FROM OPTIMALITY CRITERIA TO SEQUENTIAL CONVEX PROGRAMMING

Pierre DUYSINX
LTAS – Automotive Engineering
Academic year 2020-2021

LAY-OUT

- Optimality Criteria
  - Fully stressed design
  - Optimality criteria with one displacement constraint

- Interpretation of OC as first order approximations
  - FSD zero order approximation
  - Berke's approximation = 1st order approximation

LAY-OUT

- Optimality Criteria
  - Optimality criteria with multiple displacement and stress constraints

- Generalized optimality criteria
  - Dual maximization to solve OC criteria with multiple constraints

- Unified approach to structural optimization

OPTIMALITY CRITERIA : STRESS AND DISPLACEMENT CONSTRAINTS

Stress and displacement constraints

- Combination of the two previous O.C.
  - Stress constraints
  - Displacement constraints

- Set of active constraints is assumed to be known
  - ň active design variables
  - m active displacement constraints

- Passive design variables: side constraints or determined by the stress constraints

$$x_i = \max\{\underline{x}_i, \tilde{x}_i\} \qquad i > \tilde{n}$$

$$\tilde{x}_i = x_i^0 \max_{l=1\ldots c}\{\frac{\sigma_{i,l}^0}{\bar{\sigma}_i}\}$$

Stress and displacement constraints

- Active design variables: ruled by displacement constraints
  - m constraints $\boxed{u_j = \bar{u}_j}$

    ➔ m virtual load cases q_j

    ➔ m Lagrange multipliers λ_j
  - Explicit approximation using virtual work

$$u_j = \sum_i \frac{c_{ij}}{x_i} = \bar{u}_j \qquad c_{ij} = x_i \, \boldsymbol{q}^T \, \boldsymbol{K}_i \, \tilde{\boldsymbol{q}}_j$$

- Lagrange function

$$L(x_i, \lambda_j) = \sum_{i=1}^{n} w_i \, x_i + \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} \frac{c_{ij}}{x_i} - \bar{u}_j \right)$$

Stress and displacement constraints

- Stationary conditions

$$\frac{\partial}{\partial x_i} L(x_i, \lambda_j) = 0$$

- After some algebra, the stationary conditions can be casted under the following form

$$w_i + \sum_{j=1}^{m} \lambda_j \frac{-c_{ij}}{x_i^2} = 0$$

- If $c_{ij}>0$

$$x_i = \left(\frac{1}{w_i} \sum_{j=1}^{m} \lambda_j \, c_{ij} \right)^{(1/2)} \qquad i = 1 \ldots \tilde{n}$$

Stress and displacement constraints

- For statically determinate case: OC are exact (because x_i and c_{ij} are constant)
  - ➔ optimum in one analysis
- For statically indeterminate case: OC are approximate
  - ➔Iterative use of the redesign formulae
    - ➔ Active variables

$$x_i^{(k+1)} = x_i^{(k)} \left(\sum_{j=1}^{m} \lambda_j^{(k)} \, \varepsilon_{ij}^{(k)} \right)^{(1/2)} \qquad i = 1 \ldots \tilde{n}$$

    - ➔ Passive variables

$$\tilde{x}_i^{(k+1)} = x_i^{(k)} \max_{l=1\ldots c} \left\{ \frac{\sigma_{i,l}^{(k)}}{\bar{\sigma}_i} \right\} \quad \text{or} \quad \tilde{x}_i^{(k+1)} = \underline{x}_i$$

Stress and displacement constraints

- Lagrange multipliers λ_j????
  - Such that the active displacement constraints are satisfied as equality

$$u_j = \sum_i \frac{c_{ij}}{x_i} = \bar{u}_j$$

  - Closed form solution only if m=1
  - Otherwise numerical schemes
    - Envelop method (intuitive extension from case m=1)
    - Intuitive formula

$$\lambda_j^{(k+1)} = \lambda_j^{(k)} \left(\frac{u_j^{(k)}}{\bar{u}_j} \right)^2$$

    - Newton Raphson applied to solve the set of nonlinear equations

$$u_j = \sum_i \frac{c_{ij}}{x_i(\lambda)} = \bar{u}_j \quad j = 1 \ldots m$$

Newton-Raphson iteration (Taig & Kerr, 1973)

- Solve the system of nonlinear equations using a Newton-Raphson method

$$\begin{cases} x_i^2 = \left(\frac{1}{w_i}\sum_{j=1}^m \lambda_j\, c_{ij}\right) & i = 1 \ldots \tilde{n} \\ \sum_{i=1}^n \frac{c_{ij}}{x_i(\lambda)} = \bar{u}_j & j = 1 \ldots m \end{cases}$$

- First set of equations enables to eliminate the primal variables in terms of the Lagrange multipliers. Newton Raphson is thus used to solve the system

$$\sum_i \frac{c_{ij}}{x_i(\lambda)} = \bar{u}_j \quad j = 1 \ldots m$$

Newton-Raphson iteration (Taig & Kerr, 1973)

- Iteration scheme on Lagrange multipliers λ only

$$\lambda^{(k+1)} = \lambda^{(k)} + [\boldsymbol{H}^{(k)}]^{-1}(\overline{\boldsymbol{u}} - \boldsymbol{u}^{(k)})$$

