CONLIN & MMA solvers

Pierre DUYSINX
LTAS – Automotive Engineering
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CONLIN METHOD

LAY-OUT

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- DUAL METHOD APPROACH FOR CONLIN SUBPROBLEMS
- SEQUENTIAL QUADRATIC PROGRAMMING IN DUAL SPACE
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THE CONLIN SUBPROBLEMS

- The general numerical optimization problem coming from structural optimization writes

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & c_0(\boldsymbol{x}) \\ \text{s.t.} \quad & c_j(\boldsymbol{x}) \leq \mathbf{0} \qquad j = 1, m \\ & \underline{x}_i \leq x_i \leq \overline{x}_i \quad i = 1, n \end{aligned}$$

- The functions c_0 and c_j are not explicit nor separable functions.

- The original problem can be solved as a sequence of *convex explicit subproblems* having a simple algebraic form.

  ➔ The convex subproblems are created using the CONvex LINearization (CONLIN) approximation method.

THE CONLIN SUBPROBLEMS

- The convex linearization at $\mathbf{x}_0$ writes :

$$c(\boldsymbol{x}) = c(\boldsymbol{x}_0) + \sum_{+} c_i^0 \, (x_i - x_i^0) - \sum_{-} (x_i^0)^2 \, c_i^0 \left(\frac{1}{x_i} - \frac{1}{x_i^0} \right)$$

- Where $c_i^0 = \left. \dfrac{\partial c}{\partial x_i} \right|_{x_i^0}$

- And Σ_+ and Σ_- denotes respectively the summation over the terms for which c_0^i is positive or negative.

THE CONLIN SUBPROBLEMS

- Normalization of the design variables

$$\tilde{x}_i = \frac{x_i}{x_i^0} \quad \Rightarrow \quad \tilde{c}_i = c_i^0 \, x_i^0$$

- The factor $(x_0^i)^2$ disappears from the CONLIN expansion, which becomes :

$$c(\tilde{\boldsymbol{x}}) = c(\boldsymbol{x}_0) + \sum_{+} \tilde{c}_i \, (\tilde{x}_i - 1) - \sum_{-} \tilde{c}_i \left(\frac{1}{\tilde{x}_i} - 1 \right)$$

- The subproblem becomes

$$\begin{array}{lll} \min_x & \sum_+ c_{i0} \, (x_i - 1) - \sum_- c_{i0} \left(\frac{1}{x_i} - 1 \right) + c_0(\boldsymbol{x}_0) & \\ \text{s.t.} & \sum_+ c_{ij} \, (x_i - 1) - \sum_- c_{ij} \left(\frac{1}{x_i} - 1 \right) \leq -c_j(\boldsymbol{x}_0) & j = 1, m \\ & \underline{x}_i \leq x_i \leq \overline{x}_i & i = 1, n \end{array}$$

THE CONLIN SUBPROBLEMS

- After normalization, the sub problem can be casted into the explicit sub problem format (in which the tilde symbol has been omitted for the sake of simplicity) :

$$
\begin{array}{lll}
\min_{\boldsymbol{x}} & \sum_{+} c_{i0}\, x_i \;-\; \sum_{-} c_{i0}/x_i \;-\; \bar{c}_0 & \\
\text{s.t.} & \sum_{+} c_{ij}\, x_i \;-\; \sum_{-} c_{ij}/x_i \;\leq\; \bar{c}_j & j = 1, m \\
 & \underline{x}_i \;\leq\; x_i \;\leq\; \overline{x}_i & i = 1, n
\end{array}
$$

- where the c_{ij} denote the normalized first derivatives of the objective and constraint functions values at $\mathbf{x}_0$ and :

$$
\bar{c}_j \;=\; \sum_i |c_{ij}| x_i^0 - c_j(\boldsymbol{x}^0) \qquad j = 0 \ldots m
$$

DUAL METHOD FOR CONLIN SUBPROBLEMS

- The explicit subproblems are convex and separable. Thus they are well suited to be solved using *dual algorithms*.

