

# CONLIN & MMA solvers

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# CONLIN METHOD

# LAY-OUT

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- CONLIN SUBPROBLEMS
- DUAL METHOD APPROACH FOR CONLIN SUBPROBLEMS
- SEQUENTIAL QUADRATIC PROGRAMMING IN DUAL SPACE
- TREATMENT OF SIDE CONSTRAINTS
- CONSTRAINT RELAXATION
- OPTIMIZATION ALGORITHM
- EXAMPLE

# THE CONLIN SUBPROBLEMS

- The general numerical optimization problem coming from structural optimization writes

$$\begin{aligned} \min_{\mathbf{x}} \quad & c_0(\mathbf{x}) \\ \text{s.t.} \quad & c_j(\mathbf{x}) \leq \mathbf{0} \quad j = 1, m \\ & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1, n \end{aligned}$$

- The functions  $c_0$  and  $c_j$  are not explicit nor separable functions.
- The original problem can be solved as a sequence of *convex explicit subproblems* having a *simple algebraic form*.
  - ➔ The convex subproblems are created using the **CONvex LINearization (CONLIN) approximation method**.

# THE CONLIN SUBPROBLEMS

- The **convex linearization** at  $\mathbf{x}_0$  writes :

$$c(\mathbf{x}) = c(\mathbf{x}_0) + \sum_{+} c_i^0 (x_i - x_i^0) - \sum_{-} (x_i^0)^2 c_i^0 \left( \frac{1}{x_i} - \frac{1}{x_i^0} \right)$$

- Where  $c_i^0 = \left. \frac{\partial c}{\partial x_i} \right|_{x_i^0}$

- And  $\Sigma_+$  and  $\Sigma_-$  denotes respectively the summation over the terms for which  $c_0^i$  is positive or negative.

# THE CONLIN SUBPROBLEMS

- Normalization of the design variables

$$\tilde{x}_i = \frac{x_i}{x_i^0} \quad \Rightarrow \quad \tilde{c}_i = c_i^0 x_i^0$$

- The factor  $(x_i^0)^2$  disappears from the CONLIN expansion, which becomes :

$$c(\tilde{\mathbf{x}}) = c(\mathbf{x}_0) + \sum_{+} \tilde{c}_i (\tilde{x}_i - 1) - \sum_{-} \tilde{c}_i \left( \frac{1}{\tilde{x}_i} - 1 \right)$$

- The subproblem becomes

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{+} c_{i0} (x_i - 1) - \sum_{-} c_{i0} \left( \frac{1}{x_i} - 1 \right) + c_0(\mathbf{x}_0) \\ \text{s.t.} \quad & \sum_{+} c_{ij} (x_i - 1) - \sum_{-} c_{ij} \left( \frac{1}{x_i} - 1 \right) \leq -c_j(\mathbf{x}_0) \quad j = 1, m \\ & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1, \dots, n \end{aligned}$$

# THE CONLIN SUBPROBLEMS

- After normalization, the sub problem can be casted into the explicit sub problem format (in which the tilde symbol has been omitted for the sake of simplicity) :

$$\begin{array}{ll}
 \min_{\mathbf{x}} & \sum_{+} c_{i0} x_i - \sum_{-} c_{i0}/x_i - \bar{c}_0 \\
 \text{s.t.} & \sum_{+} c_{ij} x_i - \sum_{-} c_{ij}/x_i \leq \bar{c}_j \quad j = 1, m \\
 & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1, n
 \end{array}$$

- where the  $c_{ij}$  denote the normalized first derivatives of the objective and constraint functions values at  $\mathbf{x}_0$  and :

$$\bar{c}_j = \sum_i |c_{ij}| x_i^0 - c_j(\mathbf{x}^0) \quad j = 0 \dots m$$

# DUAL METHOD FOR CONLIN SUBPROBLEMS

- The explicit subproblems are convex and separable. Thus they are well suited to be solved using *dual algorithms*.
- Dual problem is a max-min procedure :

$$\begin{array}{ll} \max_{\lambda} & \ell(\lambda) \\ \text{s.t.} & \lambda_j \geq 0 \end{array}$$