- Gradient matrix H is given by

$$H_{jk} = \frac{\partial u_j}{\partial \lambda_k} = -\frac{1}{2} \sum_i \frac{c_{ij} c_{ik}}{w_i x_i^3}$$

Newton-Raphson iteration (Taig & Kerr, 1973)

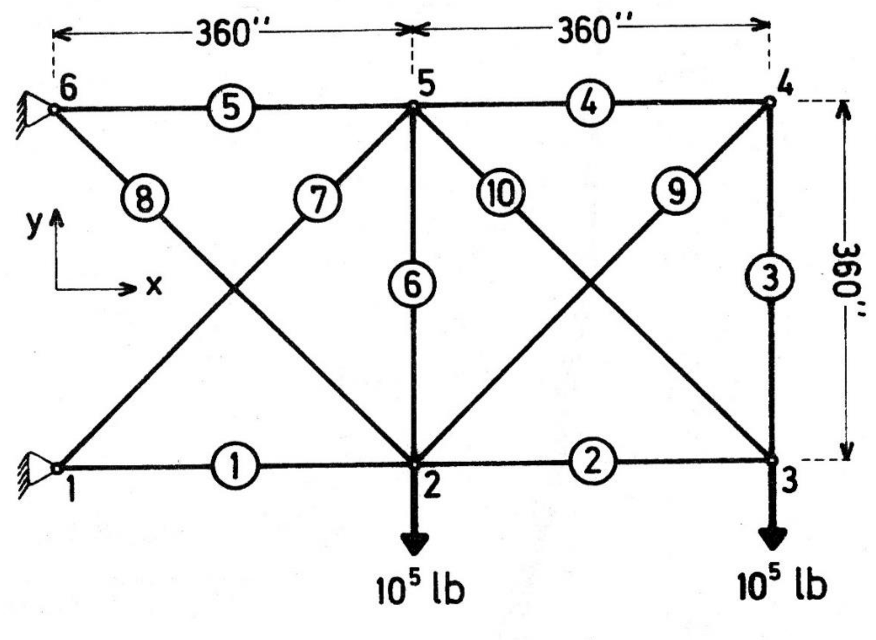
- Difficulties
  - Select an appropriate initial dual set $\lambda^{(0)}$
  - Find correct set of active / passive design variables
  - Identify the set of estimated active behavior constraints (i.e. nonzero λ_j's)
  - H might become singular at some stage of the process

- Solution: dual methods ➔ Generalized Optimality Criteria
  - H is indeed the Hessian matrix of the dual function
  - Dual maximization: determination of the Lagrange variable values
  - Set of active constraints = Non zero optimum values of the Lagrange multipliers

Ten-bar-truss example

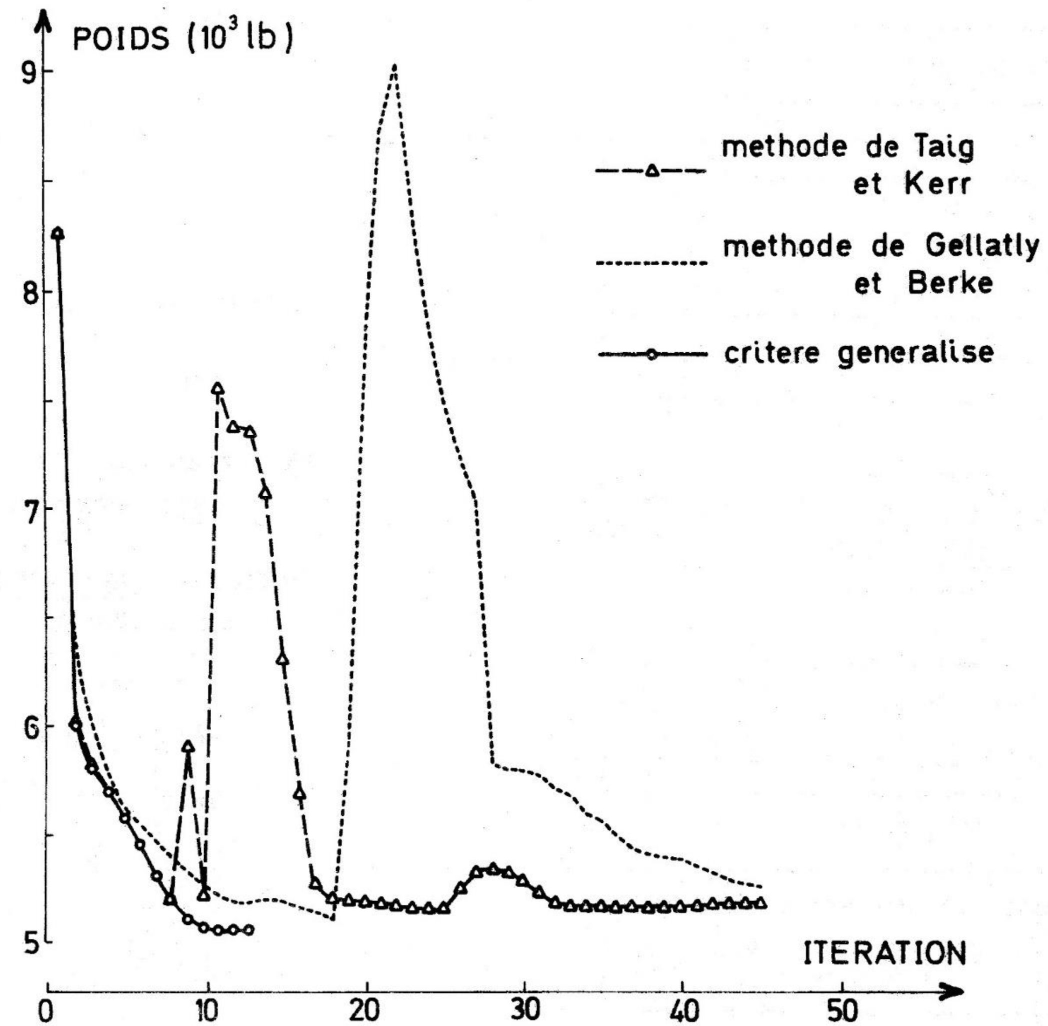
- The stress-ratioïng itself tends to increase the design variable with the smallest stress limit