- Dual problem is a max-min procedure :

$$\begin{aligned} \max_{\lambda} \quad & \ell(\lambda) \\ \text{s.t.} \quad & \lambda_j \geq 0 \end{aligned}$$

- where the dual function ℓ(λ) results from minimizing the Lagrangian function:

$$L(\boldsymbol{x}, \lambda) = \sum_{j=0}^{m} \lambda_j \left(\sum_{+} c_{ij} x_i - \sum_{-} \frac{c_{ij}}{x_i} - \bar{c}_j \right)$$

DUAL METHOD FOR CONLIN SUBPROBLEMS

- Minimization of Lagrangian function over the acceptable primal variables for a given λ:

$$\boldsymbol{x}(\lambda) = \min_{\underline{x}_i \le x_i \le \overline{x}_i} L(\boldsymbol{x}, \lambda)$$

- Because of separability the minimization of the Lagrangian problem takes a very simple and efficient form. The problem can be broken into n separate one-dimensional problems

$$\begin{aligned} \min \quad & L_i(x_i) = a_i x_i + \frac{b_i}{x_i} \\ \text{s.t.} \quad & \underline{x}_i \le x_i \le \overline{x}_i \end{aligned}$$

- where the coefficients a_i and b_i remain non negative

$$a_i = \sum_{+} c_{ij}\lambda_j \ge 0 \qquad b_i = -\sum_{-} c_{ij}\lambda_j \ge 0$$

DUAL METHOD FOR CONLIN SUBPROBLEMS

- The optimality of Lagrangian problem yields the primal-dual relationships :

$$L'(x_i) = a_i - \frac{b_i}{x_i^2} = 0$$

- It comes the fully explicit primal-dual relationships **x** = **x**(λ)

$$x_i = (b_i/a_i)^{1/2} \quad \text{if} \quad \underline{x}_i^2 \leq b_i/a_i \leq \bar{x}_i^2$$

$$x_i = \bar{x}_i \quad \text{if} \quad b_i/a_i \leq \underline{x}_i^2$$

$$x_i = \overline{x}_i \quad \text{if} \quad \overline{x}_i^2 \leq b_i/a_i$$

DUAL METHOD FOR CONLIN SUBPROBLEMS

- The dual problem can be expressed in closed form :

$$\begin{aligned} \max_{\lambda} \quad & \ell(\lambda) = \sum_{j=0}^{m} \lambda_j \left[\sum_{+} c_{ij}\, x_i(\lambda) - \sum_{-} \frac{c_{ij}}{x_i(\lambda)} - \bar{c}_j \right] \\ \text{s.t.} \quad & \lambda_j \geq 0 \end{aligned}$$

- A fundamental property of the dual function is that its first derivatives are simply given by the primal constraint values :

$$g_j = \frac{\partial \ell}{\partial \lambda_j} = \sum_{+} c_{ij} x_i(\lambda) - \sum_{-} \frac{c_{ij}}{x_i(\lambda)} - \bar{c}_j$$

DUAL METHOD FOR CONLIN SUBPROBLEMS

- The second derivatives of the dual function

$$H_{jk} = \frac{\partial^2 \ell}{\partial \lambda_j \partial \lambda_k}$$

- can also be written in closed form. Deriving a second time the first derivatives, it comes :

$$H_{jk} = \sum_{+} c_{ij} \frac{\partial x_i(\lambda)}{\partial \lambda_k} + \sum_{-} \frac{c_{ij}}{x_i^2(\lambda)} \frac{\partial x_i(\lambda)}{\partial \lambda_k}$$

- Differentiating the primal-dual relationships it comes for the free variables. :

$$\frac{\partial x_i(\lambda)}{\partial \lambda_k} = -\frac{c_{ik} x_i}{2a_i^2} \quad \text{if} \quad c_{ik} > 0$$

$$\frac{\partial x_i(\lambda)}{\partial \lambda_k} = -\frac{c_{ik}}{2x_i a_i^2} \quad \text{if} \quad c_{ik} < 0$$

- For the fixed variables these derivatives are obviously zero.

