

- <u>Transformation methods:</u>

    - The solution cost of the transformed problem growths with its size.
    - The number of transformed problems to build and solve depends on the quality (accuracy, precision…) of the transformation

    - Two main groups of approaches
        - Unconstrained minimization methods
        - Methods based on approximations

Introduction: Mathematical Programming approach

- <u>Transformation methods:</u>
  - Unconstrained minimization methods
    - Interior penalty methods
    - Exterior penalty methods
    - Extended interior penalty methods
    - Augmented Lagrangian method

  - Methods based on approximations
    - Linear Sequential Programming
    - Quadratic Sequential Programming
    - Sequential Convex Linearization (CONLIN, MMA…)

Introduction: Mathematical Programming approach

- In any cases,
  - Mathematical Programming approach is generally reliable
  - The sensitivity analysis is necessary because MP are based on the derivatives (at least first order derivatives, the gradients)
  - Older methods are not applicable to structural optimization because of their poor performance

- Research since the 60ies
  - Reduction of the cost of sensitivity analysis
    - Automatic differentiation, iterative methods, approximate re-analysis
  - Development of optimality criteria methods
  - Development of approximation concepts

Introduction: Optimality Criteria methods

- OC are based on the (strong) hypothesis that we know the set of active constraints at optimum
- One makes use of KKT conditions to draw the redesign rules.
- Developments are generally based on isostatic cases and then extended to indeterminate structures

- The simplest optimality criterion: the Fully Stressed Design (FSD)

Introduction

- Structural optimization applied to sizing (weight minimization) problem

$$
\begin{array}{lll}
\min\limits_{x} & W(x) = \sum_{i=1}^{n} w_i\, x_i & \\
\text{s.t.:} & g_j(x) \;\leq\; 0 & (j = 1 \ldots m) \\
 & \underline{x}_i \;\leq\; x_i \;\leq\; \overline{x}_i & (i = 1 \ldots n)
\end{array}
$$

  - Finite element model
  - Design variables are the transverse sizes of the structural members (Fixed geometry and material properties)
  - Design restrictions

$$
g_j(x) \;\equiv\; \begin{array}{l} u_j(x) - \overline{u}_j \;\leq\; 0 \\ \sigma_j(x) - \overline{\sigma}_j \;\leq\; 0 \\ \underline{\omega}_j^2 - \omega_j^2(x) \;\leq\; 0 \\ \underline{\lambda}_j^2 - \lambda_j^2(x) \;\leq\; 0 \end{array}
$$

Introduction

- Design constraints $g_j(x)<0$
  - Implicit functions
  - Nonlinear functions
  - One constraint evaluation requires a complete FE analysis
- Side constraints: simple and explicit
  - Fabrication / technological / physical constraints
  - Treated separately in most methods
- Iterative process ➔ HIGH COST

INTRODUCTION

- Optimality criteria techniques (OC)
  - Highly specific
  - Intuitive techniques, simple
  - Convergence to a design that is not necessarily optimal (KKT conditions)
  - Difficulties in identifying the set of active constraints
  - Convergence instabilities
  - Small number of reanalyses, independent of the number of design variables

Résumé
  - Low cost
  - But uncertainty convergence

INTRODUCTION

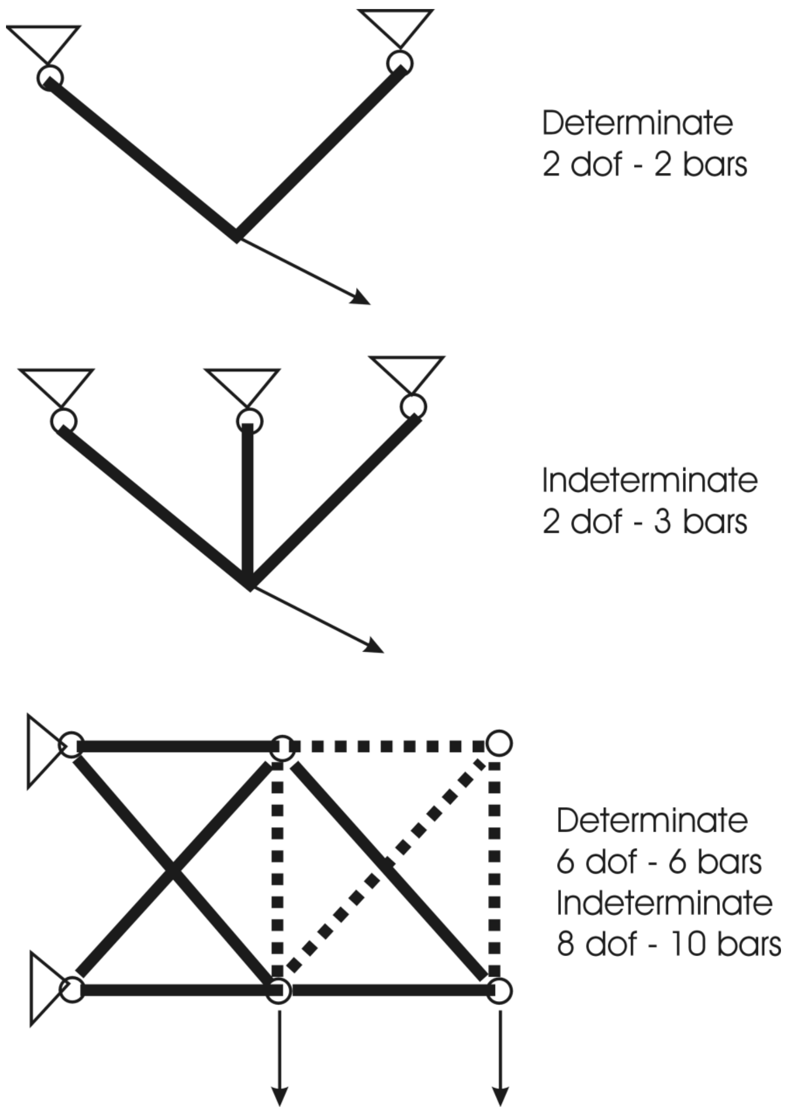
- Pure Mathematical Programming methods
  – Very general
  – Rigorous methods, quite elaborated
  – Convergence to a local minimum
  – Stable and monotonic convergence
  – Large number of reanalyzes, growing with the number of design variables

Résumé

  – Rigorous framework & guaranteed convergence
  – High cost (Growing computational cost with the size of the problem)

ANALYSIS USING
PRINCIPE OF VIRTUAL WORK
AND
FINITE ELEMENT

STATICALLY DETERMINATE AND INDETERMINATE STRUCTURES



- Statically determinate structures
  - #Unknowns = #Equilibrium equations
  - Equilibrium determines completely the problem
- Indeterminate structures
  - #Unknowns > #Equilibrium equations
  - Elastic redistribution of internal loads with the stiffness
  - Principle of minimum energy to determine unknowns

PRINCIPLE OF VIRTUAL WORK

- Let's define two unrelated states for the body:
  - The σ-state : This shows external surface forces t, body forces f, and internal stresses σ in equilibrium.
  - The ε-state : This shows continuous displacements u* and consistent strains ε^*.
  - The superscript * emphasizes that the two states are unrelated.

- The principle of virtual work then states: *External virtual work is equal to internal virtual work when equilibrated forces and stresses undergo unrelated but consistent displacements and strains*.

$$\int_V \epsilon^{*\,T} \sigma^T \, dV = \int_V \mathbf{u}^{*\,T} \mathbf{f} \, dV + \int_S \mathbf{u}^{*\,T} \mathbf{t} \, dS$$

PRINCIPLE OF VIRTUAL WORK

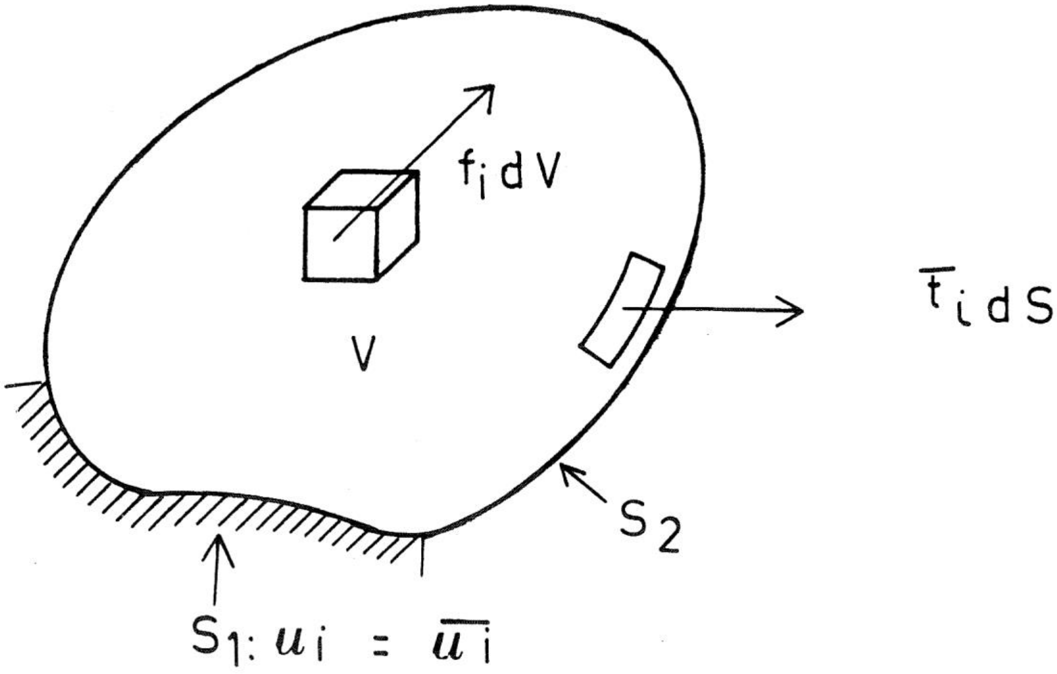
- We may specialize the virtual work equation and derive the principle of virtual displacements in variational notations :
  - Virtual displacements and strains as variations of the real displacements and strains using variational notation such as $\delta u = u^*$ and $\delta\varepsilon = \varepsilon^*$;
  - Virtual displacements be zero on the part of the surface that has prescribed displacements, and thus the work done by the reactions is zero. There remains only external surface forces on the part S_t that do work.

- The virtual work equation then becomes the principle of virtual displacements:

$$\int_V \delta\epsilon^T \sigma \, dV = \int_V \delta\mathbf{u}^T \mathbf{f} \, dV + \int_S \delta\mathbf{u}^T \mathbf{t} \, dS$$

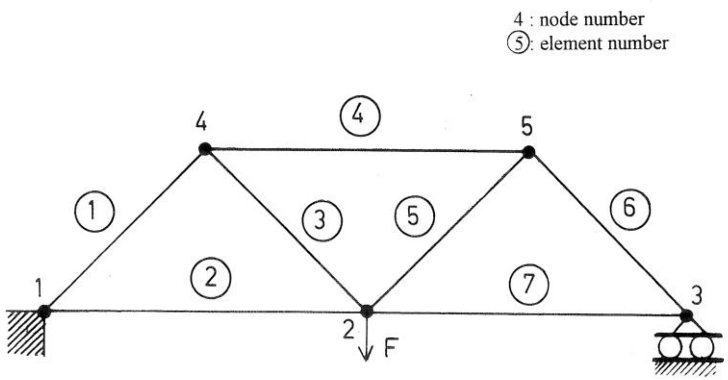
PRINCIPLE OF VIRTUAL WORK

- This relation is equivalent to the set of equilibrium equations written for a differential element in the deformable body as well as of the stress boundary conditions on the part S_t of the surface.



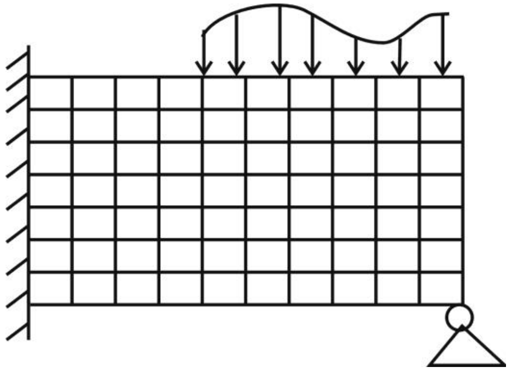
ANALYSIS OF FE DISCRETIZED STRUCTURES

- Let's suppose that the body is discrete in nature, for instance truss structure, or it is discretized into finite elements, i.e. continuum structure.



- The continuous displacement field **u**(x) in the elements can be approximated using local shape functions **N**(x) while the unknowns are the nodal displacements, which can be collected in the unknown vector **q**.

$$\mathbf{u}(\mathbf{X}) = \mathbf{N}(\mathbf{X})\,\mathbf{q}$$



ANALYSIS OF FE DISCRETIZED STRUCTURES

- The compatibility equations relates the displacements u to the strain components ε.

$$\varepsilon = \partial \mathbf{N}\,\mathbf{q} = \mathbf{B}\,\mathbf{q}$$

- The constitutive equations relate the stresses and the strains.
  - For a linear elastic behavior, the stress-strain relation is linear and writes in terms of the Hook coefficients:

$$\sigma = \mathbf{D}\,\varepsilon$$

- Inserting the strain matrix, one can calculate the stress in terms of the nodal displacements

$$\sigma = \mathbf{D}\,\mathbf{B}\,\mathbf{q} = \mathbf{T}\,\mathbf{q}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Let’s write the discretized form of the Principle of Virtual Work using finite element approximation:

$$\int_V \delta\epsilon^T \, \sigma \, dV \; = \; \int_V \delta\mathbf{u}^T \, \mathbf{f} \, dV \; + \; \int_S \delta\mathbf{u}^T \, \mathbf{t} \, dS$$

- Let's consider a variation of the displacement field δu.

$$\delta\mathbf{u} = \mathbf{N} \, \delta\mathbf{q}$$

- It is consistent with the strain field $\delta\varepsilon$.

$$\delta\epsilon = \mathbf{B} \, \delta\mathbf{q}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Internal virtual work:

$$
\begin{aligned}
\int_V \delta\epsilon^T \, \sigma \, dV &= \sum_{e=1}^{NE} \int_{V_e} \delta\mathbf{q}_e^T \, \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{q}_e \, dV \\
&= \sum_{e=1}^{NE} \delta\mathbf{q}_e^T \, \left[\int_{V_e} \mathbf{B}^T \, \mathbf{D} \, \mathbf{B} \, dV \right] \, \mathbf{q}_e \\
&= \sum_{e=1}^{NE} \delta\mathbf{q}^T \, \left[\, \mathbf{L}_e^T \, \mathbf{K}_e \, \mathbf{L}_e \, \right] \, \mathbf{q} \\
&= \delta\mathbf{q}^T \, \mathbf{K} \, \mathbf{q}
\end{aligned}
$$