- where the dual function  $\ell(\lambda)$  results from minimizing the Lagrangian function:

$$L(\mathbf{x}, \lambda) = \sum_{j=0}^m \lambda_j \left( \sum_{+} c_{ij} x_i - \sum_{-} \frac{c_{ij}}{x_i} - \bar{c}_j \right)$$



# DUAL METHOD FOR CONLIN SUBPROBLEMS

- Minimization of Lagrangian function over the acceptable primal variables for a given  $\lambda$ :

$$\mathbf{x}(\lambda) = \min_{\underline{x}_i \leq x_i \leq \bar{x}_i} L(\mathbf{x}, \lambda)$$

- **Because of separability** the minimization of the Lagrangian problem takes a very simple and efficient form. The problem can be broken into n separate one-dimensional problems

$$\begin{aligned} \min \quad & L_i(x_i) = a_i x_i + \frac{b_i}{x_i} \\ \text{s.t.} \quad & \underline{x}_i \leq x_i \leq \bar{x}_i \end{aligned}$$

- where the coefficients  $a_i$  and  $b_i$  remain non negative

$$a_i = \sum_{+} c_{ij} \lambda_j \geq 0 \qquad b_i = - \sum_{-} c_{ij} \lambda_j \geq 0$$

# DUAL METHOD FOR CONLIN SUBPROBLEMS

- The optimality of Lagrangian problem yields the primal-dual relationships :

$$L'(x_i) = a_i - \frac{b_i}{x_i^2} = 0$$

- It comes the fully explicit primal-dual relationships  $\mathbf{x} = \mathbf{x}(\lambda)$

$$x_i = (b_i/a_i)^{1/2} \quad \text{if} \quad \underline{x}_i^2 \leq b_i/a_i \leq \bar{x}_i^2$$

$$x_i = \bar{x}_i \quad \text{if} \quad b_i/a_i \leq \underline{x}_i^2$$

$$x_i = \bar{x}_i \quad \text{if} \quad \bar{x}_i^2 \leq b_i/a_i$$

# DUAL METHOD FOR CONLIN SUBPROBLEMS

- The **dual problem** can be expressed in closed form :

$$\begin{aligned} \max_{\lambda} \quad & \ell(\lambda) = \sum_{j=0}^m \lambda_j \left[ \sum_{+} c_{ij} x_i(\lambda) - \sum_{-} \frac{c_{ij}}{x_i(\lambda)} - \bar{c}_j \right] \\ \text{s.t.} \quad & \lambda_j \geq 0 \end{aligned}$$

- A fundamental property of the dual function is that its **first derivatives** are simply given by the primal constraint values :

$$g_j = \frac{\partial \ell}{\partial \lambda_j} = \sum_{+} c_{ij} x_i(\lambda) - \sum_{-} \frac{c_{ij}}{x_i(\lambda)} - \bar{c}_j$$

# DUAL METHOD FOR CONLIN SUBPROBLEMS

- The **second derivatives** of the dual function

$$H_{jk} = \frac{\partial^2 \ell}{\partial \lambda_j \partial \lambda_k}$$

- can also be written in closed form. Deriving a second time the first derivatives, it comes :

$$H_{jk} = \sum_{+} c_{ij} \frac{\partial x_i(\lambda)}{\partial \lambda_k} + \sum_{-} \frac{c_{ij}}{x_i^2(\lambda)} \frac{\partial x_i(\lambda)}{\partial \lambda_k}$$

- Differentiating the primal-dual relationships it comes for the free variables. :

$$\frac{\partial x_i(\lambda)}{\partial \lambda_k} = -\frac{c_{ik} x_i}{2a_i^2} \quad \text{if } c_{ik} > 0$$

$$\frac{\partial x_i(\lambda)}{\partial \lambda_k} = -\frac{c_{ik}}{2x_i a_i^2} \quad \text{if } c_{ik} < 0$$

- For the fixed variables these derivatives are obviously zero.

