FROM OPTIMALITY CRITERIA TO STRUCTURAL APPROXIMATIONS

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INTRODUCTION & MOTIVATION

Introduction

- Structural optimization applied to sizing (weight minimization) problem

$$
\begin{array}{lll}
\min\limits_{x} & W(x) = \sum_{i=1}^{n} w_i \, x_i & \\
\text{s.t.:} & g_j(x) \;\leq\; 0 & (j = 1 \ldots m) \\
 & \underline{x}_i \;\leq\; x_i \;\leq\; \overline{x}_i & (i = 1 \ldots n)
\end{array}
$$

  - Finite element model
  - Design variables are the transverse sizes of the structural members (Fixed geometry and material properties)
  - Design restrictions

$$
\begin{aligned}
g_j(x) \;\equiv\; & u_j(x) - \overline{u}_j \;\leq\; 0 \\
& \sigma_j(x) - \overline{\sigma}_j \;\leq\; 0 \\
& \underline{\omega}_j^2 - \omega_j^2(x) \;\leq\; 0 \\
& \underline{\lambda}_j^2 - \lambda_j^2(x) \;\leq\; 0
\end{aligned}
$$

Introduction

- Design constraints $g_j(x)<0$
  - Implicit functions
  - Non linear functions
  - One constraint evaluation requires a complete FE analysis
- Side constraints: simple and explicit
  - Fabrication / technological / physical constraints
  - Treated separately in most methods
- Iterative process ➔ HIGH COST

INTRODUCTION

- Optimality criteria techniques (OC)
  - Highly specific
  - Intuitive techniques, simple
  - Convergence to a design that is not necessarily optimal (KKT conditions)
  - Difficulties in identifying the set of active constraints
  - Convergence instabilities
  - Small number of reanalyses, independent of the number of design variables

Résumé

- Low cost
- But uncertainty convergence

INTRODUCTION

- Pure Mathematical Programming methods
  - Very general
  - Rigorous methods, quite elaborated
  - Convergence to a local minimum
  - Stable and monotonic convergence
  - Large number of reanalyses, growing with the number of design variables

Résumé

- Rigorous framework & guaranteed convergence
- High cost (Growing with the size of the problem)

BERKE’S APPROXIMATION

BERKE'S APPROXIMATION

- Theorem of virtual work,

$$T.V. = \tilde{\mathbf{q}}^T \, \mathbf{g} = \mathbf{q}^T \, \tilde{\mathbf{g}} = \tilde{\mathbf{q}}^T \, \mathbf{K} \, \tilde{\mathbf{q}}$$

- Applying a virtual load vector (unit load vector) in the direction of under the displacement u:

$$\tilde{\mathbf{g}} = (0\,0\,0 \ldots 1 \ldots 0\,0)^T$$

- The Principle of Virtual Work yiels

$$T.V. = \mathbf{q}^T \, \tilde{\mathbf{g}} \; = \; u \; \times \; 1 \; = \; \mathbf{q}^T \, \mathbf{K} \, \tilde{\mathbf{q}}$$

- With

$$\mathbf{K} \; = \; \sum_i \mathbf{L}_i^T \mathbf{K}_i \, \mathbf{L}_i$$

BERKE'S APPROXIMATION

- For truss and plate design variables, the stiffness matrix takes the interesting form:

$$\boldsymbol{K}_e = x_e \, \bar{\boldsymbol{K}}_e$$

  - Truss structures $x_e = A_e$
  - Plate structures $x_e = t_e$

- One can decompose the contribution of each element:

$$\begin{aligned} u &= \boldsymbol{q}^T \boldsymbol{K} \, \tilde{\boldsymbol{q}} = \sum_e \boldsymbol{q}_e^T \boldsymbol{K}_e \, \tilde{\boldsymbol{q}}_e \\ &= \sum_e (x_e \, \boldsymbol{q}_e^T \bar{\boldsymbol{K}} \, \tilde{\boldsymbol{q}}_e) \end{aligned}$$

BERKE'S APPROXIMATION

- The flexibility coefficients are constant for statically determinate structures

$$\boldsymbol{q}_e^T \boldsymbol{K}_e \, \tilde{\boldsymbol{q}}_e \, x_e = c_e$$

- For other structures, one can also assume a moderate redistribution of the internal loads around the current design point and has also constant value.

- Considering that the coefficients c_e are constant, Berke's criterion provides an explicit expression of the displacement u in terms of the design variables

$$\tilde{u} = \sum_e \frac{c_e^0}{x_e} \quad \text{with} \quad c_e^0 = (\boldsymbol{q}_e^{0T} \, \boldsymbol{K}_e^0 \, \tilde{\boldsymbol{q}}_e^0) \, x_e^0$$

BERKE’S APPROXIMATION
ARE FIRST ORDER APPROXIMATIONS

A first order explicit approximation of displacement

- The Berke's expansion provide a first order explicit approximation of the displacement around x^0

$$\tilde{u} = \sum_i \frac{c_i^0}{x_i} \quad \text{with} \quad c_i^0 = (\boldsymbol{q}_i^{0T} \boldsymbol{K}_i^0 \tilde{\boldsymbol{q}}_i^0)\, x_i^0$$

  - In general (indeterminate structures) the c_i^0 are not constant
  - The expression is exact for statically determinate structures, but for statically indeterminate structures, it is only exact in the current point x^0

- The Berke's expression is an approximation of the displacement u around the current design point x^0 in terms of the design variables.