- The element and the global stiffness matrices

$$
\mathbf{K}_e = \int_{V_e} \mathbf{B}^T \, \mathbf{D} \, \mathbf{B} \, dV \qquad \qquad \mathbf{K} = \sum_{e=1}^{NE} \mathbf{L}_e^T \, \mathbf{K}_e \, \mathbf{L}_e
$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- We have now to express the element degrees of freedom in terms of the degrees of freedom of the whole structure. Formally, the element displacement vector can be extracted from la the structural displacement vector by using a localization matrix $\mathbf{L}_e$ made of a few identity terms placed at the terms to be extracted.

$$\mathbf{q}_e = \mathbf{L}_e \, \mathbf{q}$$

- with

$$\mathbf{L}_e = \begin{bmatrix} 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 \end{bmatrix}$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Virtual work of external loads:

$$
\begin{aligned}
\int_V \delta \mathbf{u}^T \, \mathbf{f} \, dV \; + \; \int_S \delta \mathbf{u}^T \, \mathbf{t} \, dS
&= \sum_{e=1}^{NE} \{ \int_{V_e} \delta \mathbf{q}_e^T \, \mathbf{N}^T \mathbf{f} \, dV_e \; + \; \int_{\Gamma_{\sigma e}} \delta \mathbf{q}_e^T \, \mathbf{N}^T \, \mathbf{t} \, d\Gamma \} \\
&= \sum_{e=1}^{NE} \delta \mathbf{q}_e^T \, \mathbf{g}_e \; = \; \delta \mathbf{q}^T \sum_{e=1}^{NE} \mathbf{L}_e^T \, \mathbf{g}_e \\
&= \delta \mathbf{q}^T \, \mathbf{g}
\end{aligned}
$$

- The element and global load vector

$$
\mathbf{g}_e^T = \int_V \mathbf{f}^T \, \mathbf{N} \, dV_e \; + \; \int_{\Gamma_{\sigma e}} \mathbf{t}^T \, \mathbf{N} \, d\Gamma \qquad \qquad \mathbf{g} = \sum_{e=1}^{NE} \mathbf{L}_e^T \, \mathbf{g}_e
$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- The principle of virtual work discretized with Finite Element approximation writes:

$$\delta\mathbf{q}^T\,\mathbf{K}\,\mathbf{q} \;=\; \delta\mathbf{g}^T\,\mathbf{q}$$

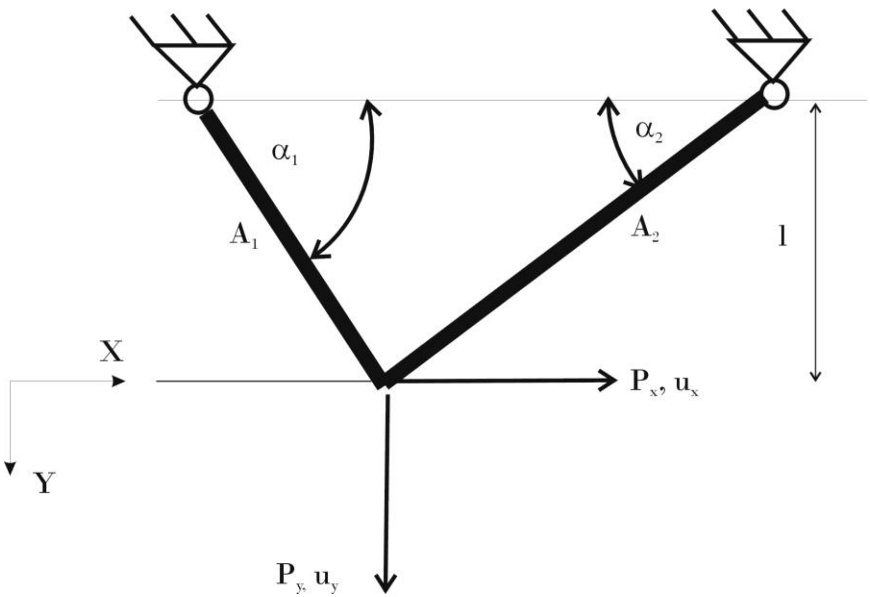
- The virtual displacement being arbitrary, the principle of virtual work yields the equilibrium equation

$$\boxed{\mathbf{K}\,\mathbf{q} \;=\; \mathbf{g}}$$

INTRODUCTION TO OPTIMALITY CRITERIA TWO BAR TRUSS

TWO-BAR TRUSS

- Let's consider the example of the two-bar truss



- **Equilibrium** between the external loads P_x, P_y and the internal efforts N_1 and N_2:

$$\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & -\cos\alpha_2 \\ \sin\alpha_1 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

- For a general structure with n bars and m node, this matrix equation becomes

$$\mathbf{P} = \mathbf{B}^T \, \mathbf{N}$$

TWO-BAR TRUSS

- Internal forces can be found by solving the matrix equations to yield

$$\mathbf{N} = (\mathbf{B}^T)^{-1}\,\mathbf{P}$$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \frac{1}{\sin(\alpha_1 + \alpha_2)} \begin{bmatrix} \sin\alpha_2 & \cos\alpha_2 \\ -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

- For statically determinate structures, the number of equilibrium equations is equal to the number of unknown member internal forces and so the matrix **B** is square and full rank. However generally speaking for indeterminate structure, the matrix **B** is rectangular, and this does not hold in general as it will be seen for the three-bar-truss

TWO-BAR TRUSS

- Thus for statically determinate structures, the internal bar forces depends only on the applied loads and of the direction cosines of the individual bars.

- **Stresses**

$$\sigma_i = \frac{N_i}{x_i}$$

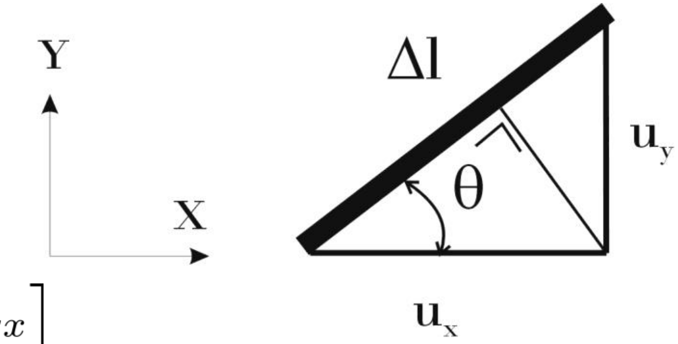
- They also depend on the applied load the geometry of the structure and the bar cross sectional areas **x***.

TWO-BAR TRUSS

- Let's write the compatibility conditions and relate nodal displacements to the applied loads.

- The elongation of the bars are related to the free node displacements

$$\Delta l_i = u_x \cos\theta + u_y \sin\theta$$



- It comes

$$\begin{bmatrix} \Delta l_1 \\ \Delta l_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\cos\alpha_2 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

- We recognize the strain matrix **B** connecting the strains to the nodal displacement.

$$\begin{bmatrix} \Delta l_1 \\ \Delta l_2 \end{bmatrix} = \mathbf{B} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

TWO-BAR TRUSS

- Bar strains

$$\epsilon_i = \frac{\Delta l_i}{l_i}$$

- Hook's law

$$\sigma_i = E\,\epsilon_i$$

- Bar forces

$$N_i = x_i\,\sigma_i = x_i\,E\,\frac{\Delta l_i}{l_i}$$

- It yields

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \frac{E_1 x_1}{l_1} & 0 \\ 0 & \frac{E_2 x_2}{l_2} \end{bmatrix} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \end{bmatrix}$$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \frac{E\,x_1}{l_1} & 0 \\ 0 & \frac{E\,x_2}{l_2} \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\cos\alpha_2 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

TWO-BAR TRUSS

- Write the applied loads in terms of the displacements.

$$\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & -\cos\alpha_2 \\ \sin\alpha_1 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} \frac{E\,x_1}{l_1} & 0 \\ 0 & \frac{E\,x_2}{l_2} \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\cos\alpha_2 & \sin\alpha_2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

- **Equation relating the applied loads to the displacements**

$$\mathbf{P} = \mathbf{B}^T\mathbf{D}\mathbf{B}\,\mathbf{u} = \mathbf{K}\,\mathbf{u}$$

- Generalized Hook matrix

$$\mathbf{D} = \begin{bmatrix} \frac{E\,x_1}{l_1} & 0 \\ 0 & \frac{E\,x_2}{l_2} \end{bmatrix}$$

- Stiffness matrix

$$\mathbf{K} = \mathbf{B}^T\mathbf{D}\mathbf{B}$$

TWO-BAR TRUSS

- Evaluation of the displacements

$$\mathbf{u} = \mathbf{K}^{-1}\,\mathbf{P}$$
$$= \mathbf{B}^{-1}\mathbf{D}^{-1}(\mathbf{B}^{T})^{-1}\,\mathbf{P} = \mathbf{B}^{-1}\mathbf{D}^{-1}\,\mathbf{N}$$

- **u** gives all the displacement at all nodes and it is more usual in optimization to be interested in a specific displacement $\mathbf{u}_j$ corresponding to the jth degrees of freedom.
- In order to extract the required components, the vector **u** can be multiplied by a vector $\mathbf{e}_j$ which contains '0' elements everywhere except for the jth component which contains a '1' at this position.

$$\mathbf{e}_{(j)}^{T} = \begin{bmatrix} 0 & 0\ldots & 1 & 0\ldots & 0 \end{bmatrix}$$

$$u_j = \mathbf{e}_{(j)}^{T}\,\mathbf{u} = \mathbf{e}_{(j)}^{T}\,\mathbf{B}^{-1}\mathbf{D}^{-1}\mathbf{N}$$

TWO-BAR TRUSS

- Remember that

$$\mathbf{N} = (\mathbf{B}^T)^{-1}\,\mathbf{P}$$

- It is interesting to remark that

$$(\mathbf{B}^T)^{-1}\mathbf{e}_{(j)} = \mathbf{n}_{(j)}$$

- It is interpreted as the set of internal forces in equilibrium with a unit load (1 N) applied on the degrees of freedom $\mathbf{u}_j$ and acting in the direction of displacement component.
- A unit load applied along degree of freedom j is called a dummy load.

- The expression of the displacement $\mathbf{u}_j$ becomes

$$u_j = \mathbf{n}_{(j)}^T\,\mathbf{D}^{-1}\mathbf{N}$$

TWO-BAR TRUSS

- The expression of the displacement $\mathbf{u}_j$

$$u_j = \mathbf{n}_{(j)}^T \, \mathbf{D}^{-1}\mathbf{N}$$

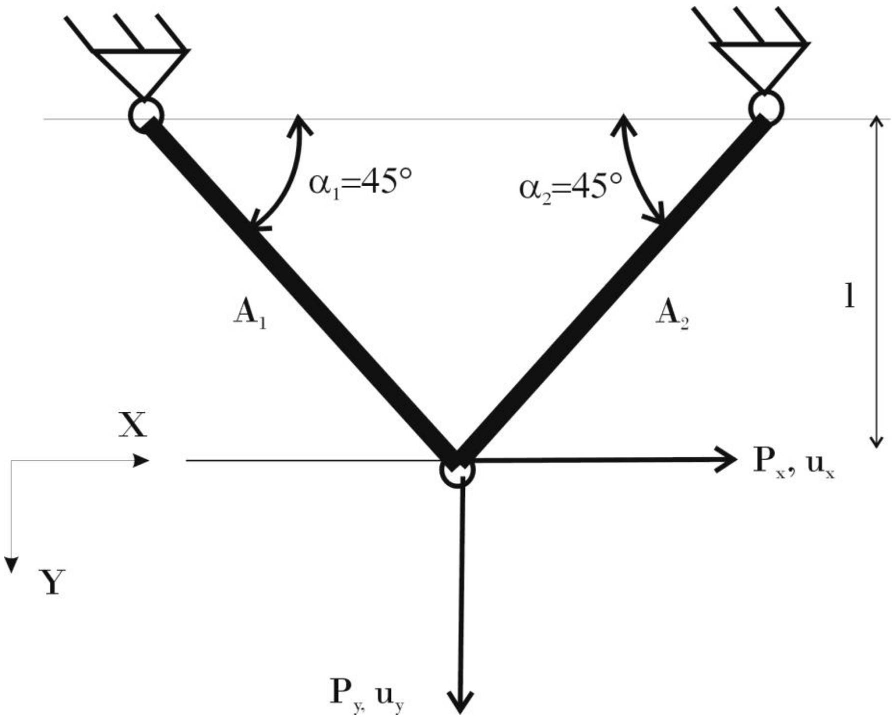
- Expanding this matrix product, we recover the familiar expression for calculating the magnitude of a specific nodal displacement in truss structures

$$u_j = \sum_{i=1}^{n} \frac{N_i \, l_i \, n_i^{(j)}}{E \, x_i}$$

- where $n_i^{(j)}$ represented components of the vector $\mathbf{n}^{(j)}$ and l_i and x_i are again the bar length and its cross-sectional areas respectively.

TWO-BAR TRUSS

- Let's consider the particular case the two-bar truss with 45° angles



$$\alpha_1 = 45^\circ \qquad \alpha_2 = 45^\circ$$

- **Equilibrium** between the external loads P_x, P_y and the internal efforts N_1 and N_2:

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \frac{1}{2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

$$\begin{cases} N_1 = \frac{\sqrt{2}}{2} P_x + \frac{\sqrt{2}}{2} P_y \\ N_2 = -\frac{\sqrt{2}}{2} P_x + \frac{\sqrt{2}}{2} P_y \end{cases}$$

TWO –BAR TRUSS

- We shall consider the following particular cases:

$$P_x = P, \qquad P_y = 0$$

- It comes

$$N_1 = \frac{\sqrt{2}}{2} P$$

$$N_2 = -\frac{\sqrt{2}}{2} P$$

- And the stresses

$$\sigma_1 = \frac{\sqrt{2}\, P}{2\, x_1}$$

$$\sigma_2 = -\frac{\sqrt{2}\, P}{2\, x_2}$$

TWO-BAR TRUSS

- We want now to evaluate the **displacement at the free node**.

- We use the dummy load case approach. Let's compute first the dummy load cases in both x and y directions at the free node.