- Example: stress limit = 25000psi except in member 8 with a variable limit from 25000 to 70000 psi



materiau	:	aluminium
tension maximale admissible	:	25 000 psi
module d'elasticite	:	10^7 psi
masse specifique	:	0.1 lb/in^3
section minimale admissible	:	0.1 in^2
deplacement maximal admissible	:	2.0 in
mise en charge	:	unique

Stress and displacement constraints



GENERALIZED OPTIMALITY CRITERIA
➔
DUAL SOLUTION OF SUBPROBLEMS

Optimality Criteria

- Explicit optimization problem:

$$\begin{array}{lll} \min\limits_{\boldsymbol{x}} & W(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_i & \\ \text{s.t.:} & \sum_i \frac{c_{ij}}{x_i} - \overline{u}_j \leq 0 & (j = 1 \dots m) \\ & \underline{x}_i \leq x_i \leq \overline{x}_i & (i = 1 \dots n) \end{array}$$

- Expression of the displacement constraints using the virtual force method

$$\tilde{u}_j = \boldsymbol{q}^T \boldsymbol{K} \tilde{\boldsymbol{q}}_j = \sum_i \boldsymbol{q}_i^T \boldsymbol{K}_i \tilde{\boldsymbol{q}}_{ij} = \sum_i \frac{c_{ij}}{x_i}$$

C_{ij} are constant in statically determinate structures

$$c_{ij} = x_i \, \boldsymbol{q}_i^T \bar{\boldsymbol{K}} \tilde{\boldsymbol{q}}_{ij}$$

Optimality Criteria

- Explicit optimality conditions of the minimum = KKT conditions

$$\frac{\partial W}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial \tilde{g}_j}{\partial x_i} = \begin{cases} = 0 & \text{if} \quad \underline{x}_i \leq x_i \leq \overline{x}_i \\ > 0 & \text{if} \quad x_i = \underline{x}_i \\ < 0 & \text{if} \quad x_i = \overline{x}_i \end{cases}$$

- That is

$$w_i + \sum_{j=1}^{m} \lambda_j \frac{-c_{ij}}{x_i^2} = \begin{cases} = 0 & \text{if} \quad \underline{x}_i \leq x_i \leq \overline{x}_i \\ > 0 & \text{if} \quad x_i = \underline{x}_i \\ < 0 & \text{if} \quad x_i = \overline{x}_i \end{cases}$$

➔ Analytical expression of the design variables in terms of the Lagrange variables (primal dual relationships)

Optimality Criteria

- Non negativity constraints of the Lagrange multipliers $\lambda_j \geq 0$
  - Active constraints $\lambda_j \geq 0 \quad \text{if} \quad \sum_i \frac{c_{ij}}{x_i(\lambda)} = \overline{u}_j$
  - Passive constraints $\lambda_j = 0 \quad \text{if} \quad \sum_i \frac{c_{ij}}{x_i(\lambda)} < \overline{u}_j$
- Optimal λ_j^* ➔ optimal x_i^* $\boxed{x_i = x_i(\lambda)}$
- Lagrange multipliers = new variables
  = dual variables

Generalized Optimality Criteria

DUAL MAXIMIZATION

- Identifying the optimal value of the Lagrange multipliers λ_j by solving the dual problem

$$\begin{aligned} \max_{\lambda_j} \quad & \ell(\lambda) \\ \text{s.t.} \quad & \lambda_j \geq 0 \qquad j = 1, \ldots, m \end{aligned}$$

- With the dual function

$$\ell(\lambda) = L(\boldsymbol{x}(\lambda), \lambda)$$

$$L(\boldsymbol{x}, \lambda) = \sum_{i=1}^{n} w_i x_i + \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} \frac{c_{ij}}{x_i} - \overline{u}_j \right)$$

- Primal dual relationships

$$\boldsymbol{x}(\lambda) = \arg \min_{\boldsymbol{x}} L(\boldsymbol{x}, \lambda)$$