# DUAL METHOD FOR CONLIN SUBPROBLEMS

- Therefore, the second order derivatives writes

$$H_{jk} = -\frac{1}{2} \sum_{i \in I} n_{ij} n_{ik} \frac{x_i}{a_i}$$

- Where

$$n_{ij} = c_{ij} \quad \text{if } c_{ij} > 0$$

$$n_{ij} = \frac{c_{ij}}{x_i^2} \quad \text{if } c_{ij} < 0$$

- And the summation is restricted to the set of active variables only

$$I = \{i \mid \underline{x}_i \leq x_i \leq \bar{x}_i\}$$

# SEQUENTIAL QUADRATIC PROGRAMMING IN DUAL SPACE

- Hessian matrix in dual space :

$$H_{jk} = -\frac{1}{2} \sum_{i \in I} n_{ij} n_{ik} \frac{x_i}{a_i}$$

- It is important to emphasize that the summation is restricted to the set  $I$  of free primal variables.

$$I = \{i \mid \underline{x}_i \leq x_i \leq \bar{x}_i\}$$

- It comes that :
  - The second derivatives of the dual function are **discontinuous**
  - There is an inherent and fundamental difficulty in using Newton type methods for solving the dual problem

# SEQUENTIAL QUADRATIC PROGRAMMING IN DUAL SPACE

- **Generalized Newton method in dual space**

- The update scheme of dual variables writes :

$$\lambda_j^+ = \lambda_j + \alpha s_j$$

- where the Newton search direction is

$$\mathbf{s} = -[\mathbf{H}]^{-1}\mathbf{g}$$

- and  $\alpha$  is the step length along the search direction  $\mathbf{s}$ .

# SEQUENTIAL QUADRATIC PROGRAMMING IN DUAL SPACE

## □ Quadratic subproblems

- The search direction  $\mathbf{s}$  is also the solution of the optimization problem :

$$\max \quad \frac{1}{2} \mathbf{s}^T \mathbf{H} \mathbf{s} + \mathbf{s}^T \mathbf{g}$$

- By selecting a unit step length  $\alpha = 1$ , it comes  $\mathbf{s} = \lambda^+ - \lambda^0$ , hence the following quadratic problem

$$\begin{aligned} \max \quad & q(\lambda) = \frac{1}{2} \lambda^T \mathbf{H} \lambda - \lambda^T \mathbf{b} \\ \text{s.t.} \quad & \lambda_j \geq 0 \end{aligned}$$

- where

$$\mathbf{b} = \mathbf{H} \lambda_0 - \mathbf{g}$$



# CONSTRAINT RELAXATION

- What do we do if the approximate feasible design domain is empty ?
- We have to introduce an additional design variable  $\delta$ . The relaxed explicit subproblem writes :

$$\begin{aligned}
 \min_{\mathbf{x}, \delta} \quad & \sum_{+} c_{i0}^0 (x_i - 1) - \sum_{-} c_{i0}^0 \left( \frac{1}{x_i} - 1 \right) + w z_0 \delta \\
 \text{s.t.} \quad & \sum_{+} c_{ij}^0 (x_i - 1) - \sum_{-} c_{ij}^0 \left( \frac{1}{x_i} - 1 \right) \leq \bar{c}_j + z_j \left( 1 - \frac{1}{\delta} \right) \quad j = 1, m \\
 & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1, n \\
 & 1 \leq \delta
 \end{aligned}$$

- where  $w$  is a user-supplied weighting factor and

$$z_j = \sum_i |c_{ij}| x_i^0 \quad j = 0 \dots m$$

# CONSTRAINT RELAXATION

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- The factors  $w$  and  $z_j$  represent increments to the functions  $c_j(\mathbf{x})$  opening the feasible design space if necessary.
- If the relaxation variables  $\delta$  hits its lower bound  $\delta = 1$  nothing is changed in the problem statement.
- If the algorithm finds a value  $\delta$  greater than unity the approximate feasible domain is artificially enlarged.