  - $P_x=1$, $P_y=0$

$$n_1 = \frac{\sqrt{2}}{2} \qquad n_2 = -\frac{\sqrt{2}}{2}$$

  - $P_x=0$, $P_y=1$

$$n_1 = \frac{\sqrt{2}}{2} \qquad n_2 = \frac{\sqrt{2}}{2}$$

TWO-BAR TRUSS

- Insert these results into the expression

$$u_j = \sum_{i=1}^{n} \frac{N_i \, l_i \, n_i^{(j)}}{E \, x_i}$$

- For a horizontal displacement:

$$\begin{aligned} u_x &= \frac{l_1}{E \, x_1} \left(\frac{\sqrt{2}}{2} P \right) \left(\frac{\sqrt{2}}{2} \right) + \frac{l_2}{E \, x_2} \left(-\frac{\sqrt{2}}{2} P \right) \left(-\frac{\sqrt{2}}{2} \right) \\ &= \frac{\sqrt{2} \, P \, l}{2 \, E} \left(\frac{1}{x_1} + \frac{1}{x_2} \right) \end{aligned}$$

- For a horizontal displacement:

$$\begin{aligned} u_y &= \frac{l_1}{E \, x_1} \left(\frac{\sqrt{2}}{2} P \right) \left(\frac{\sqrt{2}}{2} \right) + \frac{l_2}{E \, x_2} \left(-\frac{\sqrt{2}}{2} P \right) \left(\frac{\sqrt{2}}{2} \right) \\ &= \frac{\sqrt{2} \, P \, l}{2 \, E} \left(\frac{1}{x_1} - \frac{1}{x_2} \right) \end{aligned}$$

TWO-BAR TRUSS

- The most elementary optimum design problem for this class of structure consists in finding a set of bar cross sectional areas which minimizes structural weight subject to limits on the allowable stresses in individual members.

$$\min \quad W = \sum_{i=1}^{2} \rho_i l_i x_i$$

$$\mathbf{s.t.}: \quad -\bar{\sigma} \leq \frac{N_i}{x_i} \leq \bar{\sigma}_i \quad i = 1, 2$$

$$0 \leq x_i \quad i = 1, 2$$

- Although the problem is in many aspects trivial, it nevertheless forms a useful model for illustrating some of the concepts which play important roles when more complex problems are considered.

TWO-BAR TRUSS

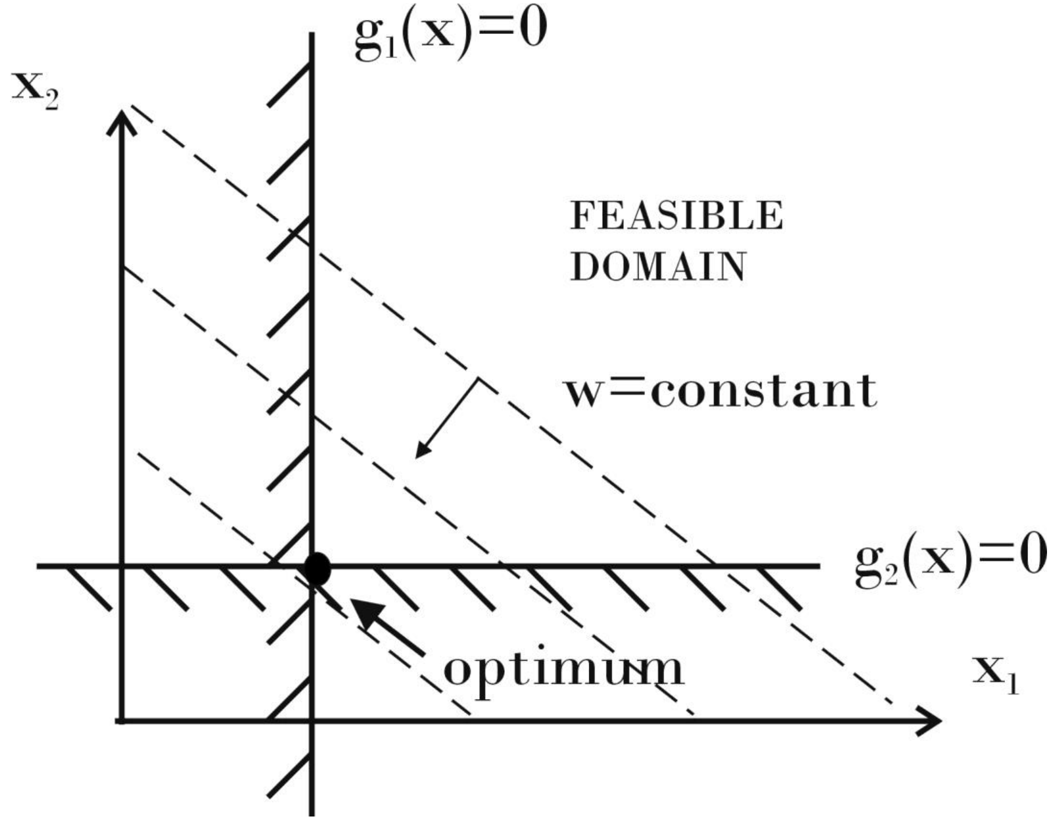
$$W = \rho l \sqrt{2}(x_1 + x_2)$$

$$\sigma_1 = \frac{\sqrt{2}\,P}{2\,x_1} \leq \bar{\sigma}$$

$$\sigma_2 = -\frac{\sqrt{2}\,P}{2\,x_2} \leq \bar{\sigma}$$

$$g_1(x_1) \equiv \frac{\sqrt{2}\,P}{2\,\bar{\sigma}} - x_1 \leq 0$$

$$g_2(x_2) \equiv \frac{\sqrt{2}\,P}{2\,\bar{\sigma}} - x_2 \leq 0$$



TWO-BAR TRUSS

- The design problems requires that we find the vector **x*** for a structure minimizing the weight subject to stress constraints σ.

- Because the structure is determinate each bar can be sized separately at the minimum value of the cross section to carry the applied loads.

- They optimized cross sectional areas are given by

$$x_i^\star = \frac{N_i}{\bar{\sigma}_i}$$

TWO-BAR TRUSS

- If we take as starting point the set of bar cross sections $\mathbf{x}^{(0)}$, and that we compute the related internal bar forces $N_i^{(0)}$ and stresses $\sigma_i^{(0)}$, the optimized cross sections that lead to reach the maximum allowable stress σ are given by the above formula:

$$x_i^\star = \frac{\sigma_i^{(0)}}{\bar{\sigma}}\, x_i^{(0)} \qquad i = 1, 2 \ldots n$$

- which we can immediately recognize as this stress-ratioing resuming reserving formula of the fully stressed design concepts, familiar in many practical application of machine design.

TWO BAR TRUSS

- Returning to the simple two bar truss problem with 45 degrees angle
- If a minimum weight design is now sought subject to limitation on the bar stresses, then the constraints imposed on the design problem becomes

$$\sigma_1 = \frac{\sqrt{2}\,P}{2\,x_1} \leq \bar{\sigma} \qquad\qquad g_1(x_1) \equiv \frac{\sqrt{2}\,P}{2\,\bar{\sigma}} - x_1 \leq 0$$

$$\sigma_2 = -\frac{\sqrt{2}\,P}{2\,x_2} \geq -\bar{\sigma} \quad \Rightarrow \quad g_2(x_2) \equiv \frac{\sqrt{2}\,P}{2\,\bar{\sigma}} - x_2 \leq 0$$

- The design restrictions are linear

TWO-BAR TRUSS

- The problem is linear, and the constraints are parallel to the axis defined by the design variables x_1, and x_2.
- It is clearly seen that each of these variable is associated with one and only one constraint and then the optimum design occurs at a vertex in design space.
- The optimum can therefore be fought by seeking to simultaneously satisfy the design constraints rather than seeking to actually minimize the objective function.

TWO-BAR TRUSS

- In later developments, we will show it is convenient to linearize the design constraints by using design variable defined as the reciprocal of the bar cross sectional areas.

$$z_i = \frac{1}{x_i} \qquad i = 1, 2 \ldots n$$

- The weight now becomes a nonlinear function

$$W = \sum_{i=1}^{n} \frac{\rho_i \, l_i}{z_i}$$

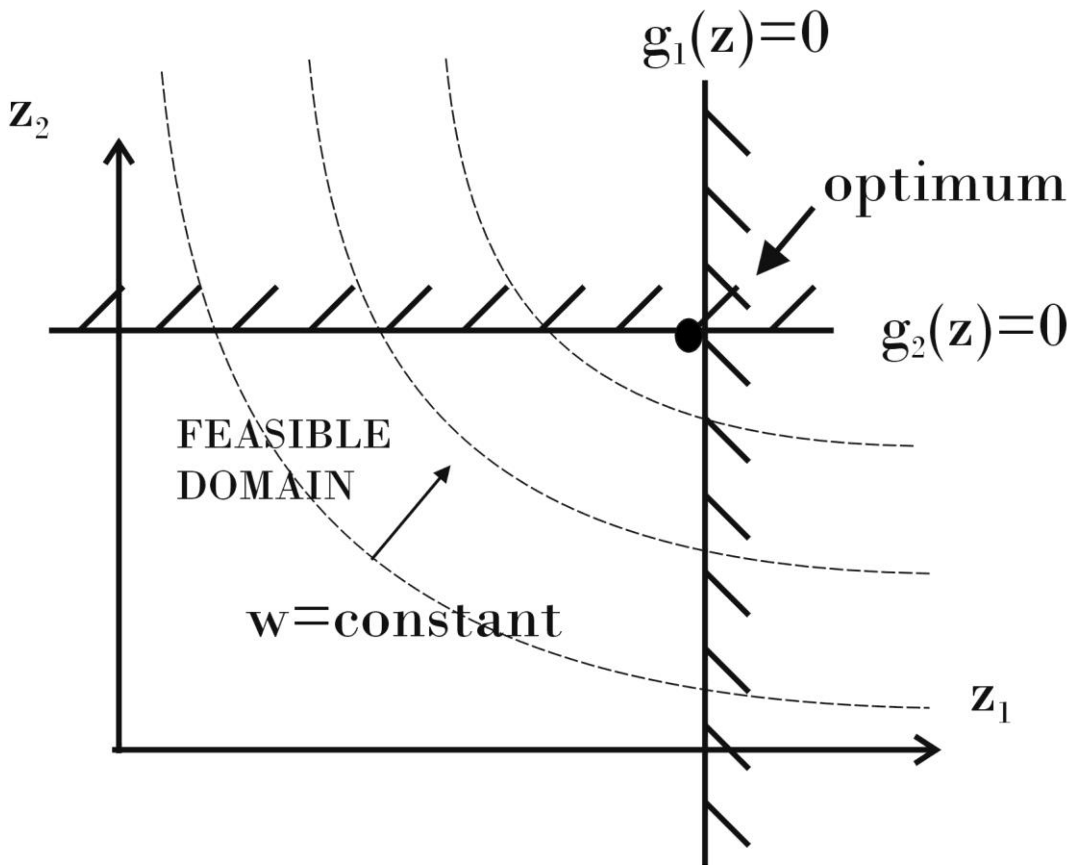
- The stress constraints remain linear function of the reciprocal variables

$$-\bar{\sigma} \leq N_i \, z_i \leq \bar{\sigma}_i$$

$$\sigma_1 = \frac{\sqrt{2} \, P \, z_1}{2} \leq \bar{\sigma}$$

$$\sigma_2 = -\frac{\sqrt{2} \, P \, z_2}{2} \geq -\bar{\sigma}$$

TWO-BAR TRUSS



$$g_1(x_1) \equiv \frac{\sqrt{2}\,P}{2}\,z_1 - \bar{\sigma} \leq 0$$

$$g_2(x_2) \equiv \frac{\sqrt{2}\,P}{2}\,z_2 - \bar{\sigma} \leq 0$$

TWO-BAR TRUSS

- We can continue our study of structural optimality theory by considering a statically determinate truss structure subject to constraints on specified nodal displacements.

- We seek for the minimum of the objective function, that is the structural weight, while satisfying to restriction over the two components of the nodal displacement.

$$\min \quad W = \sum_{i=1}^{2} \rho_i l_i x_i$$

$$\textbf{s.t.}: \quad u_x \leq \bar{u}_x$$

$$u_y \leq \bar{u}_y$$

$$0 \leq x_i \quad i = 1, 2$$

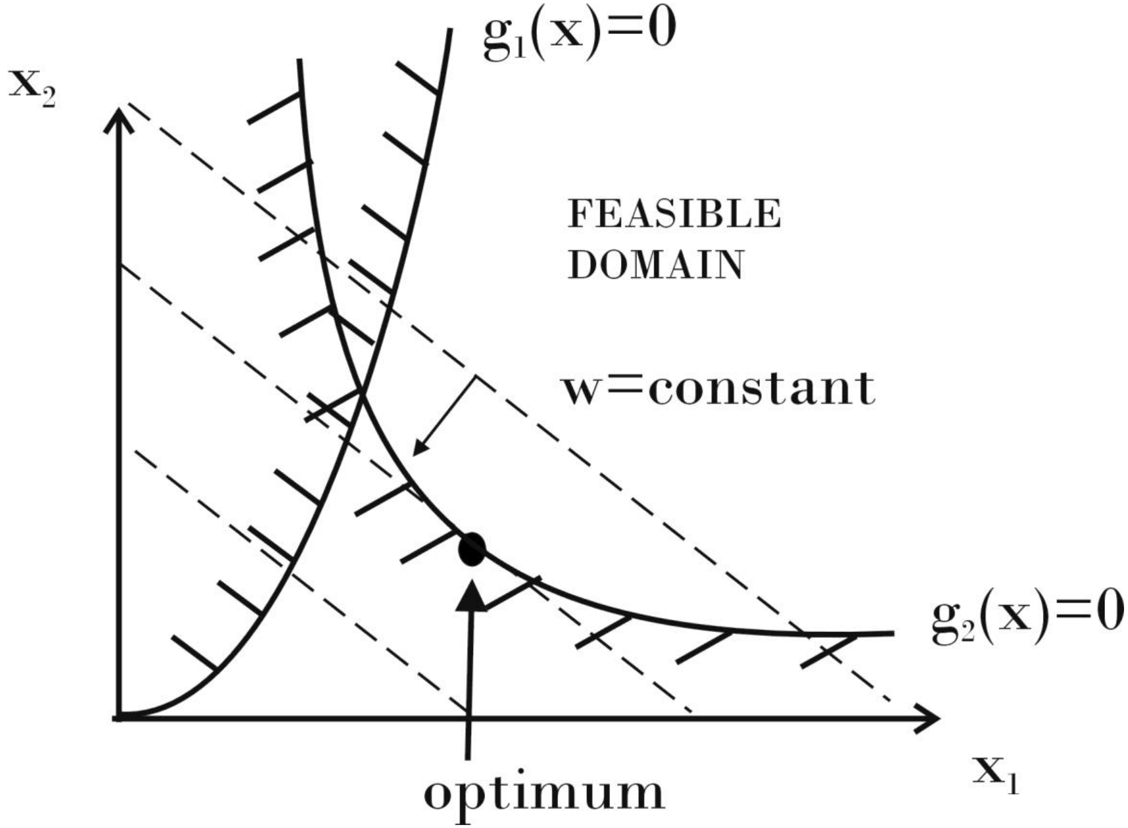
TWO-BAR TRUSS

- If we assume that the same material is used in each bar, the problem statement reads in the case of the two-bar truss with 45 degree:

$$
\begin{aligned}
\min \quad & W = \sqrt{2}\, l\, \rho\, (x_1 + x_2) \\
\textbf{s.t.}: \quad & g_1(x_1, x_2) \equiv \frac{\sqrt{2}\, P\, l}{2\, E} \left(\frac{1}{x_1} + \frac{1}{x_2} \right) - \bar{u}_x \leq 0 \\
& g_2(x_1, x_2) \equiv \frac{\sqrt{2}\, P\, l}{2\, E} \left(\frac{1}{x_1} - \frac{1}{x_2} \right) - \bar{u}_y \leq 0 \\
& 0 \leq x_i \quad i = 1, 2
\end{aligned}
$$

TWO-BAR TRUSS

- If the cross-sectional areas are taken as design variables, the problem may not be convex as it is usually illustrated by returning to the two-bar example.