A first order explicit approximation of displacement

- The value of the approximation is exact in x_i^0

$$\tilde{u}(\boldsymbol{x}_i^0) = \sum_i \frac{c_i^0}{x_i^0} = \sum_i \frac{(\boldsymbol{q}_i^{0T}\ \boldsymbol{K}_i^0\ \tilde{\boldsymbol{q}}_i^0)\ x_i^0}{x_i^0} = \sum_i (\boldsymbol{q}_i^{0T}\ \boldsymbol{K}_i^0\ \tilde{\boldsymbol{q}}_i^0) = u^0$$

  - As c_i^0 remains constant only along $D(x^0)$. It is also true for all points along the scaling line

- The derivatives of the approximations are exact in x_i^0

$$\left.\frac{\partial \tilde{u}}{\partial x_i}\right|_{\boldsymbol{x}^0} = \left.\frac{\partial u}{\partial x_i}\right|_{\boldsymbol{x}^0}$$

APPROXIMATION IS A FIRST ORDER TAYLOR EXPANSION IN THE RECIPROCAL VARIABLE SPACE

Constraint linearization in reciprocal space

- Berke's approximation of the displacement *u*.

$$\tilde{u} = \sum_i \frac{c_i^0}{x_i} \quad \text{with} \quad c_i^0 = (\mathbf{q}_i^{0T}\, \mathbf{K}_i^0\, \tilde{\mathbf{q}}_i^0)\, x_i^0$$

- Suggest to use the reciprocal variables

$$z_i = \frac{1}{x_i}$$

- The Berke's approximation can take the simple form

$$\tilde{u}_j(\mathbf{z}) = \sum_i c_{ij}\, z_i$$

Constraint linearization in reciprocal space

- We now show that the Berke's approximation is in fact the first order Taylor expansion of the displacement in the reciprocal space.

$$u(\mathbf{z}) \; = \; u(\mathbf{z}^0) + \sum_i \left(\frac{\partial u_j}{\partial z_i}\right)_0 \; (z_i - z_i^0) \; \leq \; \bar{u}$$

- In order to prove that one has to show that
  - C_{ij} are the derivative of u with respect to z_i:

$$c_i = \left.\frac{\partial u}{\partial z_i}\right|_{\mathbf{z}^0}$$

  - The constant terms in '0' cancel each other

$$u(\mathbf{z}^0) \; = \sum_i \; c_i \; z_i^0$$

Constraint linearization in reciprocal space

- It comes

$$c_i = \left.\frac{\partial u}{\partial z_i}\right|_{\mathbf{z}^0}$$

$$u(\mathbf{z}) = u(\mathbf{z}^0) + \sum_i \left(\frac{\partial u_j}{\partial z_i}\right)_0 (z_i - z_i^0) \leq \bar{u}$$

$$u(\mathbf{z}) = u(\mathbf{z}^0) + \sum_i c_i \, (z_i - z_i^0) \leq \bar{u}$$

$$u(\mathbf{z}^0) = \sum_i c_i \, z_i^0$$

0

$$u(\mathbf{z}) = [u_j(\mathbf{z}^0) - \sum_i c_i z_i^0] + \sum_i c_i \, z_i = \sum_i c_i \, z_i$$

- Finally

$$\boxed{\tilde{u} = \sum_i c_i \, z_i = \sum_i \frac{c_i}{x_i}}$$

Constraint linearization in reciprocal space

- Let's show that the virtual energy densities c_i are the first derivatives (gradients) of the constraints with respect to the reciprocal variables $z_i = 1/x_i$ that is

$$c_i = \left. \frac{\partial u}{\partial z_i} \right|_{\mathbf{z}^0}$$

- **For the approximation** obviously, we have :

$$\left. \frac{\partial \tilde{u}}{\partial z_i} \right|_{\mathbf{z}^0} = c_i = x_i^0 \, \mathbf{q}_i^{0T} \mathbf{K}_i \, \tilde{\mathbf{q}}_i^0$$

Constraint linearization in reciprocal space

- **Derivative of the (real) displacement** with respect to the reciprocal design variable $z_i = 1/x_i$. It is clear that

$$\frac{\partial u}{\partial z_i} = \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial z_i}$$

- Since

$$\frac{\partial x_i}{\partial z_i} = \frac{\partial}{\partial z_i}\left(\frac{1}{z_i}\right) = -\frac{1}{z_i^2} = -x_i^2$$

- It comes

$$\begin{aligned} \left.\frac{\partial u}{\partial z_i}\right|_{\mathbf{z}^0} &= \left.\frac{\partial u}{\partial x_i}\right|_{\mathbf{x}^0} (-x_i^{0\,2}) \\ &= -\frac{\mathbf{q}_i^{0T}\mathbf{K}_i\tilde{\mathbf{q}}_i^0}{x_i^0} (-x_i^{0\,2}) \\ &= \mathbf{q}_i^{0T}\mathbf{K}_i\tilde{\mathbf{q}}_i^0 \, x_i^0 = c_i^0 = \left.\frac{\partial \tilde{u}}{\partial z_i}\right|_{\mathbf{z}^0} \end{aligned}$$

Constraint linearization in reciprocal space

- We have to prove

$$u(\mathbf{z}^0) \;=\; \sum_i c_i \; z_i^0$$

- This is performed by writing the expression of u($\mathbf{z}_0$)

$$u(\mathbf{z}^0) \;=\; u(\mathbf{x}^0) \;=\; \sum_i \mathbf{q}_i^{0T} \mathbf{K}_i \; \tilde{\mathbf{q}}_i^0 \;=\; \sum_i \mathbf{q}_i^{0T} \mathbf{K}_i \; \tilde{\mathbf{q}}_i^0 \frac{x_i^0}{x_i^0} \;=\; \sum_i \frac{c_i^0}{x_i^0} \;=\; \sum_i c_i^0 z_i^0$$

- It comes that the approximation writes

0

$$u(\mathbf{z}) \;=\; [u_j(\mathbf{z}^0) - \sum_i c_i z_i^0] + \sum_i \; c_i \; z_i \;=\; \sum_i \; c_i \; z_i$$