Dual methods

- Primal problem

$$\min_{x \in X} \quad \sum_{i=1}^{n} w_i \, x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \frac{c_{ij}}{x_i} - \overline{u}_j \le 0 \qquad j = 1, \ldots, m$$

$$X = \{x_i \mid \underline{x}_i \le x_i \le \overline{x}_i \; ; \; i = 1, \ldots, n\}$$

- Lagrange function of the problem

$$L(\boldsymbol{x}, \lambda) = \sum_{i=1}^{n} w_i x_i + \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} \frac{c_{ij}}{x_i} - \overline{u}_j\right)$$

- KKT conditions

$$\frac{\partial W}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial \tilde{g}_j}{\partial x_i} = \begin{cases} = 0 & \text{if} \quad \underline{x}_i \le x_i \le \overline{x}_i \\ > 0 & \text{if} \quad x_i = \underline{x}_i \\ < 0 & \text{if} \quad x_i = \overline{x}_i \end{cases}$$

Dual methods

- Primal dual relations $$\min_{\underline{x}_i \le x_i \le \overline{x}_i} L(x,\lambda)$$

$$x(\lambda) = \arg \min_{x} L(x,\lambda)$$

- Because of the separabilty: solution of n 1D problems

$$\min_{\underline{x}_i \le x_i \le \overline{x}_i} [w_i x_i + \frac{1}{x_i} \sum_{j=1}^{m} \lambda_j \, c_{ij}]$$

- Gives the explicit relation of primal dual variables

$$x_i = \left(\frac{1}{w_i} \sum_{j=1}^{m} c_{ij} \lambda_j \right)^{1/2} \quad \text{if} \quad w_i \underline{x}_i^2 < \sum_{j=1}^{m} c_{ij} \lambda_j < w_i \overline{x}_i^2$$

$$x_i = \underline{x}_i \quad \text{if} \quad \sum_{j=1}^{m} c_{ij} \lambda_j \le w_i \underline{x}_i^2 \qquad\qquad x_i = \overline{x}_i \quad \text{if} \quad \sum_{j=1}^{m} c_{ij} \lambda_j \ge w_i \overline{x}_i^2$$

Dual methods

- Dual function

$$L(\boldsymbol{x}, \lambda) = \sum_{i=1}^{n} w_i x_i + \sum_{j=1}^{m} \lambda_j (\sum_{i=1}^{n} \frac{c_{ij}}{x_i} - \overline{u}_j)$$

- Dual problem

$$\begin{aligned} \max_{\lambda_j} \quad & \ell(\lambda) = \sum_{i=1}^{n} w_i x_i(\lambda) + \sum_{j=1}^{m} \lambda_j (\sum_{i=1}^{n} \frac{c_{ij}}{x_i(\lambda)} - \overline{u}_j) \\ \text{s.t.} \quad & \lambda_j \ \geq 0 \qquad j = 1, \ldots, m \end{aligned}$$

Dual methods

- PROPERTIES OF DUAL FUNCTION: Derivative of dual function

$$\frac{\partial \ell(\lambda)}{\partial \lambda_k} = g_k(\boldsymbol{x}(\lambda))$$

$$\frac{\partial \ell}{\partial \lambda_j} = \tilde{g}_j[\boldsymbol{x}(\lambda)] = \sum_{i=1}^{n} \frac{c_{ij}}{x_i(\lambda)} - \overline{u}_j$$

- Optimality conditions of the dual function

$$\lambda_j > 0 \quad \Rightarrow \quad \frac{\partial \ell}{\partial \lambda_j} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} \frac{c_{ij}}{x_i} = \overline{u}_j$$

$$\lambda_j = 0 \quad \Rightarrow \quad \frac{\partial \ell}{\partial \lambda_j} < 0 \quad \Rightarrow \quad \sum_{i=1}^{n} \frac{c_{ij}}{x_i} < \overline{u}_j$$

Dual methods

- Algorithms for maximizing the dual function based on the Newton's methods = Rigorous Update procedure of the Lagrange Multipliers

$$\lambda^{+} = \lambda + \alpha\, s$$

- Ascent direction

$$\boldsymbol{s} = -[\nabla^2 \ell]^{-1} \nabla \ell = -\boldsymbol{H}^{-1} \tilde{\boldsymbol{h}}$$

- Hessian of the dual function

$$\frac{\partial^2 \ell}{\partial \lambda_j \partial \lambda_k} = \frac{\partial \tilde{g}_k}{\partial \lambda_j} = -\frac{1}{2} \sum_{i=1}^{\tilde{n}} \frac{c_{ij}\, c_{ik}}{w_i\, x_i^3}$$

➔ Discontinuous because of the modification of **ñ,** the set of active design variables

Dual methods

- Second order discontinuous planes:

$$\sum_{j=1}^{m} c_{ij}\,\lambda_j = w_i \underline{x}_i^2$$

$$\sum_{j=1}^{m} c_{ij}\,\lambda_j = w_i \overline{x}_i^2$$

$$\sum_{j=1}^{m} c_{ij}\,\lambda_j > w_i \underline{x}_i^2 \quad \Rightarrow \quad x_i > \underline{x}_i$$

$$\sum_{j=1}^{m} c_{ij}\,\lambda_j \leq w_i \underline{x}_i^2 \quad \Rightarrow \quad x_i = \underline{x}_i$$