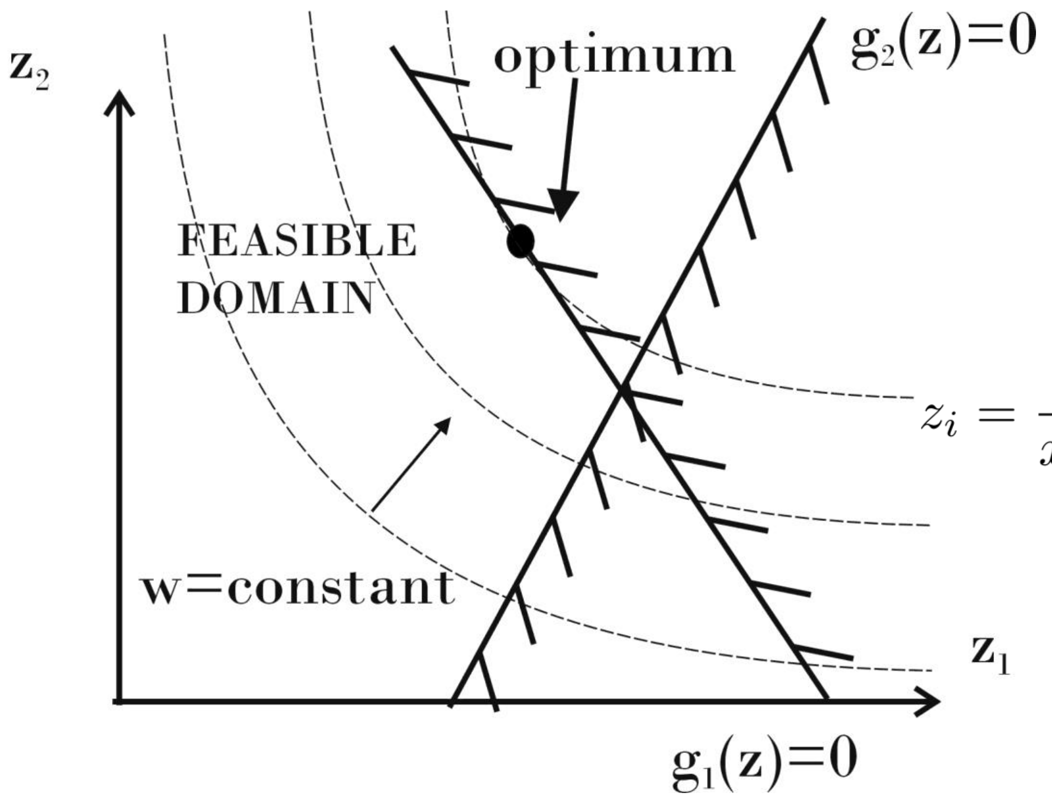
TWO-BAR TRUSS

- To circumvent this difficulty, we can take the hint given in the previous section and use the reciprocal of the cross-sectional areas as design variables.

- The two-bar truss displacement constraint problem now becomes

$$
\begin{aligned}
\min \quad & W = \sqrt{2}\, l\, \rho\, (\frac{1}{z_1} + \frac{1}{z_2}) \\
\textbf{s.t.}: \quad & g_1(z_1, z_2) \equiv \frac{\sqrt{2}\, P\, l}{2\, E}(z_1 + z_2) - \bar{u}_x \leq 0 \\
& g_2(z_1, z_2) \equiv \frac{\sqrt{2}\, P\, l}{2\, E}(z_1 - z_2) - \bar{u}_y \leq 0 \\
& 0 \leq z_i \quad i = 1, 2
\end{aligned}
$$

TWO-BAR TRUSS



$$z_i = \frac{1}{x_i} \qquad i = 1, 2 \ldots n$$

OPTIMALITY CRITERIA

- Principle of Virtual Work and Finite Element notations
- Optimality criteria for fully stressed design
- Berke’s approximation of displacement
- Optimality criteria for a single displacement constraint
- Optimality criteria for several displacement and stress constraints

OPTIMALITY CRITERIA

- General approach adopted by Optimality Criteria applied to the mass minimization problems

- Write a priori the conditions that must be satisfied by the optimal design.
  - KKT conditions of the optimum design problem
  - Based on isostatic problems

- Deduce a recursive relation to be iteratively applied to obtain the optimal design
  - "Primal design variables" (sizing variables) are given in term of the "dual variables" (Lagrange multipliers)
  - Update of "dual variables" (Lagrange multipliers) to satisfy the active constraints (and KKT conditions)

OPTIMALITY CRITERIA

- 1/ Optimality conditions are derived for isostatic (determinate) structures.
  - ➔ exact solution in that particular case
  - ➔ convergence in 1 iteration

- 2/ Extension to the general case of hyperstatic (indeterminate) structures
  - ➔ approximate solution
  - ➔ iterative scheme

OPTIMALITY CRITERIA

- Identified difficulties
  - Select a priori the set of active constraints that will be used in the optimality conditions
  - Convergence to design points which are not necessarily KKT point for the general case

OPTIMALITY CRITERIA

- Primal optimization problem with constraints

$$(P_{\text{Primal}}) \qquad \begin{array}{ll} \min\limits_{\mathbf{x}} & f(\mathbf{x}) \\ \text{s.t.:} & g_j(\mathbf{x}) \leq 0 \quad j = 1 \ldots m \end{array}$$

- Karush Kuhn Tucker optimality conditions

$$\nabla_x f(x^\star) + \sum_{j=1}^{m} \lambda_j \, \nabla_x g_j(x^\star) = 0$$

$$g_j(x^\star) \leq 0$$

$$\lambda_j \geq 0$$

$$\lambda_j^\star . g_j(x^\star) = 0$$

FULLY STRESSED DESIGN

FULLY STRESSED DESIGN

- The first considered optimality criteria is the most famous one: Fully Stressed Design
- It is founded on the intuitive hypothesis, but non analytically justified, that all components in the optimized structure reach simultaneously their maximum allowable stress, generally calculated based on a linear elastic analysis.
- It is simple, easy to implement, fast convergent
- Often used by engineers in practice for structures subject to stress restrictions only.

FULLY STRESSED DESIGN

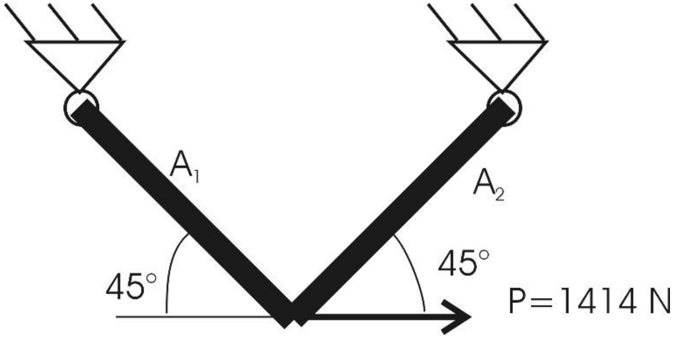
- Fully stressed design criterion (FSD)

  "The maximum allowable stress is attained in each member under at least one of the applied load cases"

- The mathematical statement of the problem of the minimum weight problem subject to stress constraints in each member i, writes as following

$$
\begin{array}{lll}
\min\limits_{\mathbf{x}} & w(\mathbf{x}) & \\
\text{s.t.:} & \sigma_{i,l}(\mathbf{x}) \ \leq \ \bar{\sigma}_i & i = 1 \ldots n; \ l = 1 \ldots c
\end{array}
$$

FSD for isostatic case: Example two-bar-truss



- Weight

$$
\begin{aligned}
W &= \rho_1\, l_1\, x_1 + \rho_2\, l_2\, x_2 \\
&= \rho\, l\, \sqrt{2}(x_1 + x_2)
\end{aligned}
$$

- Stress

$$
\sigma_1 = \frac{Q_1}{x_1} \qquad \sigma_2 = \frac{Q_2}{x_2}
$$

Efforts

$$
Q_1 = \frac{P}{\sqrt{2}} \qquad Q_2 = -\frac{P}{\sqrt{2}}
$$

- Optimization problem

$$
\begin{aligned}
\min_{x_1,\, x_2} \quad & \rho\, l\, \sqrt{2}(x_1 + x_2) \\
\text{s.t.:} \quad & \sigma_1 = \frac{P}{\sqrt{2}x_1} \le \bar{\sigma}_t \\
& \sigma_2 = \frac{P}{\sqrt{2}x_2} \le \bar{\sigma}_c
\end{aligned}
$$

FSD for isostatic case: Example two-bar-truss

- Optimum (analytical solution)

$$x_1^\star = \frac{P}{\sqrt{2}\bar{\sigma}_t} \quad x_2^\star = \frac{P}{\sqrt{2}\bar{\sigma}_c}$$

- Redesign formula

$$x_i^\star = \frac{Q_i}{\bar{\sigma}_i}$$

- If one performs a first analysis (which is not optimal) with the set of design variables $x^{(0)}$:

$$x_i^\star = \frac{\sigma_i^{(0)}}{\bar{\sigma}}\, x_i^{(0)} \qquad i = 1, 2 \ldots n$$

FSD for isostatic case: Example two-bar-truss

- FSD is exact in this case, because the two-bar-truss is a determinate structure.
- It is true for all statically determinate structures, because the internal efforts are constant and do not depend on the stiffness distribution.
- If one adds one bar (three bar truss), the truss is indeterminate, and the efforts depends on all the design variables. ➔ FSD becomes an approximation

FULLY STRESSED DESIGN

- Stress in member 'i' under load case 'l'

$$\sigma_{il} = \frac{Q_{il}}{x_i}$$

With x_i the cross section and Q_i the member force

- Bounding the maximum stress in member i for any load case l writes

$$\max_l \sigma_{il} \leq \overline{\sigma}_i \Leftrightarrow \max_l \frac{Q_{il}}{x_i} \leq \overline{\sigma}_i$$

$$\Leftrightarrow x_i \geq \max_l \frac{Q_{il}}{\overline{\sigma}_i}$$

- Iteration scheme: stress ratio formula

$$x_i^{(k+1)} = x_i^{(k)} \max_{l=1\ldots c} \{ \frac{\sigma_{i,l}^{(k)}}{\bar{\sigma}_i} \}$$

FULLY STRESSED DESIGN

- Stress ratio formula

$$x_i^{(k+1)} = x_i^{(k)} \max_{l=1\ldots c}\{\frac{\sigma_{i,l}^{(k)}}{\bar{\sigma}_i}\}$$

- The formula is rigorous for one single load case, one material.
- For statically indeterminate structures, the FSD is approximate.

- FSD can be extended to other elements than truss structures, for instance by considering the von Mises stress (e.g. in plane stress plate) in the stress ratio formula

$$\sigma_{i,l} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\,\sigma_y + 3\tau_{xy}^2}$$

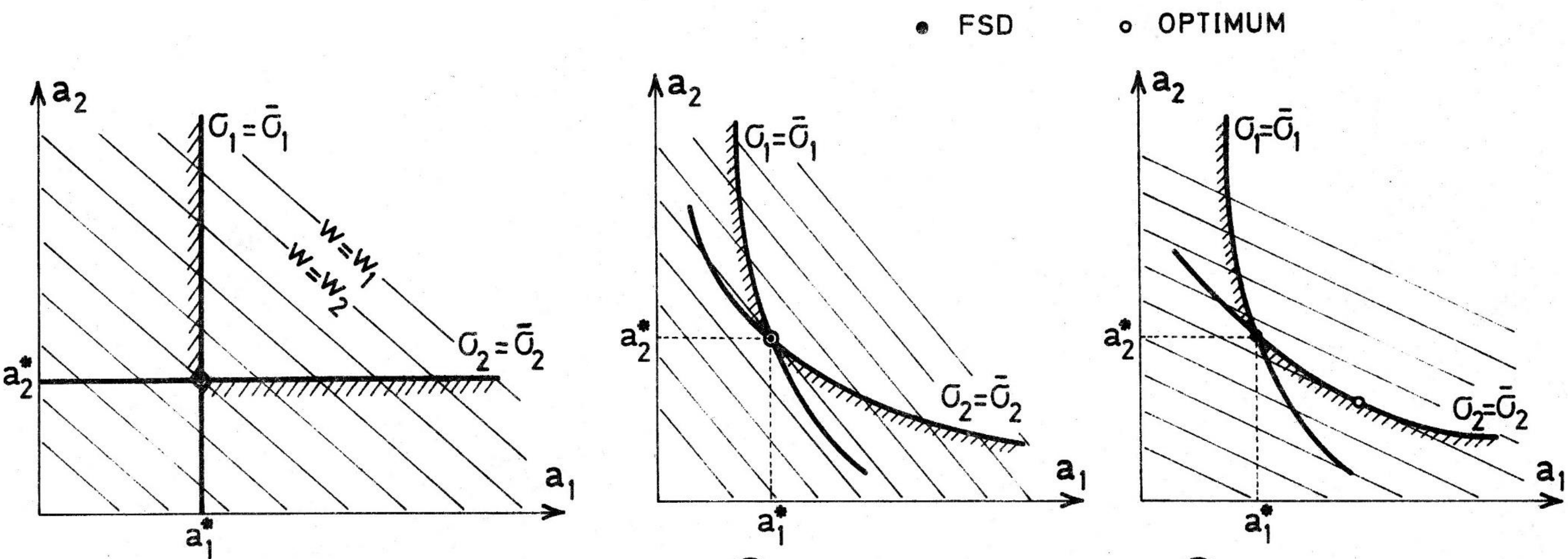
FULLY STRESSED DESIGN

- Fast convergence (When it converges!)
- The number of reanalysis steps (F.E. calculations) is small and weakly dependent to the number of design variables
- No sensitivity analysis is required
- Simple criteria.
  - Easy to implement in structural analysis computer codes.
  - Independent of the F.E. code.
- Minimum size gauge can be added

$$x_i^\star = \max\{\underline{x}_i, \frac{\sigma_{i,l}}{\bar{\sigma}} x_i^0\}$$

FULLY STRESSED DESIGN

- FSD Leads to a vertex of the design space.