First order explicit approximation of the stress constraints

- Previous interpretation of FSD and Berke's approximation suggests to generalize the first order approximation approach and to build the same high-quality approximations for stress and displacement constraints.
- For truss: stress constraints can be equivalent to relative displacements. But in general

$$\sigma = \mathbf{T}\,\mathbf{q}$$

$$\sigma_k = \mathbf{t}_k^T\,\mathbf{q}$$

- Apply "virtual load case" t_k

$$\mathbf{K}\,\tilde{\mathbf{q}}_k = \tilde{\mathbf{t}}_k$$

- The first order generalized "Berke" approximation of the stress

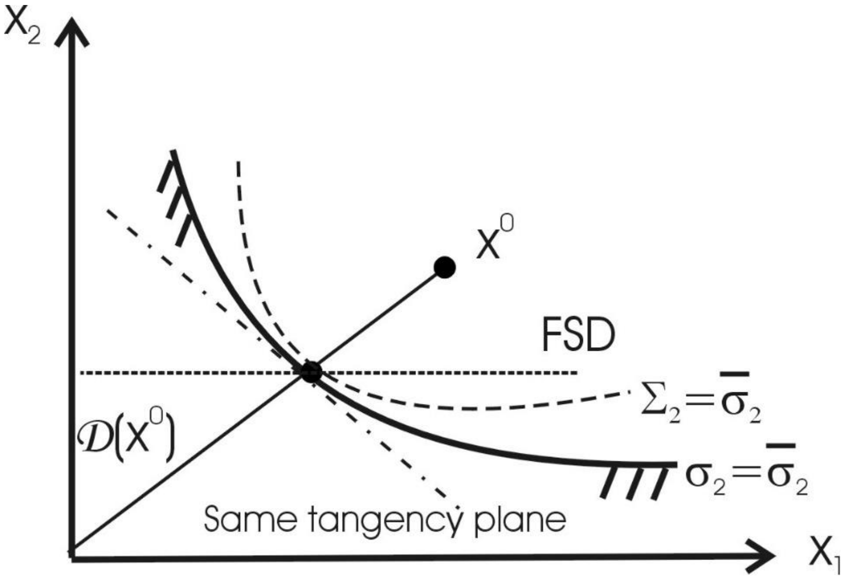
$$\tilde{\sigma}_k = \sum_{i=1}^{n} \frac{d_{ik}}{x_i} \qquad d_{ik} = (\mathbf{q}_i^T\,\mathbf{K}_i\,\tilde{\mathbf{q}}_{ik})^0\,x_i^0$$

First order explicit approximation of the stress constraints

- Property:

$$\left.\frac{\partial \sigma_k}{\partial x_i}\right|_{\mathbf{x}^0} = -\frac{d_{ik}}{x_i^2} = \left.\frac{\partial \tilde{\sigma}_k}{\partial x_i}\right|_{\mathbf{x}^0}$$

- First order approximation on the scaling line



Constraint linearization in reciprocal space

- Approximation concept approach:
  - Linearization of the stress constraints

$$\sigma_k \leq \bar{\sigma}_k \quad \Rightarrow \quad \sum_i \frac{d_{ik}}{x_i} \leq \bar{\sigma}_k$$

  - First order explicit approximation

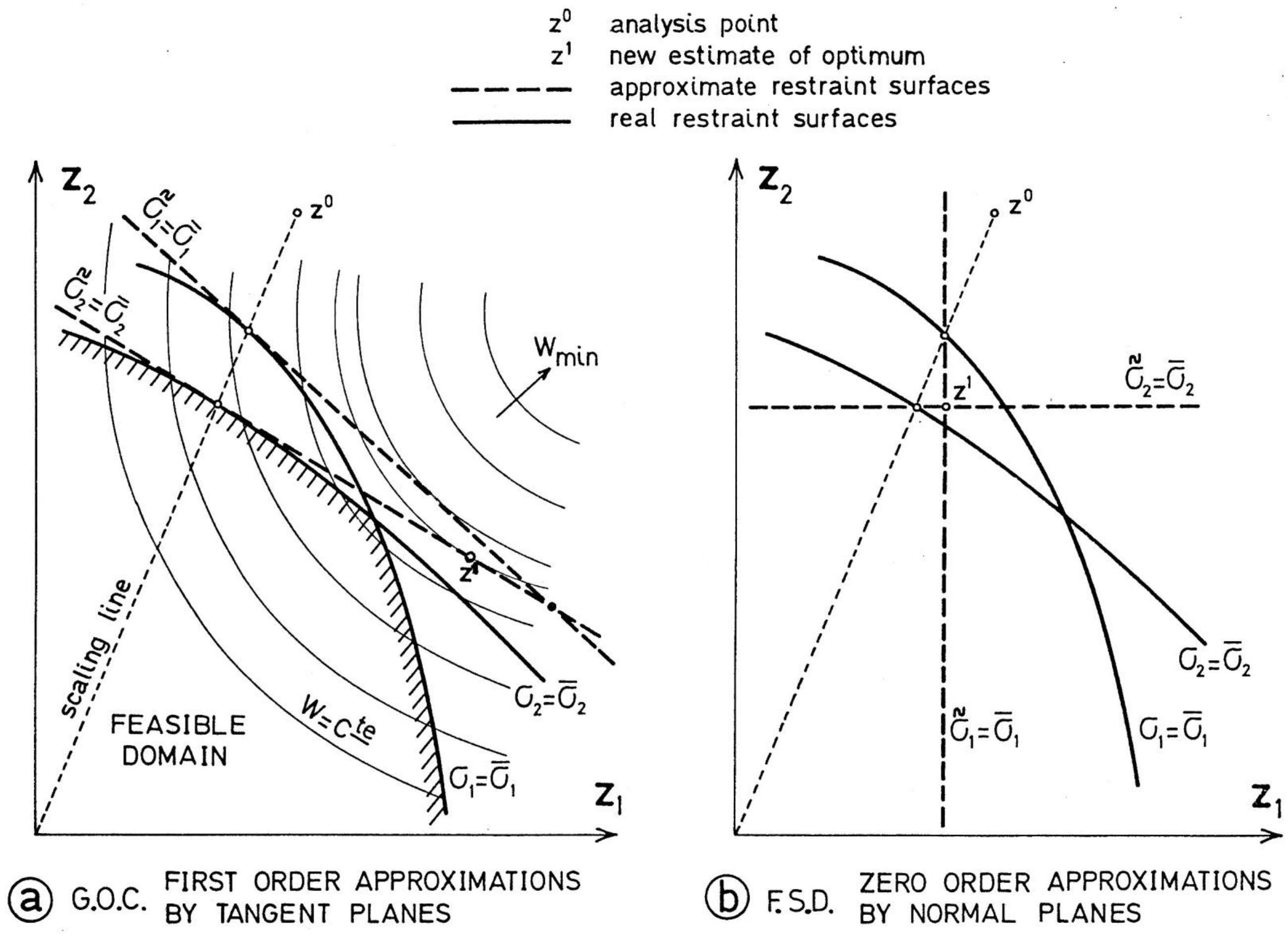
- Conventional OC: stress ratio formula
  - Fully stressed design (FSD) philosophy