# CONSTRAINT RELAXATION

- The Lagrangian problem related to the relaxation variable  $\delta$  :

$$\min_{\delta \geq 1} \quad wz_0\delta - \left(1 - \frac{1}{\delta}\right) \sum \lambda_j z_j$$

- It comes that  $\delta$  is given in terms of the dual variables by the relations :

$$\delta = \left(\sum \lambda_j z_j / wz_0\right)^{1/2} \quad \text{if} \quad \sum \lambda_j z_j > wz_0$$

$$\delta = 1 \quad \text{if} \quad \sum \lambda_j z_j \leq wz_0$$

# CONSTRAINT RELAXATION

- New dual gradient vector and Hessian matrix are :

$$g_j = \sum_{+} c_{ij} x_i(\lambda) - \sum_{-} \frac{c_{ij}}{x_i(\lambda)} - \bar{c}_j - z_j \left(1 - \frac{1}{\delta}\right)$$

$$H_{jk} = -\frac{1}{2} \sum_{i \in I} n_{ij} n_{ik} \frac{x_i}{a_i} - \frac{1}{2} \frac{z_j z_k}{w \delta^3}$$

# OPTIMIZATION ALGORITHM

- **Step 1** : Initialization : the set  $M$  of active constraints

$$M = \{j \mid c_j \text{ is potentially active}\}$$

- and define the set  $N$  of active variables

$$N = \{i \mid x_i \text{ is imposed to be free}\}$$

- **Step 2**: Compute the free primal variables in subspace  $N$  using

$$x_i(\lambda) = (b_i/a_i)^{1/2} \quad i \in N$$

- Compute the relation factor  $\delta$  using

$$\delta = (\sum \lambda_j z_j / w z_0)^{1/2} \quad \text{if } \sum \lambda_j z_j > w z_0$$

$$\delta = 1 \quad \text{if } \sum \lambda_j z_j \leq w z_0$$

# OPTIMIZATION ALGORITHM

- **Step 3:** Evaluate the dual gradient vector in subspace  $M$

$$\frac{\partial \ell}{\partial \lambda_j} = g_j(\mathbf{x}(\lambda)) \quad j \in M$$

- **Step 4:** Check optimality in subspace  $N$  if

$$g_j = 0 \text{ for } \lambda_j > 0$$

$$g_j < 0 \text{ for } \lambda_j = 0$$

Then then goto step 7

- **Step 5:** Evaluate the dual Hessian matrix in the subspace  $M$

$$H_{jk}(\mathbf{x}) \quad j, k \in M$$

# OPTIMIZATION ALGORITHM

- **Step 6:** Solve the quadratic subproblem (8.21) and return to step 2.

$$\begin{aligned} \max \quad & q(\lambda) = \frac{1}{2} \lambda^T \mathbf{H} \lambda - \lambda^T \mathbf{b} \\ \text{s.t.} \quad & \lambda_j \geq 0 \end{aligned}$$

- **Step 7:** Update of the set N
  - For  $i \in N$  if  $x_i < \underline{x}_i$  or  $x_i > \bar{x}_i$  then remove  $i$  from the set N.
  - For  $i \notin N$ , evaluate  $x_i = \sqrt{b_i/a_i}$ .
  - If  $\underline{x}_i < x_i < \bar{x}_i$  then add  $i$  to the set N.

If the set N has been modified, then go back to step 2

# OPTIMIZATION ALGORITHM

□ **Step 8:** Update the set  $M$

– For  $j \in M$ , if  $\lambda_j = 0$  then remove  $j$  from the set  $M$ .

– For  $j \notin M$ , evaluate the primal constraint values  $g_j$  using eq. (8.17). If  $g_j \leq 0 \forall j \in M$ , then go to step 9.

Otherwise add one or more active constraints to the set  $M$ .

□ **Step 9:** The maximum of the dual function has been obtained :

– Lagrange multipliers are  $\lambda^*$  for  $j \in M$ . (other ones being zero.

– Primal design variables  $x^*_i$  for  $i \in N$ . (other ones are fixed).