CRITERE DU FSD POUR UNE STRUCTURE ISOSTATIQUE

(a) FSD OPTIMAL

(b) FSD NON OPTIMAL

CRITERE DU FSD POUR UNE STRUCTURE HYPERSTATIQUE

FSD for general case (hyperstatic)

- FSD replaces the stress restrictions by hyperplanes parallel to the axes

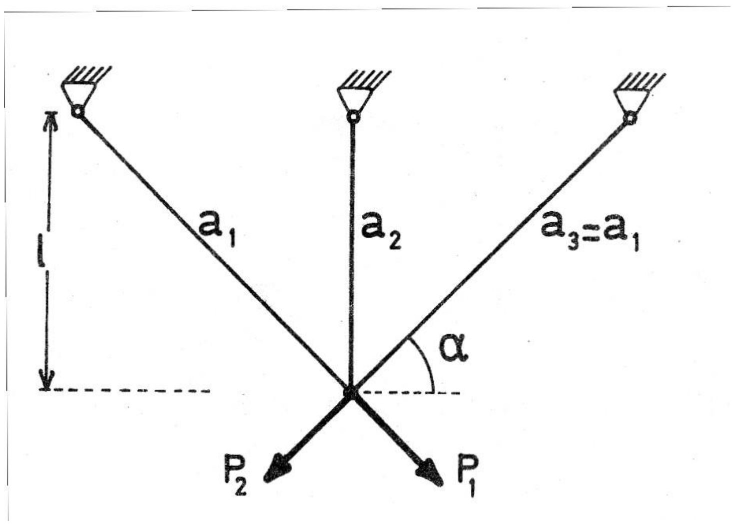
$$\sigma_i = \frac{Q_i}{x_i} \leq \bar{\sigma} \quad \Rightarrow \quad g_i(x_i) \equiv \frac{Q_i}{\bar{\sigma}} - x_i \leq 0$$

- The objective function disappears from the formulation
  - Provided that all coefficient in the objective function are positive

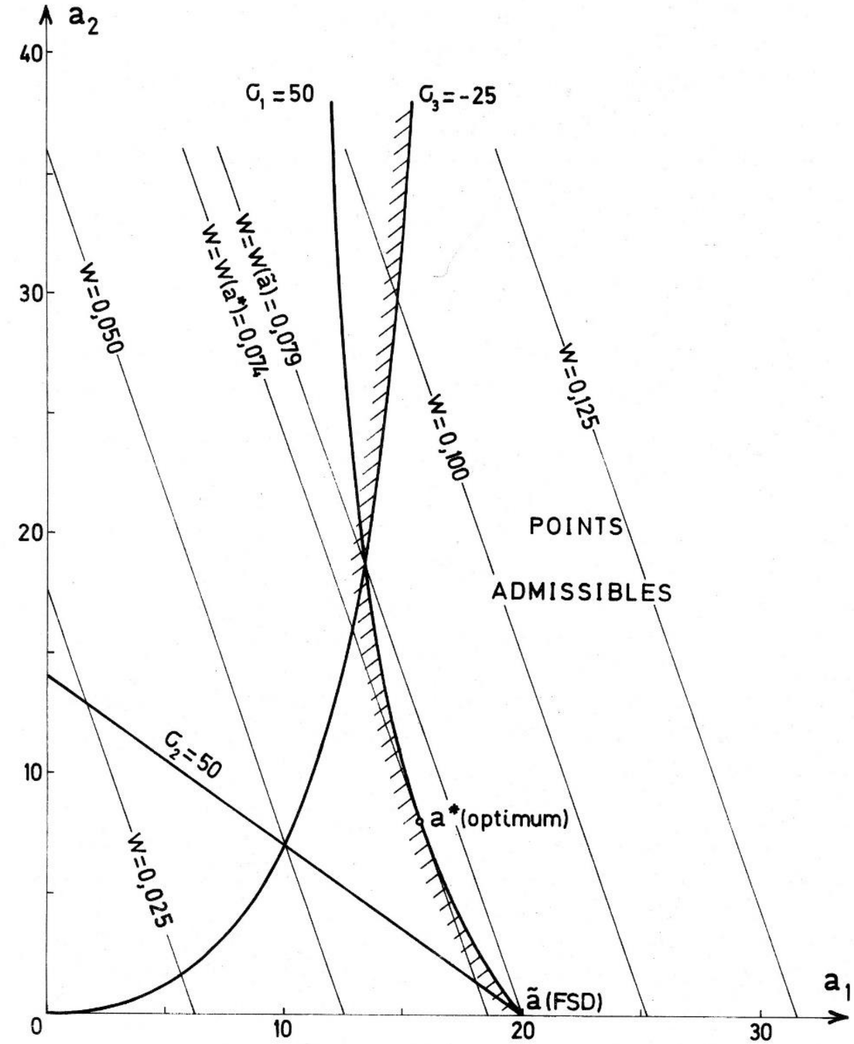
- Solution is always located in a vertex of the design space
  - Not always the case when strongly hyperstatic problems with redistribution of the internal loads
  - In these cases, it can lead to non optimal solutions and oscillatory convergence processes.

Example FSD

- Three bar truss structure



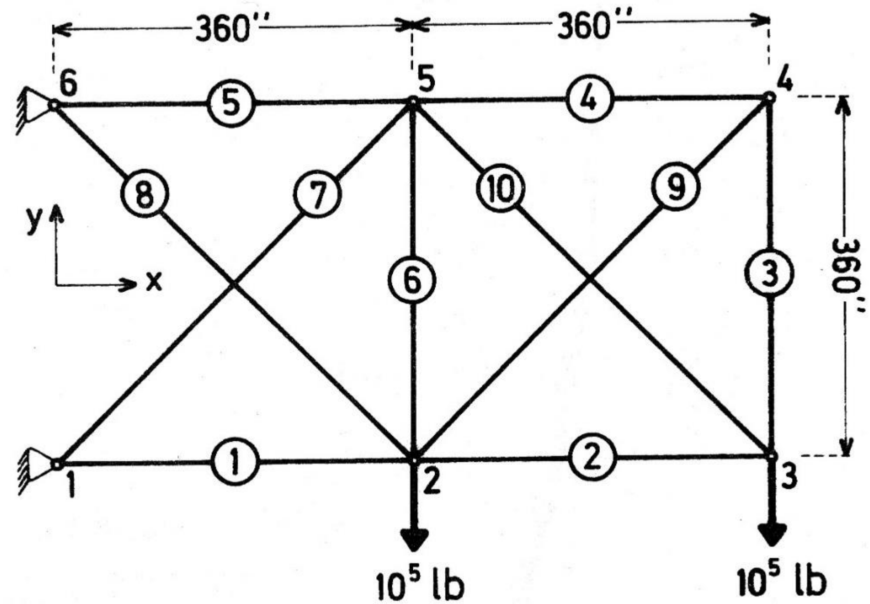
FSD is not always the optimum



Ten-bar-truss example

- The stress-ratioïng itself tends to increase the design variable with the smallest stress limit

- Example: stress limit = 25000psi except in member 8 with a variable limit from 25000 to 70000 psi



materiau	:	aluminium
tension maximale admissible	:	25 000 psi
module d'elasticite	:	10^7 psi
masse specifique	:	0.1 lb/in³
section minimale admissible	:	0.1 in²
deplacement maximal admissible	:	2.0 in
mise en charge	:	unique

Ten-bar-truss example

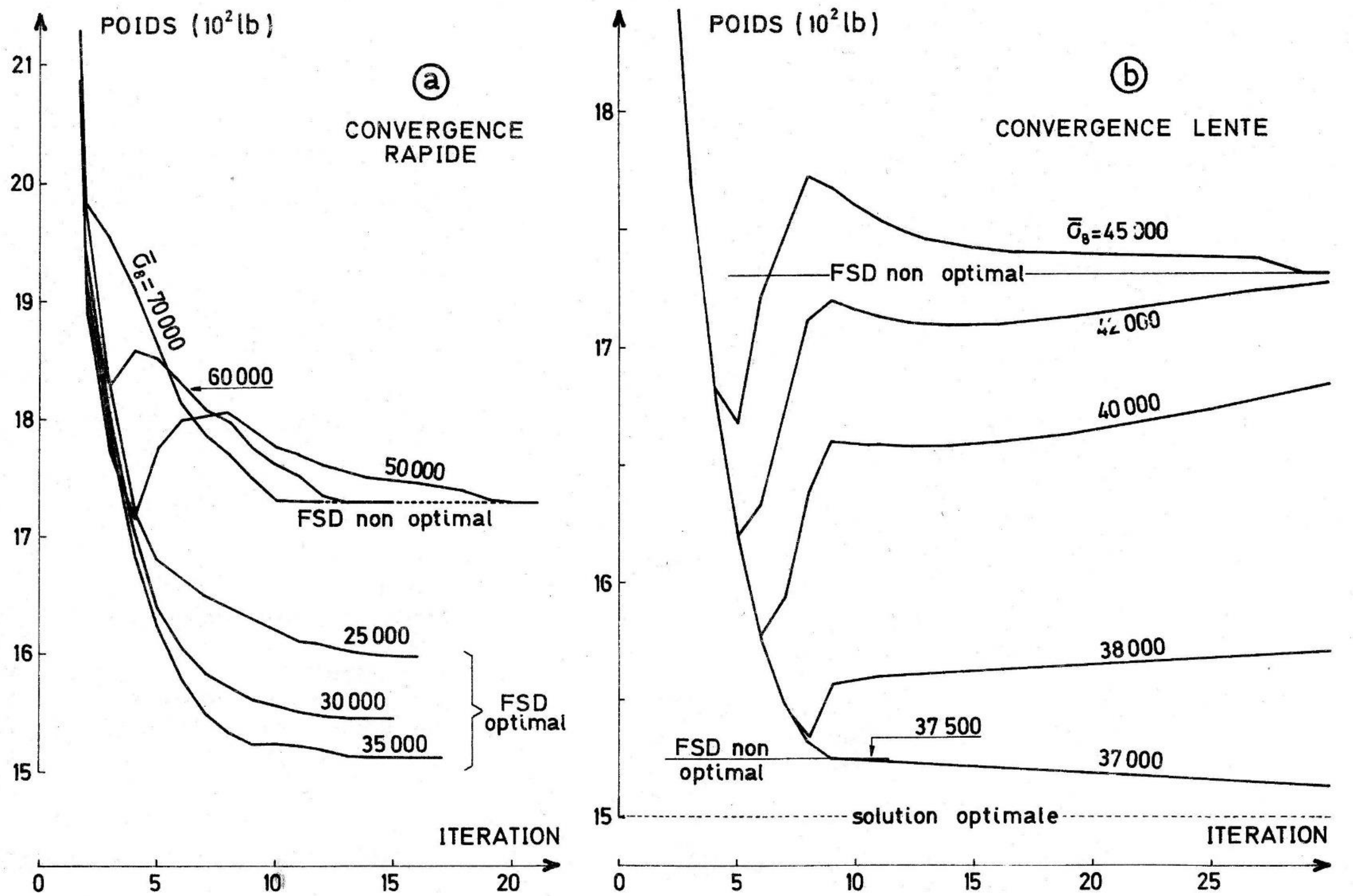
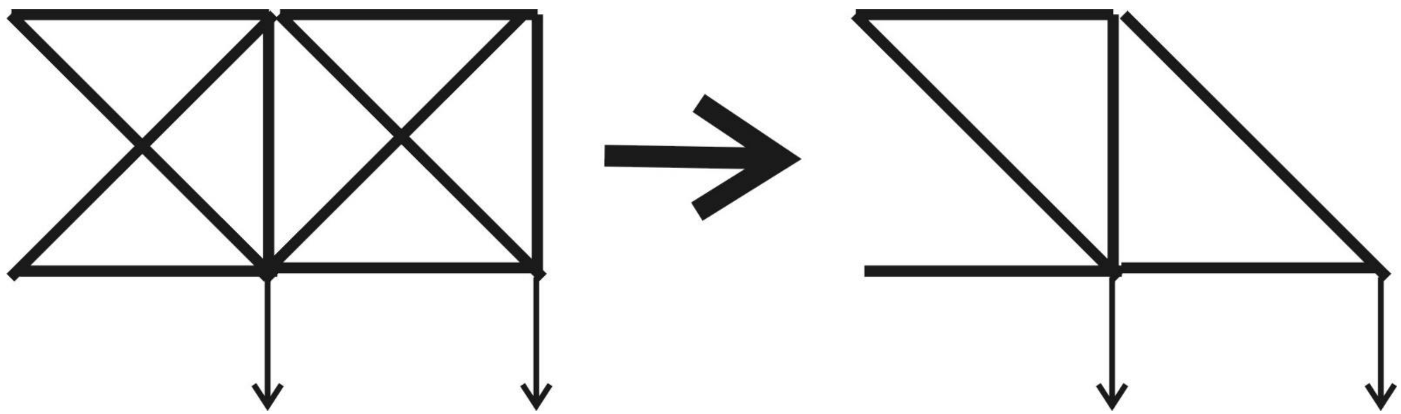


FIG. 3.7 TREILLIS 10 BARRES - CRITERE DU FSD (pour diverses valeurs de $\bar{\sigma}_8$)

FULLY STRESSED DESIGN

- FSD leads to a statically determinate structure extracted from the initial truss structure.