$$\sigma_k \leq \bar{\sigma}_k \quad \Rightarrow \quad x_k \geq \underline{\tilde{x}}_k$$

$$\underline{\tilde{x}}_k = \underline{\tilde{x}}_k^{(0)} \frac{\sigma_k^{(0)}}{\bar{\sigma}_k}$$

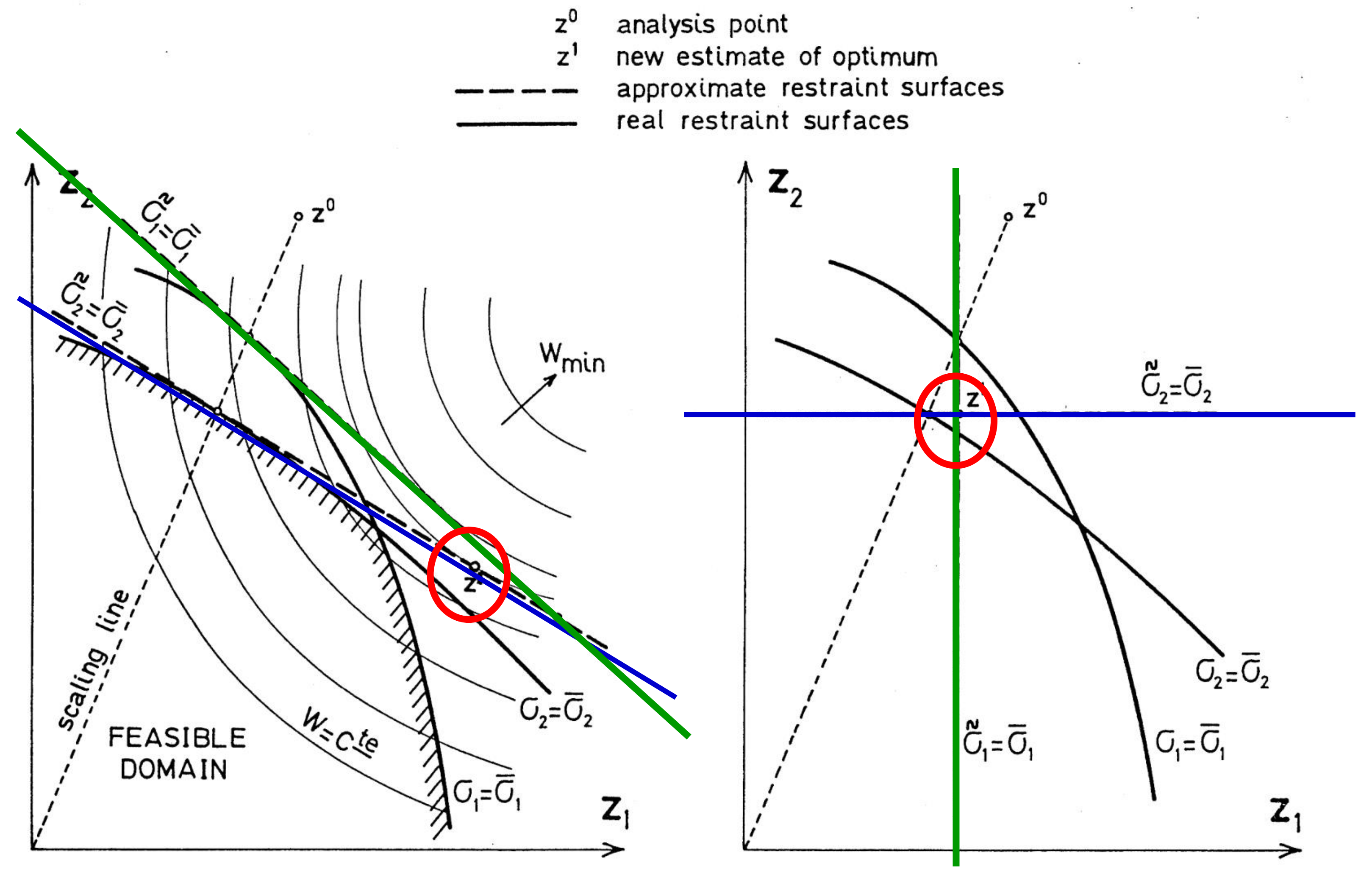
  - Zero order approximation

Constraint linearization in reciprocal space



$$\tilde{\sigma}_k(\mathbf{z}) = \sum_i d_{ij}\, z_i \text{ with } d_{ij} = \left.\frac{\partial \sigma_k}{\partial z_i}\right|_{\mathbf{z}^0} = -\frac{d_{ik}}{x_i^2}$$