Generalized Optimality Criteria

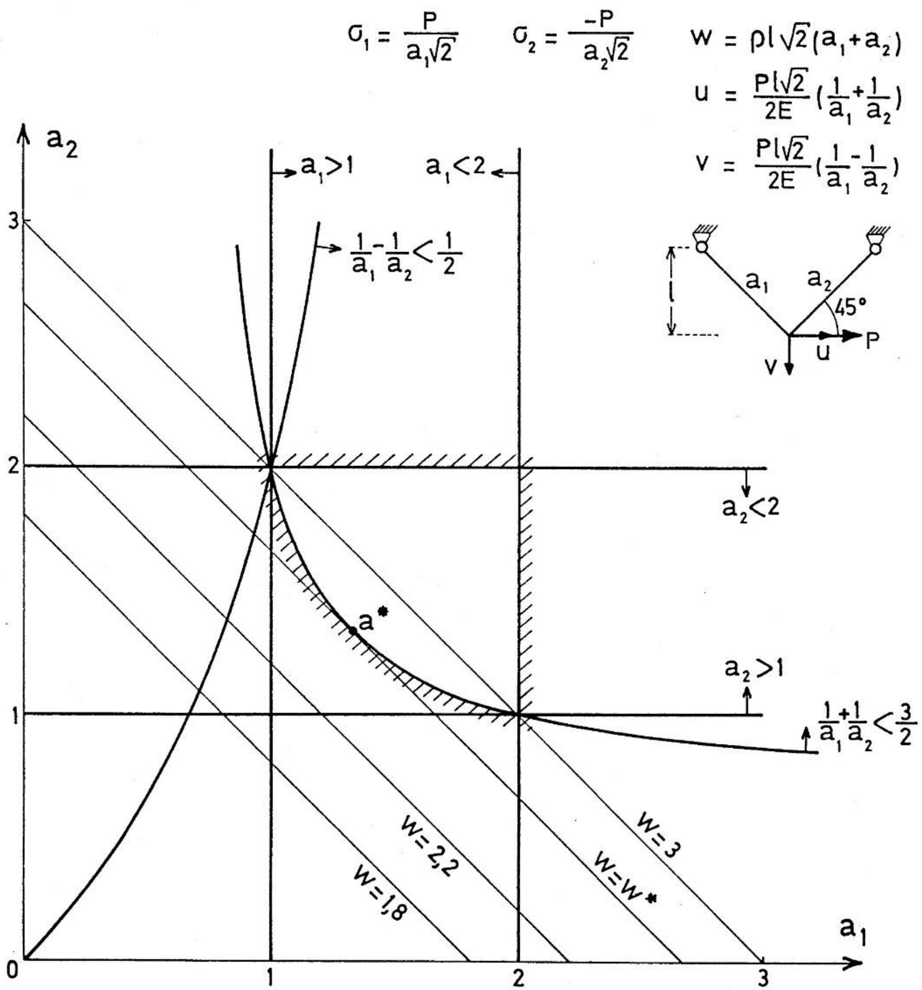
- Dual problems:
  - Explicit
  - Quasi unconstrained
  - Gradient directly available

Easy to solve by standard methods

- Dual algorithms
  - Low computational cost (like OC techniques)
  - Reliable (mathematical basis)
  - Can handle large numbers of inequality constraints
  - Automatically find the active set of constraints and the corresponding active passive design variables

Dual methods

- Two-bar truss



$$\min_{a_1,\ a_2} \quad W = \rho\, l\, \sqrt{2}(a_1 + a_2)$$

$$\text{s.t.:} \quad u = \frac{Pl\sqrt{2}}{2E}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) \leq \bar{u}$$

$$v = \frac{Pl\sqrt{2}}{2E}\left(\frac{1}{a_1} - \frac{1}{a_2}\right) \leq \bar{v}$$

$$\sigma_1 = \frac{P}{\sqrt{2}a_1} \leq \bar{\sigma}_t$$

$$\sigma_2 = \frac{P}{\sqrt{2}a_2} \leq \bar{\sigma}_c$$

$$\min_{a_1,\ a_2} \quad W = a_1 + a_2$$

$$\text{s.t.:} \quad u = \frac{1}{a_1} + \frac{1}{a_2} \leq \frac{3}{2}$$

$$v = \frac{1}{a_1} - \frac{1}{a_2} \leq \frac{1}{2}$$

$$1 \leq a_1 \leq 2$$

$$1 \leq a_2 \leq 2$$

Dual methods

- GOC relations

$$L = a_1 + a_2 + \lambda_1 \left(\frac{1}{a_1} + \frac{1}{a_2} - \frac{3}{2}\right) + \lambda_2 \left(\frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{2}\right)$$

- KKT conditions ➔ primal dual relations

$$\frac{\partial L}{\partial a_1} = 0 \quad \Leftrightarrow \quad a_1 = \sqrt{\lambda_1 + \lambda_2} \quad \text{if} \quad 1 < \lambda_1 + \lambda_2 < 4$$
$$\frac{\partial L}{\partial a_2} = 0 \quad \Leftrightarrow \quad a_2 = \sqrt{\lambda_1 - \lambda_2} \quad \text{if} \quad 1 < \lambda_1 - \lambda_2 < 4$$