Interpretation of FSD

- Stress ratio formula:

$$\tilde{x}_i = x_i^0 \frac{\sigma_i^0}{\overline{\sigma}_i} = \frac{Q_i}{\overline{\sigma}_i}$$

- The real stress constraint is implicit.

$$\sigma_i(\boldsymbol{x}) \leq \overline{\sigma}_i$$

- It is replaced with an explicit approximation of the stress constraint:

$$\boxed{\begin{aligned} \tilde{\sigma}_i(\boldsymbol{x}) &\leq \overline{\sigma}_i \\ \tilde{\sigma}_i(\boldsymbol{x}) &= \sigma_i^0 \frac{x_i^0}{x_i} = \frac{Q_i^0}{x_i} \leq \overline{\sigma}_i \end{aligned}}$$

Interpretation of FSD

- The value of the approximation is exact in x^0

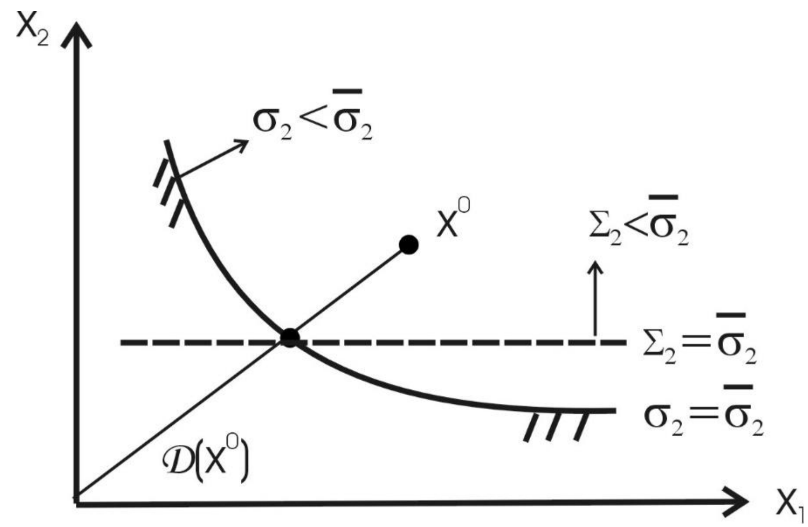
$$\tilde{\sigma}_i(\boldsymbol{x}^0) = \sigma_i^0 \frac{x_i^0}{x_i^0} = \sigma_i^0$$

- It is also exact along the scaling line

$$\tilde{\sigma}_i(\boldsymbol{x}) = \overline{\sigma}_i \quad \forall \boldsymbol{x} \in \mathcal{D}(\boldsymbol{x}^0)$$

- The derivatives are not respected

- FSD ➔ Zero order approximation in x^0 (and also along the scaling line)



Berke’s approximation

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Let's consider a system with known actual deformations ε, which are supposedly consistent, giving rise to displacements u throughout the system.
- For example, a point P has moved to P', and one wants to compute the displacement u_P of P in a considered direction **n**.
- For this particular purpose, we choose the following virtual unit force system:
  - The unit force $F^{(1)}$ is located at P and acts in the direction of **n** so that the external virtual work done by $F^{(1)}$ is, noting that the displacement in P along direction **n**.

$$F^{(1)} \times u_P = 1 \times u_P$$

  - The internal virtual work done by the virtual stresses is

$$\int_V \sigma^{(1)\,T}\, \epsilon \, dV$$

ANALYSIS OF FE DISCRETIZED STRUCTURES

- Equating the two work expressions gives the desired displacement:

$$1 \times u_P = \int_V \sigma^{(1)\,T} \, \epsilon \, dV$$

- Let's consider the unit load (1 N) applied on the considered displacement along the positive direction n

$$\tilde{\mathbf{g}} = (0\,0\,0 \ldots 1 \ldots 0\,0)^T$$

- The internal displacement field which leads to equilibrium while satisfying compatibility equations is solution

$$\mathbf{K}\tilde{\mathbf{q}} = \tilde{\mathbf{g}}$$

VIRTUAL WORK

- Use the virtual work,

$$1 \times u_P = \int_V \sigma^{(1)\,T}\,\epsilon\,dV$$

- It comes

$$\tilde{\mathbf{g}}^T\,\mathbf{q} = \int_V \tilde{\mathbf{q}}^T\,\mathbf{T}\mathbf{B}\,\mathbf{q}dV = \tilde{\mathbf{q}}^T\left[\int_V \mathbf{B}^T\mathbf{D}\mathbf{B}\,dV\right]\mathbf{q}$$
$$= \tilde{\mathbf{q}}^T\,\mathbf{K}\,\mathbf{q} = \tilde{\mathbf{q}}^T\,\mathbf{g}$$

- One gets

$$\mathbf{q}^T\,\tilde{\mathbf{g}} = u \times 1 = \mathbf{q}^T\,\mathbf{K}\,\tilde{\mathbf{q}}$$

- With

$$\mathbf{K} = \sum_{i=1}^{NE}\mathbf{L}_i^T\mathbf{K}_i\,\mathbf{L}_i$$

VIRTUAL WORK

- For many design variables, the stiffness matrix takes the interesting form:

$$\mathbf{K}_i = x_i \, \bar{\mathbf{K}}_i$$

- For instance:
  - Truss structures $x_i = A_i$
  - Plate structures $x_i = t_i$
  - Beam structures $x_i = h_i^3$
  - Shell structures $x_i = t_i^3$

VIRTUAL WORK

- One can decompose the contribution of each element:

$$u = \mathbf{q}^T \, \mathbf{K} \, \tilde{\mathbf{q}} = \sum_{i=1}^{NE} \mathbf{q}^T \mathbf{L}_i^T \mathbf{K}_i \, \mathbf{L}_i \, \tilde{\mathbf{q}}$$

$$= \sum_{i=1}^{NE} \mathbf{q}_i^T \mathbf{K}_i \, \tilde{\mathbf{q}}_i = \sum_{i=1}^{NE} x_i \, \mathbf{q}_i^T \bar{\mathbf{K}}_i \, \tilde{\mathbf{q}}$$

- It is usual to define the *flexibility coefficients*:

$$c_i = x_i^2 \, \mathbf{q}_i^T \bar{\mathbf{K}}_i \, \tilde{\mathbf{q}}_i = x_i \, \mathbf{q}_i^T \mathbf{K}_i \, \tilde{\mathbf{q}}_i$$

- So that the expression of displacement writes

$$u = \sum_{i=1}^{NE} x_i^2 \, \mathbf{q}_i^T \bar{\mathbf{K}}_i \, \tilde{\mathbf{q}}_i \, \frac{1}{x_i} = \sum_{i=1}^{NE} \frac{c_i}{x_i}$$

VIRTUAL WORK

$$\boxed{c_i = x_i^2\, \mathbf{q}_i^T \bar{\mathbf{K}}_i\, \tilde{\mathbf{q}}_i \;=\; x_i\, \mathbf{q}_i^T \mathbf{K}_i\, \tilde{\mathbf{q}}_i}$$

- For isostatic structures, we will show that these flexibility coefficient c_i are constant.

- One can intuitively understand the result. If the external load remains constant, increasing the sizing variables will reduce the element displacements as the inverse of variables. In the proposed flexibility coefficient, the denominators and the numerators both evolves as x_i^2 and cancels each other. Of course in case of indeterminate structures, there is a redistribution of the load and the flexibility coefficient do not remain strictly constant so the assuming that the c_i coefficients are constant is only a local approximation around the current design point.

VIRTUAL WORK

- Let's now investigate the physical interpretation of the Berke's expression using truss structures.

- Indeed in this particular case, it is easy to express the formula in terms of the forces. It comes:

$$
\begin{aligned}
u &= \mathbf{q}^T\,\mathbf{K}\,\tilde{\mathbf{q}} = \mathbf{g}^T\,\mathbf{K}^{-1}\,\tilde{\mathbf{g}} \\
&= \sum_i \mathbf{g}^T \mathbf{L}_i^T \mathbf{K}_i^{-1}\,\mathbf{L}_i\,\tilde{\mathbf{g}} \\
&= \sum_i \mathbf{g_i}^T \mathbf{K}_i^{-1}\,\tilde{\mathbf{g}}_i
\end{aligned}
$$

VIRTUAL WORK

- For truss structures, the compliance matrix (inverse of stiffness matrix) and the element load vectors have simple expressions since they are simple scalars:

$$\begin{aligned} \mathbf{g_i} &= Q_i \\ \mathbf{K}_i^{-1} &= \frac{l_i}{E_i\, x_i} \end{aligned}$$

- Applying a unit dummy load case generates a system of internal loads $\tilde{Q}_i$ which are in equilibrium

- It comes

$$u \times 1 = \sum_i \frac{Q_i\, \tilde{Q}_i\, l_i}{E_i\, x_i}$$

VIRTUAL WORK

- Therefore the flexibility coefficients writes

$$c_i = x_i \, \mathbf{q}_i^T \mathbf{K}_i \, \tilde{\mathbf{q}}_i = \sum_i \mathbf{g_i}^T \mathbf{K}_i^{-1} \, \tilde{\mathbf{g}}_i = \frac{Q_i \, \tilde{Q}_i \, l_i}{E_i \, x_i} \, x_i$$

- For isostatic trusses, c_i is obviously constant since the element loads Q_i and $\tilde{Q}_i$ remain independent of the sizing variables!

- For indeterminate structures, the c_i's are nor constant and the Berke's explicit expression is in fact first order approximations of the real displacement. This approximation is equivalent to a first order Taylor expansion using a change of design variables, i.e. after using intermediate reciprocal variables.

VIRTUAL WORK

- For indeterminate structures, the load redistribution is generally weak and the c_i are nearly constant:

$$c_i \;=\; x_i\,\mathbf{q}_i^T\mathbf{K}_i\,\tilde{\mathbf{q}}_i \;=\; \sum_i x_i^2\mathbf{q_i}^T\bar{\mathbf{K}}_i\,\tilde{\mathbf{q}}_i \;=\; \frac{Q_i\,\tilde{Q}_i\,l_i}{E_i\,x_i}\,x_i$$

- And the following expression is generally a very good expression of the displacement u:

$$\boxed{u \;=\; \sum_{i=1}^{NE}\frac{c_i}{x_i}}$$

Berke's expression is a first order approximation

- The approximation is exact in x_i^0

$$\tilde{u}(\boldsymbol{x}_i^0) = \sum_i \frac{c_i^0}{x_i^0} = \sum_i \frac{(\boldsymbol{q}_i^{0T}\, \boldsymbol{K}_i^0\, \tilde{\boldsymbol{q}}_i^0)\, x_i^0}{x_i^0} = \sum_i (\boldsymbol{q}_i^{0T}\, \boldsymbol{K}_i^0\, \tilde{\boldsymbol{q}}_i^0) = u^0$$

  - As c_i^0 remains constant only along $D(x^0)$. It is also true for all points along the scaling line

- The derivatives of the approximations are exact in x_i^0

$$\left.\frac{\partial \tilde{u}}{\partial x_i}\right|_{\boldsymbol{x}^0} = \left.\frac{\partial u}{\partial x_i}\right|_{\boldsymbol{x}^0}$$

Berke’s expression is a first order approximation

- The derivatives of the Berke’s approximation in x^0:

$$\left.\frac{\partial \tilde{u}}{\partial x_i}\right|_{\boldsymbol{x}^0} = -\frac{c_i^0}{(x_i^0)^2}$$

- If one remembers the definition of the mutual energy coefficients

$$c_i^0 = x_i^0\, \boldsymbol{q}_i^{0T} \boldsymbol{K}_i\, \tilde{\boldsymbol{q}}_i^0$$

- It comes

$$\left.\frac{\partial \tilde{u}}{\partial x_i}\right|_{\boldsymbol{x}^0} = -\frac{\boldsymbol{q}_i^{0T} \boldsymbol{K}_i\, \tilde{\boldsymbol{q}}_i^0}{(x_i^0)}$$

Berke’s expression is a first order approximation

- The true derivative of the displacement function with respect to x_i can be calculated as follows

$$
\begin{aligned}
\frac{\partial u}{\partial x_i} &= \frac{\partial \boldsymbol{q}^T}{\partial x_i}(\boldsymbol{K}\tilde{\boldsymbol{q}}) + \boldsymbol{q}^T \frac{\partial \boldsymbol{K}}{\partial x_i}\tilde{\boldsymbol{q}} + \boldsymbol{q}^T \boldsymbol{K} \frac{\partial \tilde{\boldsymbol{q}}}{\partial x_i} \\
&= \frac{\partial}{\partial x_i}(\boldsymbol{q}^T \tilde{\boldsymbol{g}}) + \boldsymbol{q}^T \frac{\partial \boldsymbol{K}}{\partial x_i}\tilde{\boldsymbol{q}} + \frac{\partial}{\partial x_i}(\boldsymbol{g}^T \tilde{\boldsymbol{q}}) \\
&= \frac{\partial}{\partial x_i} u + \boldsymbol{q}^T \frac{\partial \boldsymbol{K}}{\partial x_i}\tilde{\boldsymbol{q}} + \frac{\partial}{\partial x_i} u
\end{aligned}
$$

- It comes

$$
\frac{\partial u}{\partial x_i} = -\boldsymbol{q}^T \frac{\partial \boldsymbol{K}}{\partial x_i}\tilde{\boldsymbol{q}}
$$

Berke's expression is a first order approximation

- Since $$\boldsymbol{K} = \sum_i \boldsymbol{L}_i^T \boldsymbol{K}_i \boldsymbol{L}_i \qquad \frac{\partial \boldsymbol{K}_i}{\partial x_i} = \frac{\boldsymbol{K}_i}{x_i}$$

- We have $$\frac{\partial u}{\partial x_i} = -\frac{1}{x_i}\,\boldsymbol{q}_i^T \boldsymbol{K}_i \tilde{\boldsymbol{q}}_i$$

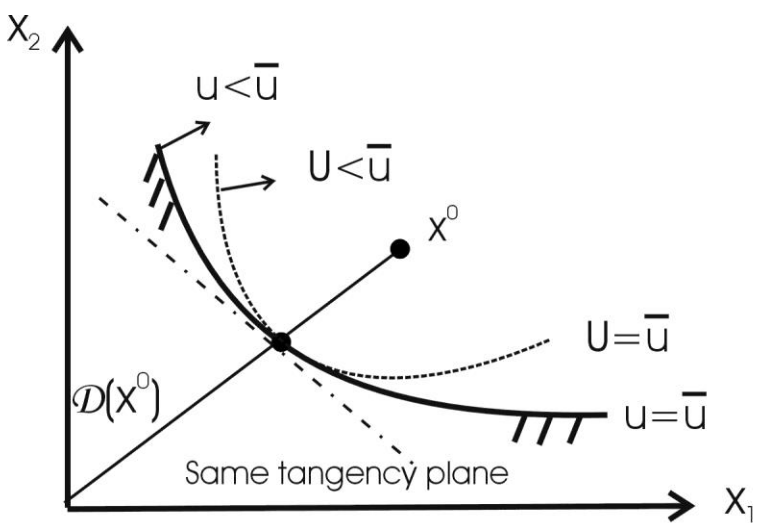
- In point x°, $$\left.\frac{\partial u}{\partial x_i}\right|_0 = -\frac{1}{x_i^0}\,\boldsymbol{q}_i^{0T} \boldsymbol{K}_i \tilde{\boldsymbol{q}}_i^0$$

- Which is exactly the same expression as the one obtained by deriving the Berke's criterion

Berke's expression is a first order approximation

Properties of Berke's criteria

- 1/ $\tilde{u}(x) = \overline{u} \quad \forall x \in \mathcal{D}(x^0)$
- 2/ $\dfrac{\partial \tilde{u}}{\partial x_i} = \dfrac{\partial u}{\partial x_i} \quad \forall x \in \mathcal{D}(x^0)$
- 3/



- u(x) is a first order approximation on the scaling line i.e. it gives exact values of the displacements on D(x^0) as well as its first derivatives.