DUAL METHOD FOR CONLIN SUBPROBLEMS

- Therefore, the second order derivatives writes

$$H_{jk} = -\frac{1}{2} \sum_{i \in I} n_{ij} n_{ik} \frac{x_i}{a_i}$$

- Where

$$n_{ij} = c_{ij} \quad \text{if} \quad c_{ij} > 0$$

$$n_{ij} = \frac{c_{ij}}{x_i^2} \quad \text{if} \quad c_{ik} < 0$$

- And the summation is restricted to the set of active variables only

$$I = \{ i \,|\, \underline{x}_i \leq x_i \leq \overline{x}_i \}$$

SEQUENTIAL QUADRATIC PROGRAMMING IN DUAL SPACE

- Hessian matrix in dual space :

$$H_{jk} = -\frac{1}{2} \sum_{i \in I} n_{ij} n_{ik} \frac{x_i}{a_i}$$

- It is important to emphasize that the summation is restricted to the set I of free primal variables.

$$I = \{ i \mid \underline{x}_i \leq x_i \leq \overline{x}_i \}$$

- It comes that :
  - The second derivatives of the dual function are discontinuous
  - There is an inherent and fundamental difficulty in using Newton type methods for solving the dual problem

SEQUENTIAL QUADRATIC PROGRAMMING IN DUAL SPACE

- **Generalized Newton method in dual space**
- The update scheme of dual variables writes :

$$\lambda_j^{+} = \lambda_j + \alpha \, s_j$$

- where the Newton search direction is

$$\boldsymbol{s} = -[\boldsymbol{H}]^{-1}\boldsymbol{g}$$

- and α is the step length along the search direction ***s***.

SEQUENTIAL QUADRATIC PROGRAMMING IN DUAL SPACE

- **Quadratic subproblems**

- The search direction **s** is also the solution of the optimization problem :

$$\max \quad \frac{1}{2}\boldsymbol{s}^T\boldsymbol{H}\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{g}$$

- By selecting a unit step length $\alpha = 1$, it comes **s** = $\lambda^+ - \lambda^\circ$, hence the following quadratic problem

$$\begin{aligned} \max \quad & q(\lambda) = \frac{1}{2}\lambda^T\boldsymbol{H}\lambda - \lambda^T\boldsymbol{b} \\ \text{s.t.} \quad & \lambda_j \geq 0 \end{aligned}$$

- where

$$\boldsymbol{b} = \boldsymbol{H}\lambda_0 - \boldsymbol{g}$$

CONSTRAINT RELAXATION

- What do we do if the approximate feasible design domain is empty ?
- We have to introduce an additional design variable δ. The relaxed explicit subproblem writes :

$$
\begin{array}{lll}
\min_{x,\delta} & \sum_{+} c_{i0}^{0}\,(x_i - 1) \; - \; \sum_{-} c_{i0}^{0} \left(\frac{1}{x_i} - 1\right) \; + \; w\, z_0 \delta & \\
\text{s.t.} & \sum_{+} c_{ij}^{0}\,(x_i - 1) \; - \; \sum_{-} c_{ij}^{0} \left(\frac{1}{x_i} - 1\right) \; \leq \; \bar{c}_j \; + \; z_j\,(1 - \frac{1}{\delta}) & j = 1, m \\
 & \underline{x}_i \; \leq \; x_i \; \leq \; \overline{x}_i & i = 1, n \\
 & 1 \; \leq \; \delta &
\end{array}
$$

- where w is a user-supplied weighting factor and

$$
z_j \; = \; \sum_i |c_{ij}|\, x_i^0 \quad j = 0 \ldots m
$$

CONSTRAINT RELAXATION

- The factors w and z_j represent increments to the functions $c_j(\mathbf{x})$ opening the feasible design space if necessary.

- If the relaxation variables δ hits its lower bound δ = 1 nothing is changed in the problem statement.
- If the algorithms finds a value δ greater than unity the approximate feasible domain is artificially enlarged.