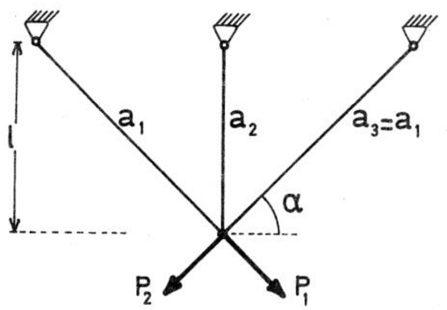
Stress ratioing



ⓐ G.O.C. FIRST ORDER APPROXIMATIONS BY TANGENT PLANES

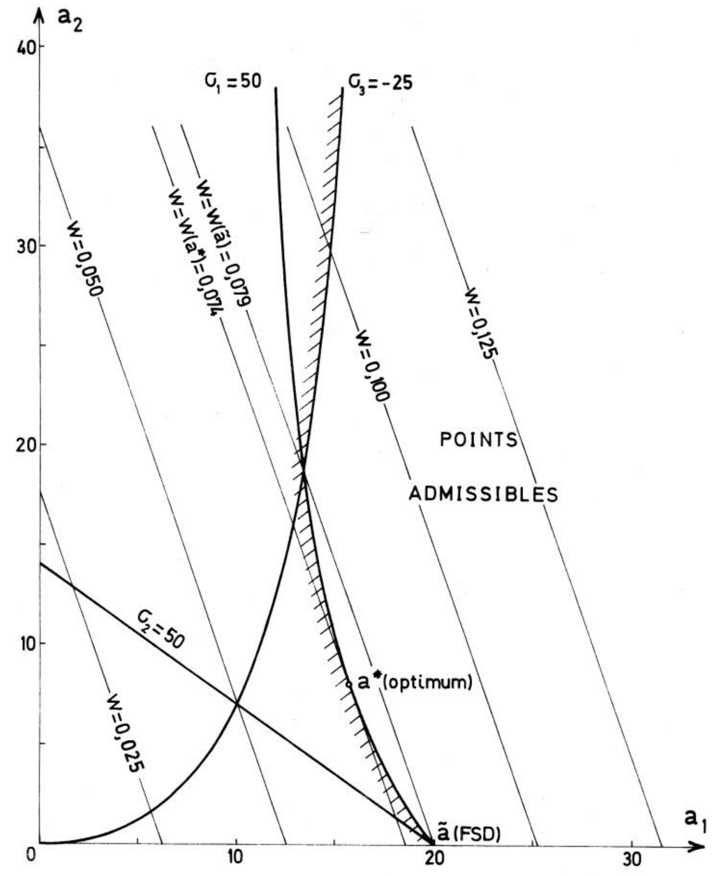
ⓑ F.S.D. ZERO ORDER APPROXIMATIONS BY NORMAL PLANES

Stress ratioing



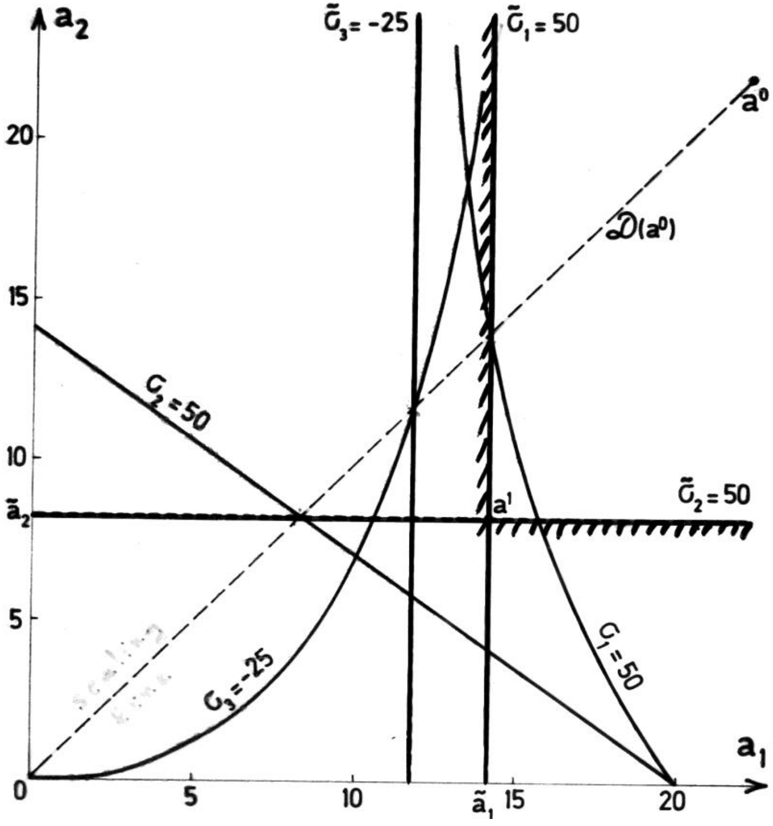
module d'Young	$E = 7000\ kg/mm^2$
masse volumique	$\rho = 2.8\ 10^{-6}\ kg/mm^3$
tension maximale admissible:	
traction	$\bar{\sigma} = 50\ kg/mm^2$
compression	$\underline{\sigma} = -25\ kg/mm^2$
hauteur du treillis	$l = 500\ mm$
angle des barres	$\alpha = 45°$

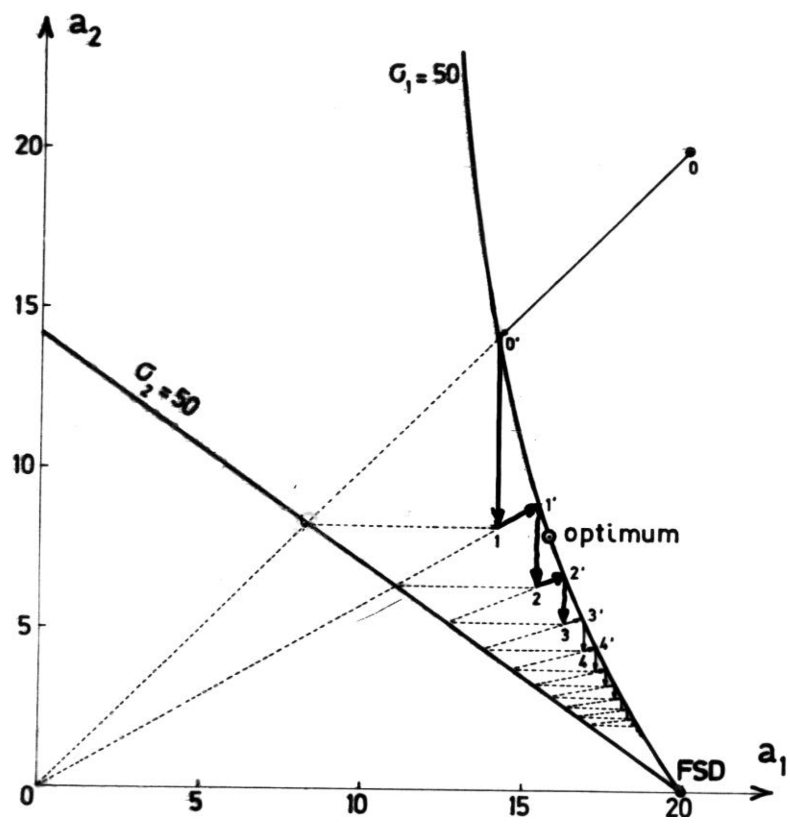
cas de charge 1 : $P_1 = 1000\ kg$
" " 2 : $P_2 = 1000\ kg$
(symetriques)