OPTIMALITY CRITERIA FOR A SINGLE DISPLACEMENT CONSTRAINT

Single displacement constraint

- Let's come back to the minimum weight design problem. One considers the problem with a single displacement constraint:

$$\boxed{\begin{array}{lll} \min\limits_{\boldsymbol{x}} & w(\boldsymbol{x}) & \\ \text{s.t. :} & u(\boldsymbol{x}) & \leq \; \bar{u} \end{array}}$$

- Virtual loading case (unit load) in the direction of the displacement u

$$u = \tilde{\mathbf{q}}^T \, \mathbf{g} = \mathbf{q}^T \, \tilde{\mathbf{g}} = \mathbf{q}^T \, \mathbf{K} \, \tilde{\mathbf{q}}$$

- Decomposition in the contributions of each element:

$$u = \sum_i \frac{c_i}{x_i} \qquad c_i = \left(\boldsymbol{q}^T \, \boldsymbol{K}_i \, \tilde{\boldsymbol{q}} \right) x_i$$

  – c_i constant for a statically determinate structure.

Single displacement constraint

- Explicit problem: $\min_{\boldsymbol{x}} \quad w(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_i$

  $\text{s.t. :} \quad u = \sum_{i=1}^{n} \frac{c_i}{x_i} \leq \bar{u}$

- Let's introduce a Lagrange multiplier λ and shape the <u>Lagrange function</u>

$$L(x_i, \lambda) = \sum_{i=1}^{n} w_i \, x_i + \lambda \left(\sum_{i=1}^{n} \frac{c_i}{x_i} - \bar{u} \right)$$

- <u>Stationary conditions</u>

$$\frac{\partial}{\partial x_i} L(x_i, \lambda) = 0 \qquad\qquad w_i + \lambda \, (c_i \frac{-1}{x_i^2}) = 0$$

Single displacement constraint

- If c_i is positive: OK!

$$x_i^2 = \lambda \frac{c_i}{w_i}$$

- If c_i is less or equal to zero: → passive variables

$$x_i = \underline{x}_i$$

- To identify the Lagrange variable λ, one substitutes the x_i by its value into the constraint that displacement an equality constraint:

$$\sum_{i=1}^{n} \frac{c_i}{x_i(\lambda)} = \bar{u}$$

- Let's define

$$u_0 = \sum_{\{i|c_i \leq 0\}} \frac{c_i}{x_i}$$

Single displacement constraint

- Let's identify the Lagrange multiplier λ:

$$\sum_{i=1}^{n} \frac{c_i}{x_i(\lambda)} = \bar{u}$$

$$\sum_{\{i|c_i>0\}} \sqrt{\frac{w_i\, c_i}{\lambda}} = \bar{u} - u_0$$

$$\boxed{\lambda = \frac{1}{(\bar{u} - u_0)^2} \left(\sum_{k=1}^{\tilde{n}} \sqrt{w_k\, c_k} \right)^2}$$

- So it comes

$$\boxed{x_i = \left[\frac{1}{\bar{u} - u_0} \sum_{k=1}^{\tilde{n}} \sqrt{w_k\, c_k} \right] \sqrt{\frac{c_i}{w_i}} \quad i = 1 \ldots \tilde{n}}$$

Single displacement constraint

- **Physical meaning of the optimality criteria**
  - Strain energy $\mathbf{q}^T \mathbf{K} \mathbf{q}$
  - Virtual strain energy $$\boldsymbol{q}^T \boldsymbol{K} \tilde{\boldsymbol{q}} = \sum_{i=1}^{n} \boldsymbol{q}^T \boldsymbol{K}_i \tilde{\boldsymbol{q}} = \sum_{i=1}^{n} \frac{c_i}{x_i}$$
- Let's define the virtual strain energy of bar 'i'

$$e_i = \boldsymbol{q}^T \boldsymbol{K}_i \tilde{\boldsymbol{q}} = \frac{c_i}{x_i}$$

- The energy density of bar 'i' is the energy of bar 'i' divided by the weight of this bar

$$\varepsilon_i = \frac{e_i}{w_i\, x_i}$$

Single displacement constraint

- **Physical meaning of the optimality criteria**
- It comes

$$\frac{1}{\lambda} = \frac{c_i}{x_i^2\, w_i} = \frac{e_i}{w_i\, x_i} = \varepsilon_i = \text{Cst}$$

- The virtual strain energy density per unit weight is the same in each element.

$$\boxed{\varepsilon_i\, \lambda = 1}$$

Single displacement constraint

- Statically determinate case: one structural analysis to reach the optimum

- Statically indeterminate case c_i is not constant: several iterations are necessary

$$
\begin{aligned}
x_i^2 &= \lambda \, \frac{c_i}{w_i} \\
&= \lambda \, \frac{c_i \, x_i^2}{w_i \, x_i^2} \\
&= x_i^2 \, \lambda \, \frac{c_i}{w_i \, x_i^2} \\
&= x_i^2 \, \lambda \, \varepsilon_i
\end{aligned}
\quad \Longrightarrow \quad
[x_i^{(k+1)}]^2 \; = \; [x_i^{(k)}]^2 \, (\lambda \, \varepsilon_i)
$$

Single displacement constraint

- Statically determinate case: one structural analysis and reach the optimum

- Statically indeterminate case c_i is not constant, one has to use an iterative scheme:
  - Active variables $c_i>0$

$$x_i^{(k+1)} = x_i^{(k)} \left(\lambda^{(k)} \varepsilon_i^{(k)}\right)^{1/2}$$

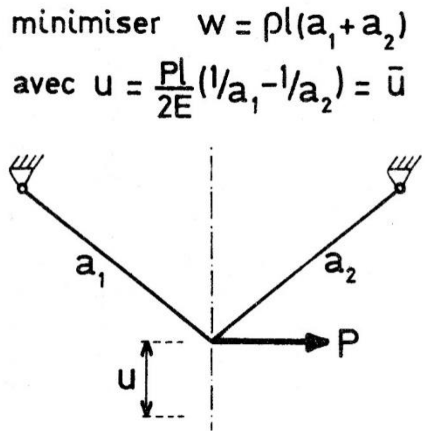
  - Passive variables $c_i<0$

$$\varepsilon_i^{(k)} \leq 0 \quad \Rightarrow \quad x_i^{(k+1)} = \underline{x}_i$$

- Fast convergence to the optimum independently of the number of design variables.

Two bar truss example

- Minimum weight design subject to a horizontal displacement constraint



$$Q_1 = \frac{P}{\sqrt{2}} \qquad Q_2 = -\frac{P}{\sqrt{2}}$$

$$\tilde{Q}_1 = \frac{1}{\sqrt{2}} \qquad \tilde{Q}_2 = -\frac{1}{\sqrt{2}}$$

- The virtual work theorem

$$u \times 1 = \frac{Q_1 \tilde{Q}_1}{E_1\, x_1}\, l_1 + \frac{Q_2 \tilde{Q}_2}{E_2\, x_2}\, l_2 = \frac{Pl\sqrt{2}}{E\; 2} \left(\frac{1}{x_1} + \frac{1}{x_2} \right)$$

- Weight of the truss

$$W = \rho_1\, l_1\, x_1 + \rho_2\, l_2\, x_2 = \rho\, l\, \sqrt{2}(x_1 + x_2)$$

Two bar truss example

- The optimization problem

$$\min_{x_1,\, x_2} \quad W = \rho\, l\, \sqrt{2}(x_1 + x_2)$$

$$\text{s.t. :} \quad u = \frac{Pl\sqrt{2}}{E\,2}\left(\frac{1}{x_1} + \frac{1}{x_2}\right) \leq \bar{u}$$

- The Langrange function

$$L(\boldsymbol{x}, \lambda) = \rho\, l\, \sqrt{2}(x_1 + x_2) + \lambda \left[\frac{Pl\sqrt{2}}{E\,2}\left(\frac{1}{x_1} + \frac{1}{x_2}\right) - \bar{u}\right]$$

- Optimality conditions

$$\frac{\partial L}{\partial x_1} = \rho\, l\sqrt{2} - \lambda\, \frac{Pl\sqrt{2}}{E\,2}\, \frac{1}{x_1^2}$$

$$\frac{\partial L}{\partial x_2} = \rho\, l\sqrt{2} - \lambda\, \frac{Pl\sqrt{2}}{E\,2}\, \frac{1}{x_2^2}$$

Two bar truss example

- The solution to these equations gives the value of the design variables in terms of the Lagrange multipliers :

$$x_1^{\star\,2} \;=\; x_2^{\star\,2} \;=\; \lambda\,\frac{P}{2\,E\,\rho}$$

- The Lagrange multiplier is determined from the equation of the constraint

$$\frac{P\,l\,\sqrt{2}}{2\,E}\,\left(\sqrt{\frac{2\,E\,\rho}{P\,\lambda}} + \sqrt{\frac{2\,E\,\rho}{P\,\lambda}}\right) \;=\; \bar{u}$$

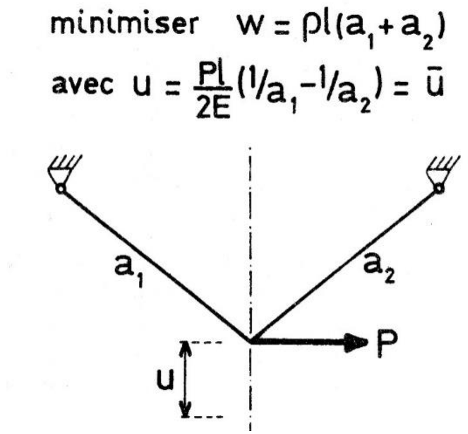
$$\sqrt{\lambda} \;=\; \frac{2\,l}{\bar{u}}\,\sqrt{\frac{P\,\rho}{E}}$$

- The optimal variables

$$x_1^{\star} \;=\; x_2^{\star} \;=\; \frac{P\,l\,\sqrt{2}}{E\,\bar{u}}$$

Two bar truss example

- Minimum weight design subject to a vertical displacement constraint



$$Q_1 = \frac{P}{\sqrt{2}} \qquad Q_2 = -\frac{P}{\sqrt{2}}$$

$$\tilde{Q}_1 = \frac{1}{\sqrt{2}} \qquad \tilde{Q}_2 = \frac{1}{\sqrt{2}}$$

- The virtual work theorem

$$v \times 1 = \frac{Q_1 \tilde{Q}_1}{E_1\, x_1}\, l_1 + \frac{Q_2 \tilde{Q}_2}{E_2\, x_2}\, l_2 = \frac{Pl\sqrt{2}}{E\, 2} \left(\frac{1}{x_1} - \frac{1}{x_2} \right)$$

- Weight of the truss

$$W = \rho_1\, l_1\, x_1 + \rho_2\, l_2\, x_2 = \rho\, l\, \sqrt{2}(x_1 + x_2)$$

Two bar truss example

- The optimization problem

$$\min_{x_1,\, x_2} \quad W = \rho\, l\, \sqrt{2}(x_1 + x_2)$$

$$\text{s.t. :} \quad v = \tfrac{Pl\sqrt{2}}{E\,2} \left(\tfrac{1}{x_1} - \tfrac{1}{x_2} \right) \leq \bar{v}$$

- The Langrange function

$$L(x, \lambda) = \rho\, l\, \sqrt{2}(x_1 + x_2) + \lambda \left[\frac{Pl\sqrt{2}}{E\,2} \left(\frac{1}{x_1} - \frac{1}{x_2} \right) - \bar{v} \right]$$

- Optimality conditions

$$\frac{\partial L}{\partial x_1} = \rho\, l\sqrt{2} - \lambda \frac{Pl\sqrt{2}}{E\,2} \frac{1}{x_1^2}$$

$$\frac{\partial L}{\partial x_2} = \rho\, l\sqrt{2} + \lambda \frac{Pl\sqrt{2}}{E\,2} \frac{1}{x_2^2}$$

Two bar truss example

$$\boxed{\frac{\partial L}{\partial x_2} > 0}$$

- This means that the variables x_1 and x_2 can be as small as we want while satisfying the constraint on the displacement constraint on v. It is the minimum gauge on x_2 which determines the optimum

$$x_2^\star = \underline{x}_2$$

Single displacement constraint

- It is easy to add the minimum size constraint
  - Selection of passive and active elements ➔ An element is passive if

$$c_i \leq 0$$

$$\text{or} \quad \lambda\, c_i / w_i \leq \underline{x}_i^2$$

- It is easy to add stress constraints in addition to the flexibility restriction
  - Stress constraints are transformed into lower bound (side constraints) using the FSD approach

$$\hat{x}_i \;=\; \max\{\underline{x}_i,\; \max_l\{\frac{\sigma_{i,l}}{\bar{\sigma}}\}\, x_i^0\}$$

  - An element is passive if

$$c_i \leq 0$$

$$\text{or} \quad \lambda\, c_i / w_i \leq \hat{x}_i^2$$

Single displacement constraint

- For isostatic structures,
  - Solution exact in one structural (Finite Element) analysis
  - Redesign criteria must may be applied iteratively when the active / passive design variable set has to be selected when restrictions are imposed on the minimum size or by stress constraints (FSD)

- For hyperstatic structures
  - C_i's are not constant because of the load redistribution
  - Redesign criteria must be applied iteratively.
  - Fast convergence.
  - Generally non convergence problems are related to the stress constraints which are not accounting (approximated) accurately

OPTIMALITY CRITERIA : MULTIPLE DISPLACEMENT AND STRESS CONSTRAINTS

Multiple displacement and stress constraints

- Combination of the two previous O.C.
  - Displacement constraints
  - Stress constraints

$$
\begin{aligned}
&\min\ W(x_i) \\
\mathbf{s.t.}:\ &u_j(\mathbf{x}) \leq \bar{u}_j \quad j = 1 \ldots m \\
&\sigma_i(\mathbf{x}) \leq \bar{\sigma}_i \quad i = 1 \ldots n \\
&\underline{x}_i \leq x_i
\end{aligned}
$$

Multiple displacement and stress constraints

- Displacement constraints (assumed to be active)

$$\boxed{u_j = \bar{u}_j}$$

  - ➔ m virtual load cases (unit load)

$$\mathbf{K}\,\tilde{\mathbf{q}}_j = \tilde{\mathbf{g}}_j^{(1)}$$

  - Explicit approximation using virtual work

$$u_j = \sum_i \frac{c_{ij}}{x_i} = \bar{u}_j \qquad c_{ij} = x_i\,\mathbf{q}^T\,\mathbf{K}_i\,\tilde{\mathbf{q}}_j$$