CONSTRAINT RELAXATION

- The Lagrangian problem related to the relaxation variable δ :

$$\min_{\delta \geq 1} \quad w z_0 \delta \; - \; (1 - \frac{1}{\delta}) \sum \lambda_j z_j$$

- It comes that δ is given in terms of the dual variables by the relations :

$$\delta = (\sum \lambda_j z_j / w z_0)^{1/2} \quad \text{if} \quad \sum \lambda_j z_j > w z_0$$

$$\delta = 1 \quad \text{if} \quad \sum \lambda_j z_j \leq w z_0$$

CONSTRAINT RELAXATION

- New dual gradient vector and Hessian matrix are :

$$g_j = \sum_{+} c_{ij} x_i(\lambda) - \sum_{-} \frac{c_{ij}}{x_i(\lambda)} - \bar{c}_j - z_j(1 - \frac{1}{\delta})$$

$$H_{jk} = -\frac{1}{2} \sum_{i \in I} n_{ij} n_{ik} \frac{x_i}{a_i} - \frac{1}{2} \frac{z_j z_k}{w \delta^3}$$

OPTIMIZATION ALGORITHM

- Step 1 : Initialization : the set M of active constraints

$$M \;=\; \{j \mid c_j \text{ is potentially active}\}$$

- and define the set N of active variables

$$N \;=\; \{i \mid x_i \text{ is imposed to be free}\}$$

- Step 2: Compute the free primal variables in subspace *N* using

$$x_i(\lambda) = (b_i/a_i)^{1/2} \quad i \in N$$

- Compute the relation factor *δ* using

$$\delta = \left(\sum \lambda_j z_j / w z_0\right)^{1/2} \quad \text{if} \quad \sum \lambda_j z_j > w z_0$$

$$\delta = 1 \qquad \text{if} \quad \sum \lambda_j z_j \leq w z_0$$

OPTIMIZATION ALGORITHM

- Step 3: Evaluate the dual gradient vector in subspace M

$$\frac{\partial \ell}{\partial \lambda_j} = g_j(\boldsymbol{x}(\lambda)) \quad j \in M$$

- Step 4: Check optimality in subspace *N* if

$$g_j = 0 \text{ for } \lambda_j > 0$$
$$g_j < 0 \text{ for } \lambda_j = 0$$

Then then goto step 7

- Step 5: Evaluate the dual Hessian matrix in the subspace M

$$H_{jk}(\boldsymbol{x}) \quad j, k \in M$$

OPTIMIZATION ALGORITHM

- Step 6: Solve the quadratic subproblem (8.21) and return to step 2.

$$\begin{aligned} \max \quad & q(\lambda) = \frac{1}{2}\lambda^T \boldsymbol{H}\lambda - \lambda^T \boldsymbol{b} \\ \text{s.t.} \quad & \lambda_j \geq 0 \end{aligned}$$

- Step 7: Update of the set N

– For $i \in N$ if $x_i < \underline{x}_i$ or $x_i > \bar{x}_i$ then remove i from the set N.

- For $i \notin N$, evaluate $x_i = \sqrt{b_i / a_i}$

If $\underline{x}_i < x_i < \bar{x}_i$ then add i to the set N.

If the set N has been modified, then go back to step 2

OPTIMIZATION ALGORITHM

- Step 8: Update the set M

– For $j \in M$, if $\lambda_j = 0$ then remove j from the set M.

– For $j \notin M$, evaluate the primal constraint values g_j using eq. (8.17). If $g_j \leq 0 \ \forall j \in M$, then go to step 9.

Otherwise add one or more active constraints to the set M.

- Step 9: The maximum of the dual function has been obtained :

– Lagrange multipliers are λ^* for $j \in M$. (other ones being zero.

– Primal design variables x^*_i for $i \in N$. (other ones are fixed).

EXAMPLE: CONLIN APPOXIMATION

- Let's consider, the optimization problem :

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1 + 4x_2 \\ \text{s.t.} \quad & x_2 - x_1 \geq 0 \\ & 3x_1 - 2x_2 \geq 1 \end{aligned}$$

- We consider the following starting point $\mathbf{x} = (3, 4)^T$ around which we have to perform the convex linearization process.