## EXAMPLE: CONLIN APPROXIMATION

- Let's consider, the optimization problem :

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1 + 4x_2 \\ \text{s.t.} \quad & x_2 - x_1 \geq 0 \\ & 3x_1 - 2x_2 \geq 1 \end{aligned}$$

- We consider the following starting point  $\mathbf{x} = (3, 4)^\top$  around which we have to perform the convex linearization process.
- First of all the problem is written under standard form :

$$\begin{aligned} \min_{x_1, x_2} \quad & c_0(\mathbf{x}) = x_1 + 4x_2 \\ \text{s.t.} \quad & c_1(\mathbf{x}) \equiv x_1 - x_2 \leq 0 \\ & c_2(\mathbf{x}) \equiv -3x_1 + 2x_2 \leq -1 \end{aligned}$$

## EXAMPLE: CONLIN APPROXIMATION

- CONLIN approximation of the objective function.

$$c_0(x_1, x_2) = x_1 + 4x_2$$

- This linear function has all positive coefficient and so positive derivatives.

$$\frac{\partial c_0}{\partial x_1} = 1 > 0 \quad \text{and} \quad \frac{\partial c_0}{\partial x_2} = 4 > 0$$

- Therefore the linear expansion is used for both variables and the objective function remains unchanged.

$$\tilde{c}_0(x_1, x_2) = x_1 + 4x_2$$

## EXAMPLE: CONLIN APPROXIMATION

- The first constraint

$$c_1(x_1, x_2) = x_1 - x_2 \leq 0$$

- has derivatives with different signs :

$$\frac{\partial c_1}{\partial x_1} = 1 > 0 \quad \text{and} \quad \frac{\partial c_1}{\partial x_2} = -1 < 0$$

- The CONLIN linearization then proceeds to a direct variable expansion for  $x_1$  and a reciprocal variable expansion in  $x_2$ .

$$\begin{aligned} \tilde{c}_1 &= c_1(\mathbf{x}^0) + \frac{\partial c_1}{\partial x_1}(x_1 - x_1^0) + \frac{\partial c_1}{\partial x_2}(-1)(x_2^0)^2 \left( \frac{1}{x_2} - \frac{1}{x_2^0} \right) \leq 0 \\ &= (-1) + (+1)(x_1 - 3) + (-1)(-16) \left( \frac{1}{x_2} - \frac{1}{4} \right) \leq 0 \end{aligned}$$

- It comes

$$\tilde{c}_1 = x_1 + \frac{16}{x_2} \leq 8$$

## EXAMPLE: CONLIN APPROXIMATION

- For the second constraint,

$$c_2(x_1, x_2) = -3x_1 + 2x_2 \leq -1$$

- The first derivative is negative while the second one is positive so that we have a reciprocal variable expansion for  $x_1$  and a direct expansion for  $x_2$  :

$$\frac{\partial c_2}{\partial x_1} = -3 < 0 \quad \text{and} \quad \frac{\partial c_2}{\partial x_2} = 2 > 0$$

- The CONLIN approximation writes

$$\begin{aligned} \tilde{c}_2 &= c_2(\mathbf{x}^0) + \frac{\partial c_2}{\partial x_1}(-1)(x_1^0)^2 \left( \frac{1}{x_1} - \frac{1}{x_1^0} \right) + \frac{\partial c_2}{\partial x_2}(x_2 - x_2^0) \leq -1 \\ &= (-1) + (-3)(-9) \left( \frac{1}{x_1} - \frac{1}{3} \right) + (2)(x_2 - 4) \leq -1 \end{aligned}$$

- And finally

$$\tilde{c}_2 = \frac{27}{x_1} + 2x_2 \leq 17$$

## EXAMPLE: DUAL SOLUTION

- The primal CONLIN problem is

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + \frac{16}{x_2} \leq 8 \\ & \frac{27}{x_1} + 2x_2 \leq 17 \end{aligned}$$

- As the subproblem is convex and separable, it can be solved using duality approach

## EXAMPLE: DUAL SOLUTION

- The Lagrange function writes

$$L(\mathbf{x}, \lambda) = x_1 + 4x_2 + \lambda_1\left(x_1 + \frac{16}{x_2} - 8\right) + \lambda_2\left(\frac{27}{x_1} + 2x_2 - 17\right)$$

- The primal dual relationships are derived from the minimum of the Lagrange function with respect to  $x_i$  variables for given dual variables  $\lambda_j$ .