- Stress constraints accounted through minimum size restrictions

$$x_i \geq \max\{\underline{x}_i, \hat{x}_i\}$$

$$\hat{x}_i = x_i^0 \max_{l=1\ldots c}\{\frac{\sigma_{i,l}^0}{\bar{\sigma}_i}\}$$

Multiple displacement and stress constraints

- Combination of the two previous O.C.
  - Stress constraints
  - Displacement constraints

- Set of active constraints is assumed to be known
  - ň active design variables
  - m active displacement constraints

- Passive design variables: side constraints or determined by the stress constraints

$$x_i = \max\{\underline{x}_i, \hat{x}_i\} \qquad i > \hat{n}$$

$$\hat{x}_i = x_i^0 \max_{l=1\ldots c}\{\frac{\sigma_{i,l}^0}{\bar{\sigma}_i}\}$$

Multiple displacement and stress constraints

- Active variables ➔ optimality conditions w.r.t. displacement constraints (assumed to be active)

$$u_j = \bar{u}_j$$

- Lagrange function
  - ➔ m Lagrange multipliers λ_j
  - Explicit approximation using virtual work

$$L(x_i, \lambda_j) = \sum_{i=1}^{n} w_i \, x_i + \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} \frac{c_{ij}}{x_i} - \bar{u}_j \right)$$

Multiple displacement and stress constraints

- Stationary conditions

$$\frac{\partial}{\partial x_i} L(x_i, \lambda_j) = 0$$

- After some algebra, the stationary conditions can be cast under the following form

$$w_i + \sum_{j=1}^{m} \lambda_j \frac{-c_{ij}}{x_i^2} = 0$$

- If $c_{ij}>0$

$$x_i = \left(\frac{1}{w_i} \sum_{j=1}^{m} \lambda_j \, c_{ij} \right)^{(1/2)} \qquad i = 1 \ldots \hat{n}$$

Multiple displacement and stress constraints

- After some algebra, the stationary conditions can be rewritten under the following form

$$\sum_{j=1}^{m} \lambda_j \, \varepsilon_{ij} \; = \; 1$$

- With the virtual (mutual) strain energy densities

$$\varepsilon_{ij} \; = \; \frac{\mathbf{q}^T \mathbf{K}_i \tilde{\mathbf{q}}_j}{w_i x_i}$$

- In optimized structure, we have in each element the same combined virtual energy density, equal to unity

Multiple displacement and stress constraints

- For statically determinate case: OC are exact (because x_i and c_{ij} are constant)
  - ➔ optimum in one analysis
- For statically indeterminate case: OC are approximate
  - ➔ Iterative use of the redesign formulae
    - ➔ Active variables

$$x_i^{(k+1)} = x_i^{(k)} \left(\sum_{j=1}^{m} \lambda_j^{(k)} \, \varepsilon_{ij}^{(k)} \right)^{(1/2)} \qquad i = 1 \ldots \hat{n}$$

    - ➔ Passive variables

$$\hat{x}_i^{(k+1)} = x_i^{(k)} \max_{l=1 \ldots c} \left\{ \frac{\sigma_{i,l}^{(k)}}{\bar{\sigma}_i} \right\} \quad \text{or} \quad \hat{x}_i^{(k+1)} = \underline{x}_i$$

Multiple displacement and stress constraints

- Lagrange multipliers λ_j????
  - Such that the displacement constraints are satisfied as equality

$$u_j = \sum_i \frac{c_{ij}}{x_i} = \bar{u}_j$$

  - Closed form solution only if m=1
  - Otherwise numerical schemes (see details in the latter)
    - Envelop method (intuitive extension from case m=1)

$$\lambda_j^{(k+1)} = \lambda_j^{(k)} \left(\frac{u_j^{(k)}}{\bar{u}_j} \right)^2$$

    - Newton Raphson applied to solve the set of nonlinear equations

$$u_j = \sum_i \frac{c_{ij}}{x_i(\lambda)} = \bar{u}_j \quad j = 1 \dots m$$

Multiple displacement and stress constraints

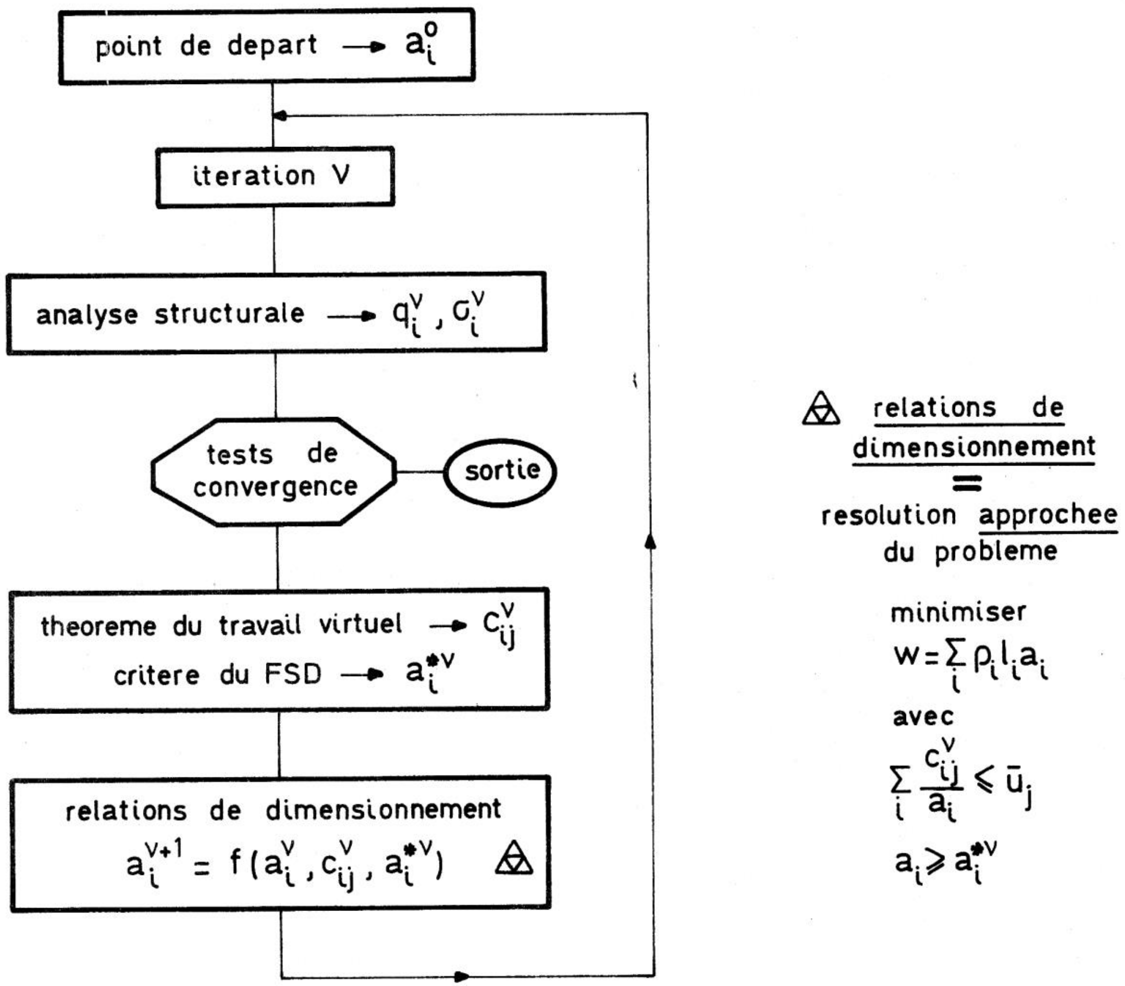


FIG. 3.13 UTILISATION ITERATIVE DES CRITERES D'OPTIMALITE

Envelop method (Gellatly & Berke, 1971)

- Intuitive method, simple use, close to FSD

- Each displacement constraint is first considered alone and independently.
- For constraint j only

$$x_i^{[j]} = \frac{1}{\bar{u}_j - u_{0j}} \left(\frac{c_{ij}}{w_i}\right)^{1/2} \sum_{k=1}^{\tilde{n}_j} (w_k\, c_{kj})^{1/2} \qquad i = 1 \ldots \hat{n}_j$$

  - and

$$x_i^{[j]} = \underline{x}_i \quad \text{or} \quad \hat{x}_i = x_i^0 \max_{l=1\ldots c} \{\frac{\sigma_{i,l}^0}{\bar{\sigma}_i}\} \quad i > \hat{n}_j$$

- Then, one takes the maximum size for all displacement constraints (envelop)

$$x_i = \max_{j=1,m} \{x_i^{[j]}\} \quad i = 1 \ldots \hat{n}$$

Envelop method (Gellatly & Berke, 1971)

- Remark : number of active variables
  - ň= total number of active variables
  - $ň_j$ = number of active variables for the jth constraint if it was the only critical one

- x_i is an active variable for the jth constraint if
  - c_{ij}> 0
  - x_i > $\underline{x}_i$ or FSD

$$x_i \geq \underline{x}_i \quad \text{or} \quad x_i \geq \hat{x}_i = x_i^0 \max_{l=1\dots c} \{ \frac{\sigma^0_{i,l}}{\bar{\sigma}_i} \}$$

  - x_i given by the jth constraint in the update formula

Envelop method (Gellatly & Berke, 1971)

- Formula must be repeated 2 or 3 times before stabilizing the sets of active design variables for each constraint
- The approach produces satisfactory results if the number of constraints is not too large
- Advantages
  - Easy implementation
  - No numerical difficulties

Newton-Raphson iteration (Taig & Kerr, 1973)

- Solve the system of nonlinear equations using a Newton-Raphson method

$$\begin{cases} x_i^2 = \left(\frac{1}{w_i} \sum_{j=1}^m \lambda_j \, c_{ij} \right) & i = 1 \ldots \tilde{n} \\ \sum_{i=1}^n \frac{c_{ij}}{x_i(\lambda)} = \bar{u}_j & j = 1 \ldots m \end{cases}$$

- First set of equations enables to eliminate the primal variables in terms of the Lagrange multipliers. Newton Raphson is thus used to solve the system

$$\sum_i \frac{c_{ij}}{x_i(\lambda)} = \bar{u}_j \quad j = 1 \ldots m$$

Newton-Raphson iteration (Taig & Kerr, 1973)

- Iteration scheme on Lagrange multipliers λ only

$$\mathbf{u}^{(k)} = \mathbf{u}^{(k-1)} + \sum_{j=1}^{m} \frac{\partial \mathbf{u}(\lambda)}{\partial \lambda_j} \left(\lambda_j^{(k)} - \lambda_j^{(k-1)} \right) = \bar{\mathbf{u}}$$

- Then new set of Lagrange multipliers are given by

$$\lambda_j^{(k)} = \lambda_j^{(k-1)} + \left[\frac{\partial \mathbf{u}(\lambda)}{\partial \lambda_j} \right]^{-1} (\bar{\mathbf{u}} - \mathbf{u}^{(k-1)})$$

- Iteration scheme on Lagrange multipliers λ

$$\lambda^{(k+1)} = \lambda^{(k)} + [\mathbf{H}^{(k)}]^{-1} (\overline{\mathbf{u}} - \mathbf{u}^{(k)})$$

Newton-Raphson iteration (Taig & Kerr, 1973)

- Iteration scheme on Lagrange multipliers λ only

$$\lambda^{(k+1)} = \lambda^{(k)} + [\boldsymbol{H}^{(k)}]^{-1}(\overline{\boldsymbol{u}} - \boldsymbol{u}^{(k)})$$

- Gradient matrix H is given by

$$\left.\begin{aligned} \frac{\partial u^{(k)}}{\partial \lambda^{(k)}} &= -\sum_{i=1}^{n} \frac{c_{ij}}{x_i^2} \frac{\partial x_i(\lambda^{(k)})}{\partial \lambda} \\ 2\,x_i \frac{\partial x_i}{\partial \lambda_k} &= \frac{c_{ik}}{w_i} \end{aligned}\right\} \Rightarrow \quad H_{jk} = \frac{\partial u_j}{\partial \lambda_k} = -\frac{1}{2} \sum_i \frac{c_{ij} c_{ik}}{w_i x_i^3}$$

Newton-Raphson iteration (Taig & Kerr, 1973)

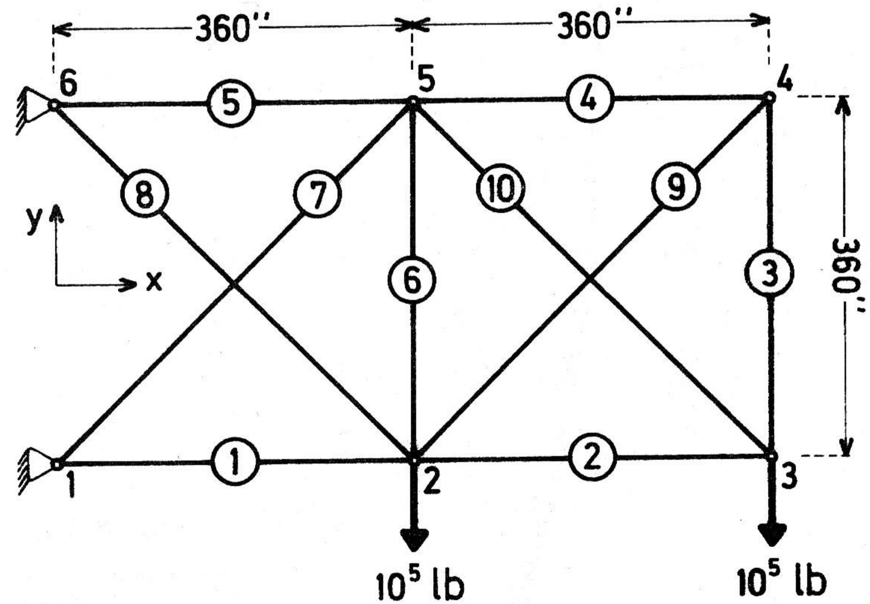
- Difficulties
  - Select appropriate initial $\lambda^{(0)}$
  - Find the correct set of active / passive design variables
  - Identify the set of estimated active behaviour constraints (i.e. nonzero λ_j's)
  - H might become singular at some stage of the process

- Solution: dual methods
  - H is indeed the Hessian matrix of the dual function
  - Iteration is the quadratic programming ascent direction in dual space

Ten-bar-truss example

- The stress-ratioïng itself tends to increase the design variable with the smallest stress limit

- Example: stress limit = 25000psi except in member 8 with a variable limit from 25000 to 70000 psi



materiau	:	aluminium
tension maximale admissible	:	25000 psi
module d'elasticite	:	10^7 psi
masse specifique	:	0.1 lb/in^3
section minimale admissible	:	0.1 in^2
deplacement maximal admissible	:	2.0 in
mise en charge	:	unique

Stress and displacement constraints