Three-bar truss

Stress ratioing

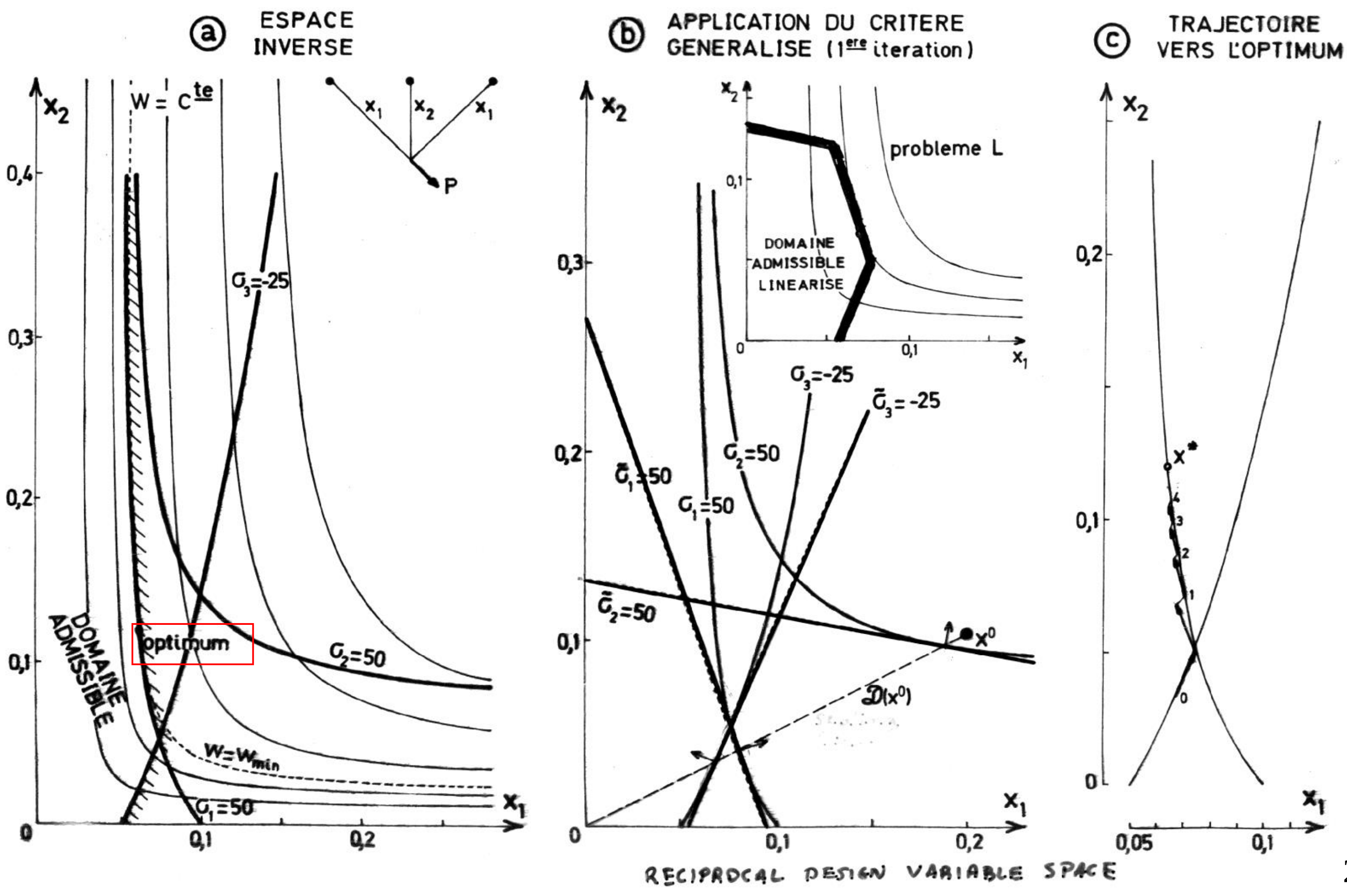




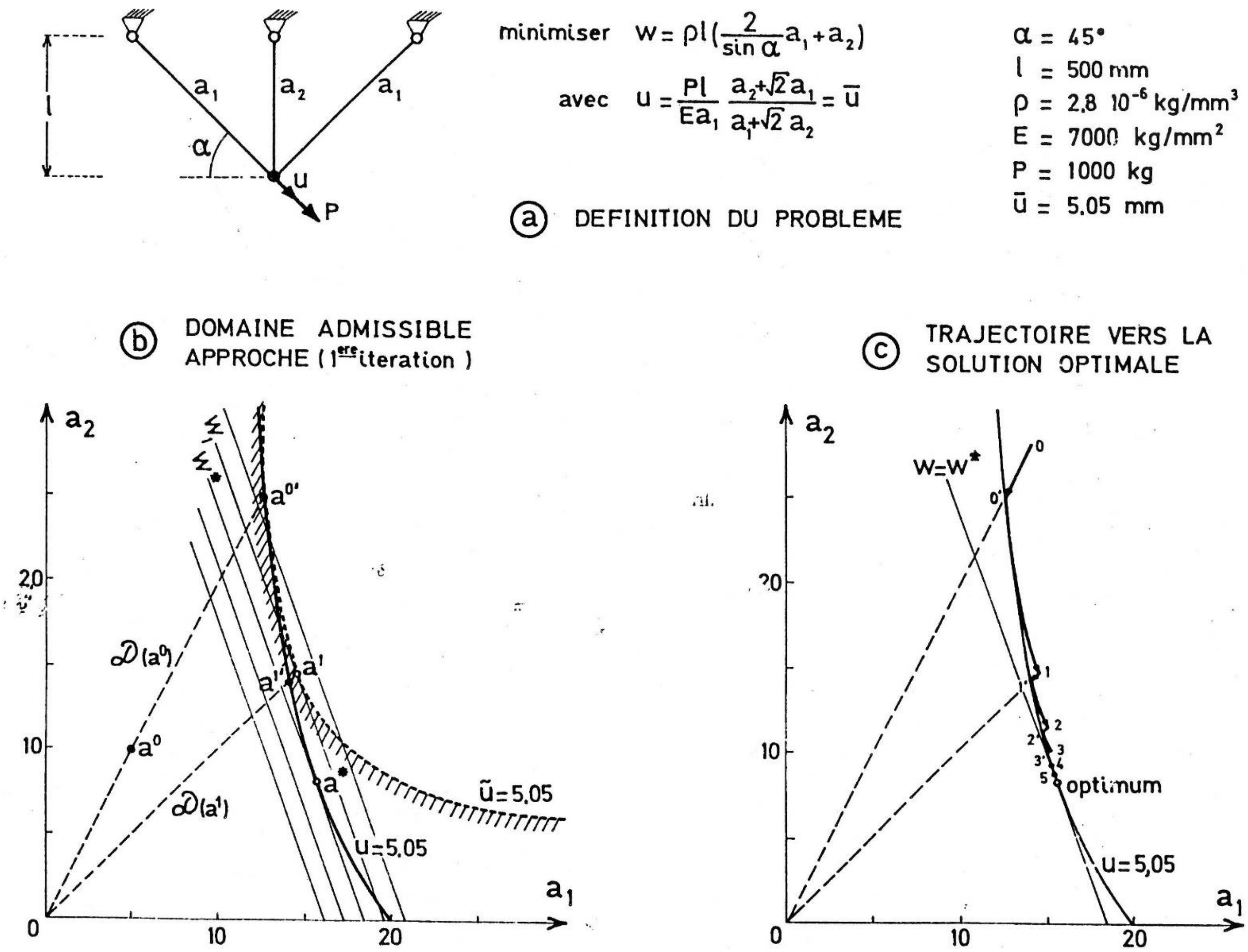
Direct design variables space

Stress ratioing

Reciprocal design variables space

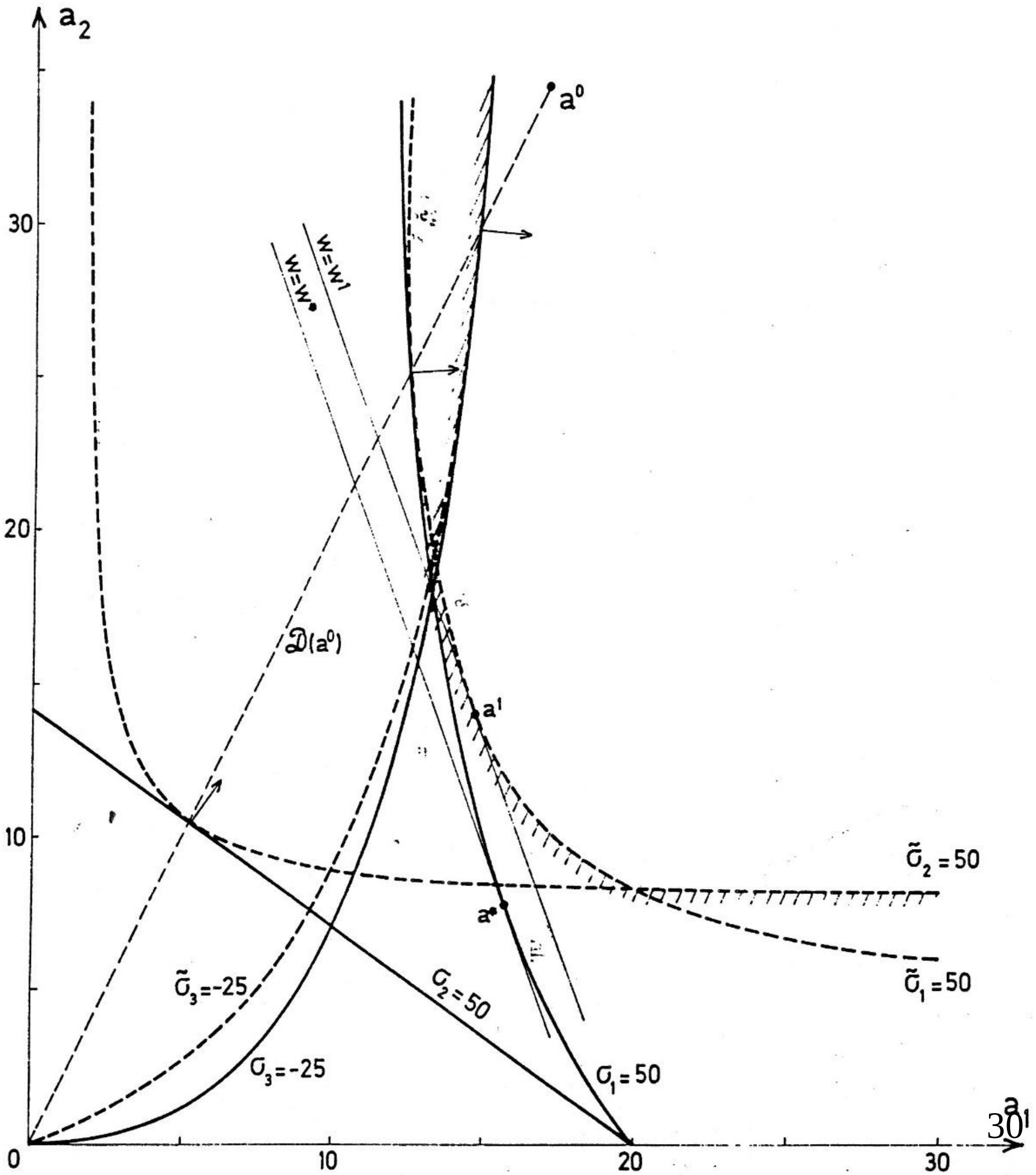


Stress ratioing



Stress ratioing

Three-bar truss:
First order approximation of stress constraints



RELATIONS BETWEEN OC AND MP

- Generalized OC = MP linearization method
- Approximation concept approach
- Constraint gradients

Generalized OC = Linearization method

- GOC : sequence of explicit subproblems where real constraints are approximated by

$$\tilde{g}_j(x) = \sum_i \frac{c_i^0}{x_i} - \bar{u}_j \leq 0$$

  – c_{ij} = virtual energy densities

$$z_i = \frac{1}{x_i}$$

$$c_{ij}^0 = \left.\frac{\partial u_j}{\partial z_i}\right|_0$$

- Approximation: sequence of linearized problems with

$$\tilde{g}_j(z) = \left[u_j^0 + \sum_i \frac{\partial u_j}{\partial z_i}(z_i - z_i^0)\right] - \bar{u}_j \leq 0$$

$$= \sum_i \left.\frac{\partial u}{\partial z_i}\right|_0 z_i - \bar{u}_j \leq 0$$

  – c_{ij} = derivatives of the response functions with respect to the reciprocal variables

Generalized OC = Linearization method

- Unified approach:
  - Sequence of explicit (separable) subproblems obtained by linearizing the behavior constraints with respect to the reciprocal variables

$$\min \quad W(z) = \sum_i \frac{w_i}{z_i}$$

$$\text{s.t.} \quad \tilde{u}_j(z) = \sum_i c_{ij}\, z_i \leq \bar{u}_j$$

$$\underline{z}_i \leq z_i \leq \bar{z}_i$$

Later: linearizing any behavior constraints with respect to the reciprocal variables!

  - Independence wrt the number of design variables!
- Solution of the explicit subproblems
  - Dual solution scheme: generalization of conventional OC techniques (GOC)
  - Primal solution scheme: mixed method: gradual transition between pure MP and OC approaches

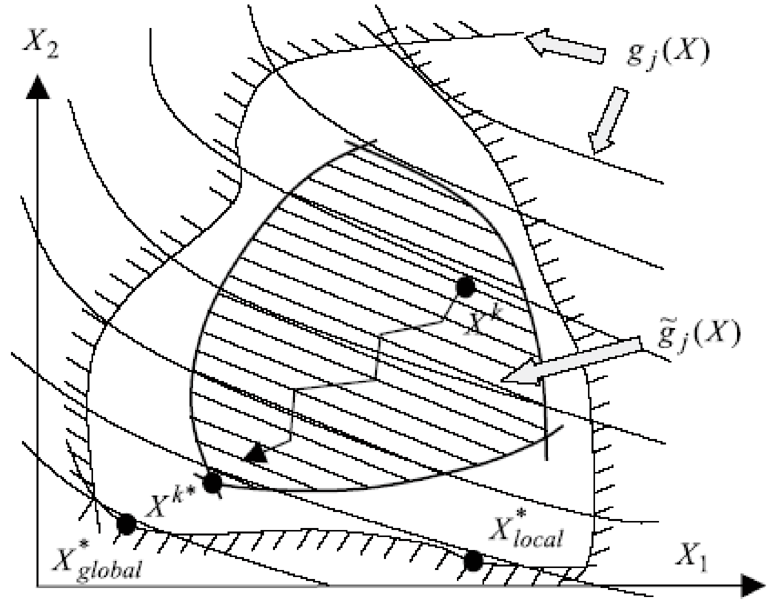
CONCLUSION

CONCLUSION

- Berke's approximation has been successful in providing high quality explicit approximations of displacement constraints
- Optimality Criteria can reduce substantially the number of function evaluation in solving costly problems in truss sizing
- They are used in building fast solution algorithms
- Berke's approximations are first order Taylor expansion of the displacement in terms of the reciprocal design variables
- How can we extend the principle to other engineering design problems?
  - Answer: Structural Approximations