- Discontinuity planes

$$\lambda_1 + \lambda_2 = 1 \qquad \lambda_1 + \lambda_2 = 4$$
$$\lambda_1 - \lambda_2 = 1 \qquad \lambda_1 - \lambda_2 = 4$$

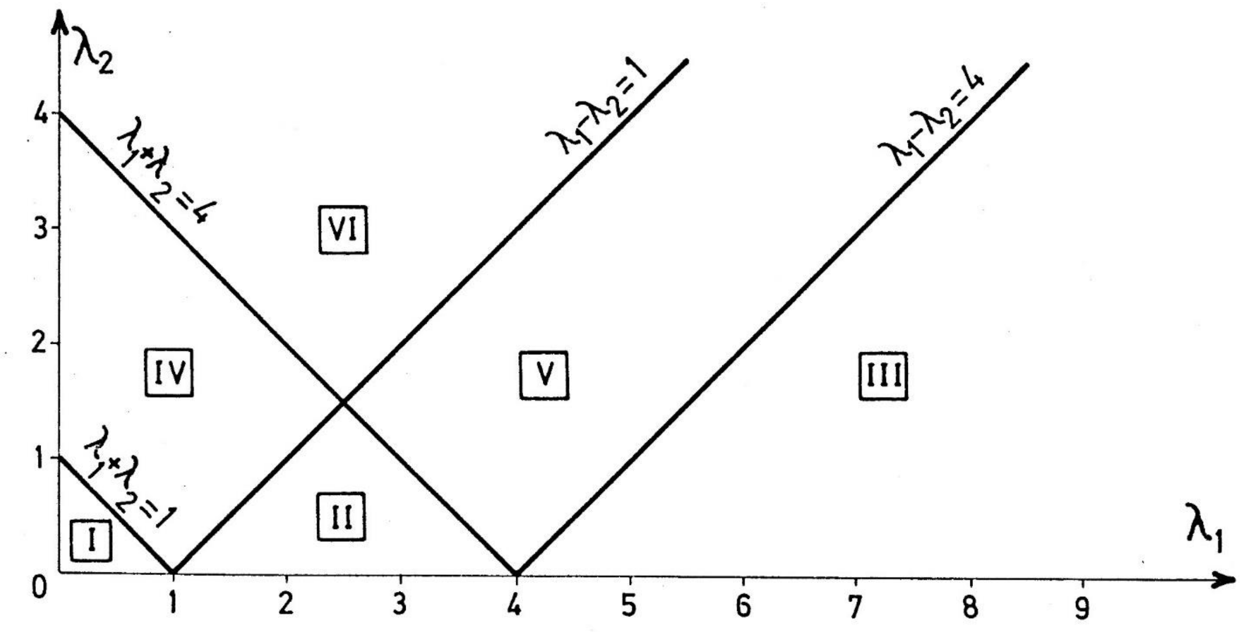
$$a_1 = 1 \quad \text{if} \quad \lambda_1 + \lambda_2 \leq 1$$
$$a_1 = 2 \quad \text{if} \quad \lambda_1 + \lambda_2 \geq 4$$
$$a_2 = 1 \quad \text{if} \quad \lambda_1 - \lambda_2 \leq 1$$
$$a_2 = 2 \quad \text{if} \quad \lambda_1 - \lambda_2 \geq 4$$

Dual methods

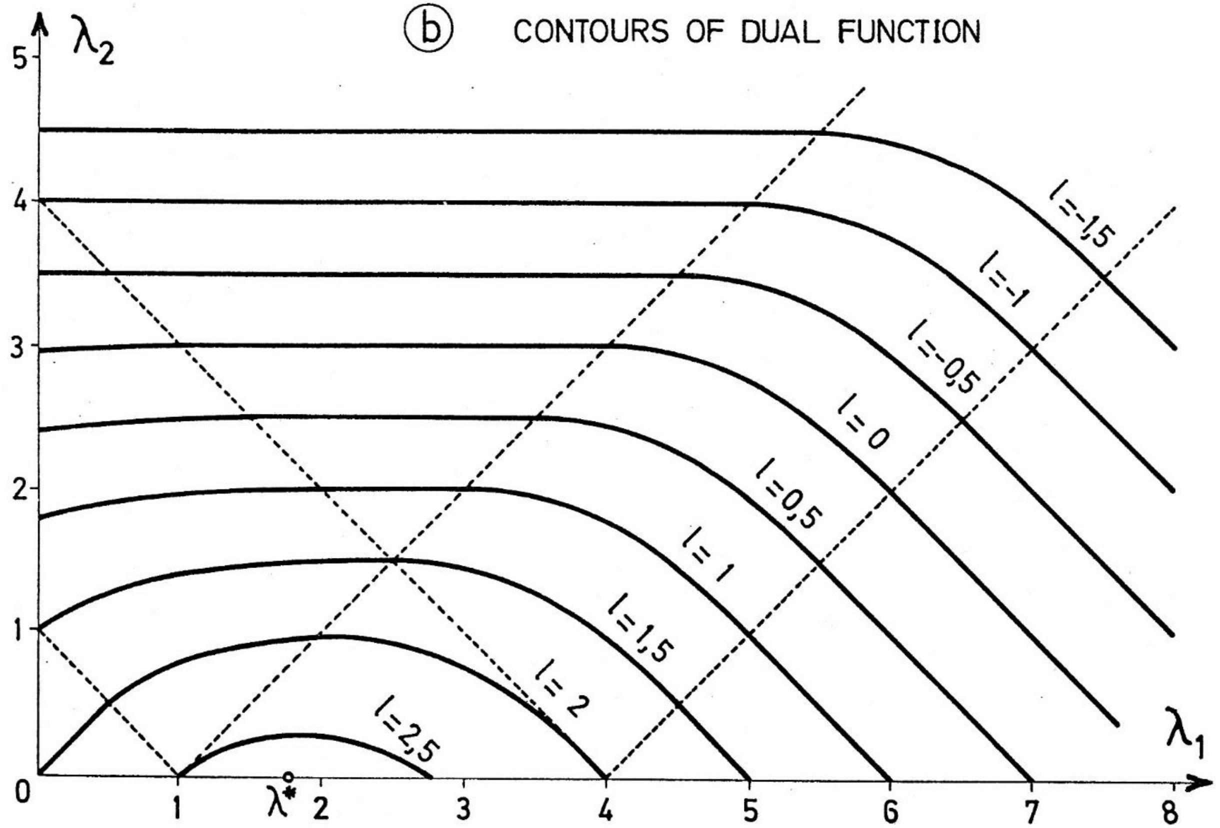
- Explicit dual function for 2-bar truss