$$\min_{\mathbf{x}} L(\mathbf{x}, \lambda)$$

- The optimality conditions gives

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= 1 + \lambda_1 - \frac{27\lambda_2}{x_1^2} = 0 \\ \frac{\partial L}{\partial x_2} &= 4 - \frac{16\lambda_1}{x_2^2} + 2\lambda_2 = 0\end{aligned}$$

## EXAMPLE: DUAL SOLUTION

- They yield the primal dual relationships

$$x_1 = \left( \frac{27\lambda_2}{1 + \lambda_1} \right)^{1/2}$$

$$x_2 = \left( \frac{8\lambda_1}{2 + \lambda_2} \right)^{1/2}$$

## EXAMPLE: DUAL SOLUTION

- The dual maximization problem then writes

$$\begin{aligned} \max_{\lambda} \quad & \ell(\lambda) = \left( \frac{27\lambda_2}{1 + \lambda_1} \right)^{1/2} + 4 \left( \frac{8\lambda_1}{2 + \lambda_2} \right)^{1/2} \\ & + \lambda_1 \left[ \left( \frac{27\lambda_2}{1 + \lambda_1} \right)^{1/2} + 16 \left( \frac{2 + \lambda_2}{8\lambda_1} \right)^{1/2} - 8 \right] \\ & + \lambda_2 \left[ 27 \left( \frac{1 + \lambda_1}{27\lambda_2} \right)^{1/2} + 2 \left( \frac{8\lambda_1}{2 + \lambda_2} \right)^{1/2} - 17 \right] \\ \text{s.t.} \quad & \lambda_1, \lambda_2 \geq 0 \end{aligned}$$



## EXAMPLE: DUAL SOLUTION

- It can be easily verified that

$$\frac{\partial \ell}{\partial \lambda_1} = \left( \frac{27\lambda_2}{1 + \lambda_1} \right)^{1/2} + 16 \left( \frac{2 + \lambda_2}{8\lambda_1} \right)^{1/2} - 8$$

$$\frac{\partial \ell}{\partial \lambda_2} = 27 \left( \frac{1 + \lambda_1}{27\lambda_2} \right)^{1/2} + 2 \left( \frac{8\lambda_1}{2 + \lambda_2} \right)^{1/2} - 17$$



# MMA METHOD

## Dual method for MMA sub-problems

- MMA subproblems (after normalization):

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i=1}^n \frac{p_{i0}}{U_{ij} - x_i} + \sum_{i=1}^n \frac{q_{i0}}{x_i - L_{ij}} \\ \text{s.t.} \quad & \sum_{i=1}^n \frac{p_{ij}}{U_{ij} - x_i} + \sum_{i=1}^n \frac{q_{ij}}{x_i - L_{ij}} \leq d_j \quad j = 1 \dots m \\ & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1 \dots n \end{aligned}$$

Remark: asymptotes  $U_{ij}$  and  $L_{ij}$  can depend on both variable index  $i$  and constraint index  $j$

- Lagrange function (where we have introduced  $\lambda_0=1$  for simplicity)

$$L(\mathbf{x}, \lambda) = \sum_{j=0}^m \lambda_j \left( \sum_{i=1}^n \frac{p_{ij}}{U_{ij} - x_i} + \sum_{i=1}^n \frac{q_{ij}}{x_i - L_{ij}} - d_j \right) \quad 35$$

# Dual method for MMA sub-problems

- The Lagrangian problem:

$$\min_{\underline{x}_i \leq x_i \leq \bar{x}_i} L(\mathbf{x}, \lambda)$$

- Because of separability, the n-dimensional problem can be split into n 1-dimensional problems:

$$\min_{\underline{x}_i \leq x_i \leq \bar{x}_i} L_i(x_i, \lambda) = \sum_{j=0}^m \frac{\lambda_j p_{ij}}{U_{ij} - x_i} + \sum_{j=0}^m \frac{\lambda_j q_{ij}}{x_i - L_{ij}}$$