SEQUENTIAL CONVEX PROGRAMMING APPROACH

Direct solution of the original optimisation problem which is generally non-linear, implicit in the design variables



$$\begin{array}{lll} \min\limits_{\boldsymbol{x}} & g_0(\boldsymbol{x}) & \\ \text{s.t.} & g_j(\boldsymbol{x}) \leq \bar{g}_j & j = 1 \ldots m \end{array}$$

is replaced by a sequence of optimisation sub-problems

$$\begin{array}{lll} \min\limits_{\boldsymbol{x}} & \tilde{g}_0(\boldsymbol{x}) & \\ \text{s.t.} & \tilde{g}_j(\boldsymbol{x}) \leq \bar{g}_j & j = 1 \ldots m \end{array}$$

by using approximations of the responses and using powerful mathematical programming algorithms

SEQUENTIAL CONVEX PROGRAMMING APPROACH

- Two basic concepts:
  - Structural approximations replace the implicit problem by an explicit optimisation sub-problem using convex, separable, conservative approximations; e.g. CONLIN, MMA
  - Solution of the convex sub-problems: efficient solution using dual methods algorithms or SQP method.
- Advantages of SCP:
  - Optimised design reached in a reduced number of iterations: 10 to 20 F.E. analyses
  - Efficiency, robustness, generality, and flexibility, small computation time
  - Large scale problems in terms of number of design constraints and variables