définition domain	primal variables $a_1(\lambda_1, \lambda_2)$	$a_2(\lambda_1, \lambda_2)$	dual function $\ell(\lambda_1, \lambda_2)$
I	1	1	$2 + \frac{\lambda_1 - \lambda_2}{2}$
II	$\sqrt{\lambda_1 + \lambda_2}$	$\sqrt{\lambda_1 - \lambda_2}$	$2(\sqrt{\lambda_1 + \lambda_2} + \sqrt{\lambda_1 - \lambda_2}) - \frac{3\lambda_1 + \lambda_2}{2}$
III	2	2	$4 - \frac{\lambda_1 + \lambda_2}{2}$
IV	$\sqrt{\lambda_1 + \lambda_2}$	1	$1 + 2\sqrt{\lambda_1 + \lambda_2} - \frac{\lambda_1 + 3\lambda_2}{2}$
V	2	$\sqrt{\lambda_1 - \lambda_2}$	$2 + 2\sqrt{\lambda_1 - \lambda_2} - \lambda_1$
VI	2	1	$3 - \lambda_2$

Dual methods



Dual methods

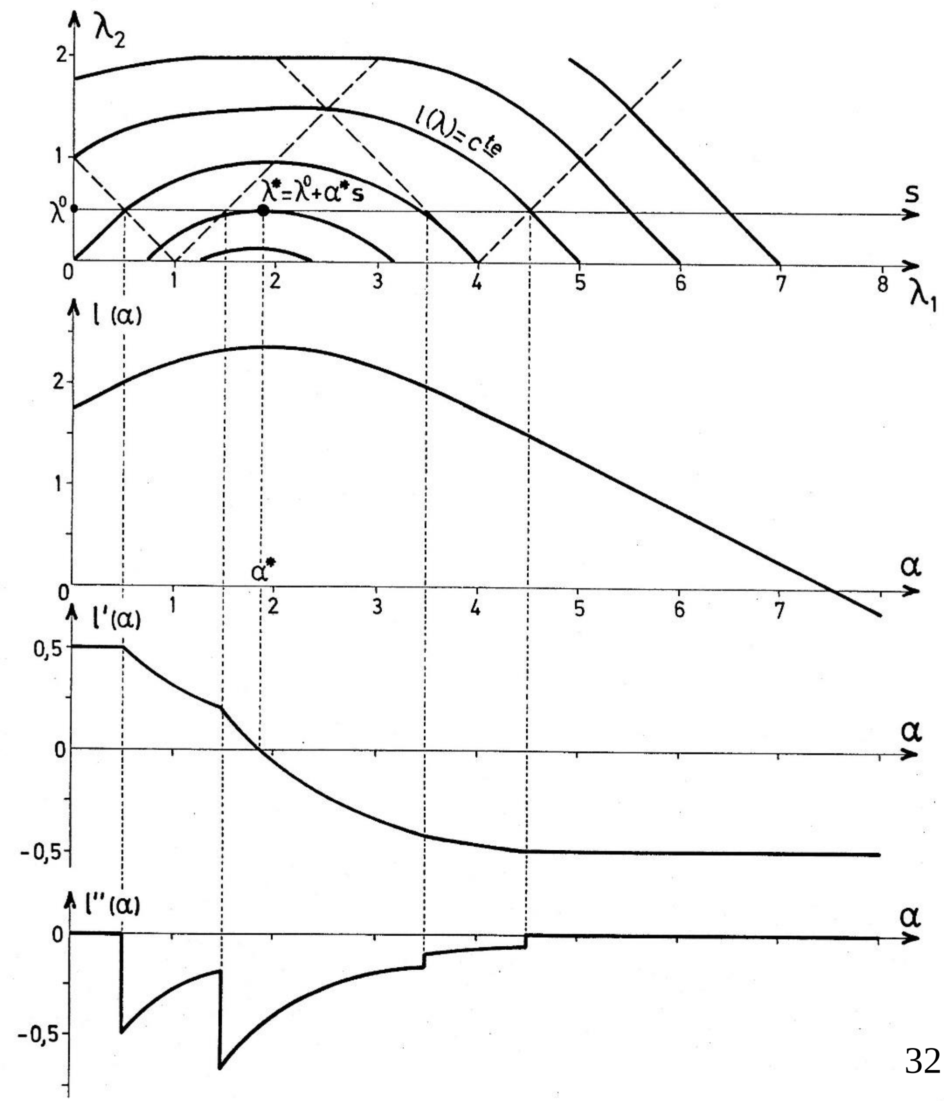


Dual methods

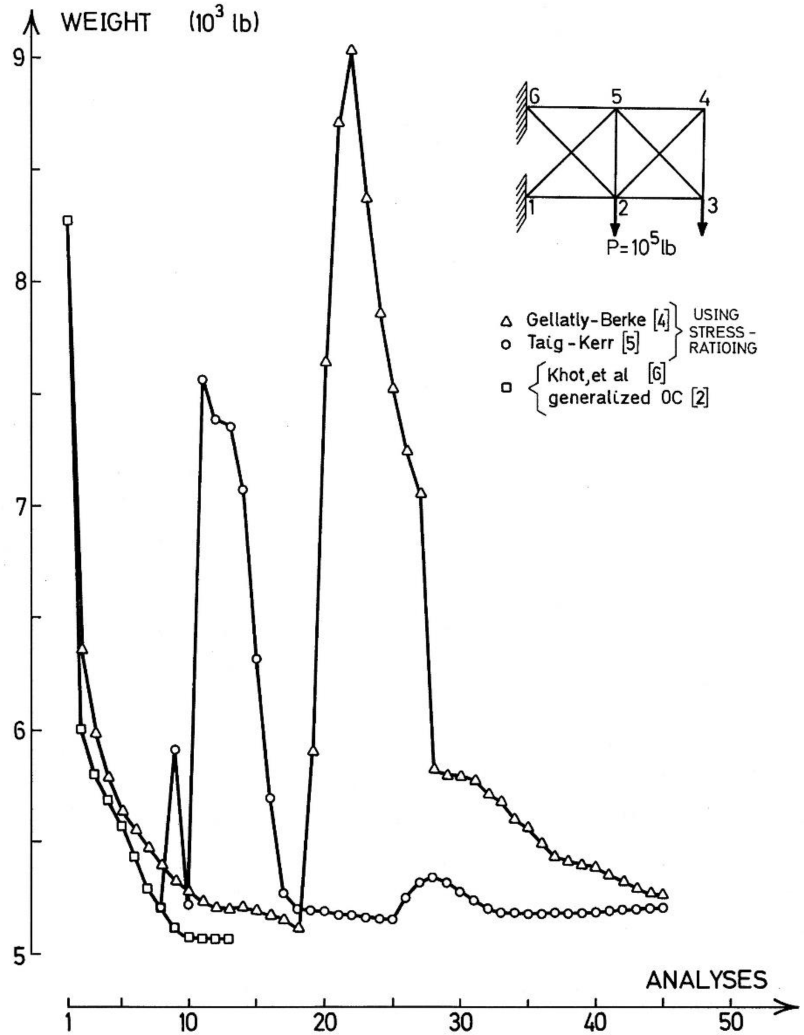
- Line search

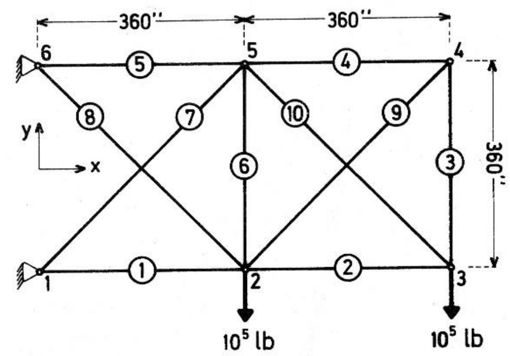
$$\lambda^{+} = \lambda^{0} + \alpha\, s$$

$$\max_{\alpha} \ \ell(\lambda(\alpha))$$



Dual methods





materiau	:	aluminium
tension maximale admissible	:	25 000 psi
module d'elasticite	:	10^7 psi
masse specifique	:	0.1 lb/in^3
section minimale admissible	:	0.1 in^2
deplacement maximal admissible	:	2.0 in
mise en charge	:	unique

10-bar truss – Conventional and Generalized OC

CONCLUSION

CONCLUSION

- Sequence of explicit subproblems
  - First Order Approximations

- Efficient solution of optimization problem
  - Primal / dual solution schemes

DUAL

- Generalized of OC
- Computationally economical but convergence instability
- Discrete design variable possible
- Reliable computer implementation
- Dual bound = monitoring

PRIMAL

- Mixed method (OC/PM)
- Control over convergence at a higher cost
- Other objective functions non separable explicit functions
- Sophisticated algorithms