## Dual method for MMA sub-problems

- For pure MMA problems, asymptotes depend only on variable index  $i$ , i.e.  $U_{ij} = U_i$  and  $L_{ij} = L_i$ , so that Lagrangian problem can be solved in closed form from optimality conditions:

$$\frac{\sum_{j=0}^m \lambda_j p_{ij}}{(U_i - x_i)^2} - \frac{\sum_{j=0}^m \lambda_j q_{ij}}{(x_i - L_i)^2} = 0$$

- Primal-dual relationships for pure MMA:

$$x_i^*(\lambda) = \frac{U_i + \eta L_i}{\eta + 1} \quad \text{with} \quad \eta = \sqrt{\frac{\sum_{j=0}^m \lambda_j p_{ij}}{\sum_{j=0}^m \lambda_j q_{ij}}}$$

## Dual method for MMA sub-problems

- For generalized MMA family, each constraint has its own set of asymptotes, and it is not possible anymore to find the closed form solution of Lagrangian problem. Solution is obtained by resorting to an iterative Newton scheme applied to optimality conditions

$$x_i(\lambda^+) = x_i(\lambda) - \frac{\partial L_i / \partial x_i}{\partial^2 L_i / \partial x_i^2}$$

with

$$\frac{\partial L_i}{\partial x_i} = \sum_{j=0}^m \frac{\lambda_j p_{ij}}{(U_{ij} - x_i)^2} - \sum_{j=0}^m \frac{\lambda_j q_{ij}}{(x_i - L_{ij})^2}$$

$$\frac{\partial^2 L_i}{\partial x_i^2} = 2 \sum_{j=0}^m \frac{\lambda_j p_{ij}}{(U_{ij} - x_i)^3} - 2 \sum_{j=0}^m \frac{\lambda_j q_{ij}}{(x_i - L_{ij})^3}$$

## Dual method for MMA sub-problems

- Primal-dual relations with treatment of side-constraints:

$$\begin{aligned}x_i(\lambda) &= x_i^* && \text{if } \underline{x}_i \leq x_i^* \leq \bar{x}_i, \\x_i(\lambda) &= \underline{x}_i && \text{if } x_i^* \leq \underline{x}_i, \\x_i(\lambda) &= \bar{x}_i && \text{if } x_i^* \geq \bar{x}_i.\end{aligned}$$

- Dual function, gradient of dual function:
  - calculate  $x = x(\lambda)$
  - compute  $f(x(\lambda))$  and  $g(x(\lambda))$
  - insert the calculated values in  $L(x, \lambda) = l(\lambda)$
  - gradient is just given by  $g(x(\lambda))$ .
- If sub-problem is too complicated, sub-problem solution is itself decomposed into a sequence of quadratic separable sub-sub-problems that can be efficiently solved by dual method.

# Annex 1: GCMMA code in Matlab by Svanberg

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# GCMMA code in Matlab by Svanberg

- For academic purpose, Svanberg wrote a Matlab version of its GCMMA solver: i.e. Globally convergent approximations + solver (IP method)

- GCMMA : mmasub function in Matlab

```
function [xmma, ymma, zmma, lam, xsi, eta, mu, zet, s, low, upp] = ...  
mmasub(m, n, iter, xval, xmin, xmax, xold1, xold2, ...  
f0val, df0dx, df0dx2, fval, dfdx, dfdx2, low, upp, a0, a, c, d);
```

- Solver : subsolv function (called from mmasub)

```
function [xmma, ymma, zmma, lamma, xsimma, etamma, mumma, zetmma,  
smma] = ...  
subsolv(m, n, epsimin, low, upp, alfa, beta, p0, q0, P, Q, a0, a, b, c, d);
```

# GCMMA code in Matlab by Svanberg

- Solve canonical problems of the form :

$$\begin{array}{ll} \min & f_0(\mathbf{x}) + z + \sum_{j=1}^m (c_j y_j + \frac{1}{2} d_j y_j^2) \\ \text{s.t.} & f_j(\mathbf{x}) - a_j z - y_j \leq 0 & j = 1 \dots m \\ & \underline{x}_i \leq x_i \leq \bar{x}_i & i = 1 \dots n \\ & y_j \geq 0 & j = 1 \dots m \\ & z \geq 0 \end{array}$$

- With
  - $f_0, f_1 \dots f_m$  real functions, continuous and differentiable
  - $a_i, c_i, d_i$  real non negative numbers, with  $a_i + d_i > 0$

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- Classic problems of non-linear programming

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{s.t.} & f_j(\mathbf{x}) \leq 0 \quad j = 1 \dots m \\ & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1 \dots n \end{array}$$

- Parameter choice:

$$\begin{array}{ll} a_j = 0 & j = 1 \dots m \\ d_j = 0 & j = 1 \dots m \\ c_j = \text{big} & j = 1 \dots m \end{array}$$

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- Least square problem

$$\begin{array}{ll}
 \min & \sum_{k=1}^p (h_k(\mathbf{x}))^2 \\
 \text{s.t.} & g_j(\mathbf{x}) \leq 0 \quad j = 1 \dots q \\
 & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1 \dots n
 \end{array}$$

- Play with the quadratic term in  $y_j$  and the coefficient  $d_j$
- Parameter choice:

$$\begin{array}{lll}
 m = 2p + q & a_j = 0 & j = 1 \dots m \\
 f_0(\mathbf{x}) = 0 & d_k = 2 & k = 1 \dots 2p \\
 f_k(\mathbf{x}) = h_k(\mathbf{x}) & k = 1 \dots p & d_{2p+j} = 0 \quad j = 1 \dots q \\
 f_{p+k}(\mathbf{x}) = -h_k(\mathbf{x}) & k = 1 \dots p & c_k = 0 \quad k = 1 \dots 2p \\
 f_{2p+j}(\mathbf{x}) = g_j(\mathbf{x}) & j = 1 \dots q & c_{2p+j} = \text{big} \quad j = 1 \dots q
 \end{array}$$

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- $L_1$  minimization problem

$$\begin{array}{ll}
 \min & \sum_{k=1}^p |h_k(\mathbf{x})| \\
 \text{s.t.} & g_j(\mathbf{x}) \leq 0 \quad j = 1 \dots q \\
 & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1 \dots n
 \end{array}$$

- Play with the linear terms in  $y_j$  and the coefficient  $c_j$
- Parameter choice:

$$\begin{array}{llll}
 m = 2p + q & & & \\
 f_0(\mathbf{x}) = 0 & & a_j = 0 & j = 1 \dots m \\
 f_k(\mathbf{x}) = h_k(\mathbf{x}) & k = 1 \dots p & d_j = 0 & j = 1 \dots m \\
 f_{p+k}(\mathbf{x}) = -h_k(\mathbf{x}) & k = 1 \dots p & c_k = 1 & k = 1 \dots 2p \\
 f_{2p+j}(\mathbf{x}) = g_j(\mathbf{x}) & j = 1 \dots q & c_{2p+j} = \text{big} & j = 1 \dots q
 \end{array}$$

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- $L_\infty$  minimization problems or min max problems

$$\begin{array}{ll}
 \min & \max_{k=1 \dots p} |h_k(\mathbf{x})| \\
 \text{s.t.} & g_j(\mathbf{x}) \leq 0 \quad j = 1 \dots q \\
 & \underline{x}_i \leq x_i \leq \bar{x}_i \quad i = 1 \dots n
 \end{array}$$

- Play with the linear terms in z and the coefficient  $a_j$
- Parameter choice:

$$\begin{array}{llll}
 m = 2p + q & & & \\
 f_0(\mathbf{x}) = 0 & & a_k = 1 & k = 1 \dots 2p \\
 f_k(\mathbf{x}) = h_k(\mathbf{x}) & k = 1 \dots p & a_{2p+j} = 0 & j = 1 \dots q \\
 f_{p+k}(\mathbf{x}) = -h_k(\mathbf{x}) & k = 1 \dots p & d_j = 0 & j = 1 \dots m \\
 f_{2p+j}(\mathbf{x}) = g_j(\mathbf{x}) & j = 1 \dots q & c_j = \text{big} & j = 1 \dots m
 \end{array}$$