- First of all the problem is written under standard form :

$$\begin{aligned} \min_{x_1,x_2} \quad & c_0(\boldsymbol{x}) = x_1 + 4x_2 \\ \text{s.t.} \quad & c_1(\boldsymbol{x}) \equiv x_1 - x_2 \leq 0 \\ & c_2(\boldsymbol{x}) \equiv -3x_1 + 2x_2 \leq -1 \end{aligned}$$

EXAMPLE: CONLIN APPOXIMATION

- CONLIN approximation of the objective function.

$$c_0(x_1, x_2) = x_1 + 4x_2$$

- This linear function has all positive coefficient and so positive derivatives.

$$\frac{\partial c_0}{\partial x_1} = 1 > 0 \qquad \text{and} \qquad \frac{\partial c_0}{\partial x_2} = 4 > 0$$

- Therefore the linear expansion is used for both variables and the objective function remains unchanged.

$$\tilde{c}_0(x_1, x_2) = x_1 + 4x_2$$

EXAMPLE: CONLIN APPOXIMATION

- The first constraint

$$c_1(x_1, x_2) \ = \ x_1 - x_2 \leq 0$$

- has derivatives with different signs :

$$\frac{\partial c_1}{\partial x_1} = 1 > 0 \qquad \text{and} \qquad \frac{\partial c_2}{\partial x_2} = -1 < 0$$

- The CONLIN linearization then proceeds to a direct variable expansion for x_1 and a reciprocal variable expansion in x_2.

$$\begin{aligned} \tilde{c}_1 &= c_1(\boldsymbol{x}^0) + \frac{\partial c_1}{\partial x_1}(x_1 - x_1^0) + \frac{\partial c_1}{\partial x_2}(-1)(x_2^0)^2 \left(\frac{1}{x_2} - \frac{1}{x_2^0} \right) \leq 0 \\ &= (-1) + (+1)(x_1 - 3) + (-1)(-16) \left(\frac{1}{x_2} - \frac{1}{4} \right) \leq 0 \end{aligned}$$

- It comes

$$\boxed{\tilde{c}_1 \ = \ x_1 + \frac{16}{x_2} \leq 8}$$

EXAMPLE: CONLIN APPOXIMATION

- For the second constraint,

$$c_2(x_1, x_2) \; = \; -3x_1 + 2x_2 \leq -1$$

- The first derivative is negative while the second one is positive so that we have a reciprocal variable expansion for x_1 and a direct expansion for x_2 :

$$\frac{\partial c_2}{\partial x_1} = -3 < 0 \qquad \text{and} \qquad \frac{\partial c_2}{\partial x_2} = 2 > 0$$

- The CONLIN approximation writes

$$\begin{aligned} \tilde{c}_2 &= c_2(\boldsymbol{x}^0) + \frac{\partial c_2}{\partial x_1}(-1)(x_1^0)^2 \left(\frac{1}{x_1} - \frac{1}{x_1^0} \right) + \frac{\partial c_2}{\partial x_2}(x_2 - x_2^0) \leq -1 \\ &= (-1) + (-3)(-9) \left(\frac{1}{x_1} - \frac{1}{3} \right) + (2)(x_2 - 4) \leq -1 \end{aligned}$$

- And finally

$$\boxed{\tilde{c}_2 \; = \; \frac{27}{x_1} x_1 + 2x_2 \leq 17}$$

EXAMPLE: DUAL SOLUTION

- The primal CONLIN problem is

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + \frac{16}{x_2} \leq 8 \\ & \frac{27}{x_1} + 2x_2 \leq 17 \end{aligned}$$

- As the subproblem is convex and separable, it can be solved using duality approach

EXAMPLE: DUAL SOLUTION

- The Lagrange function writes

$$L(\boldsymbol{x}, \lambda) \;=\; x_1 + 4x_2 + \lambda_1(x_1 + \frac{16}{x_2} - 8) + \lambda_2(\frac{27}{x_1} + 2x_2 - 17)$$

- The primal dual relationships are derived from the minimum of the Lagrange function with respect to x_i variables for given dual variables λ_j .

$$\min_{\boldsymbol{x}} L(\boldsymbol{x}, \lambda)$$

- The optimality conditions gives

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 1 + \lambda_1 - \frac{27\lambda_2}{x_1^2} = 0 \\ \frac{\partial L}{\partial x_2} &= 4 - \frac{16\lambda_1}{x_2^2} + 2\lambda_2 = 0 \end{aligned}$$

EXAMPLE: DUAL SOLUTION

- They yield the primal dual relationships

$$x_1 = \left(\frac{27\lambda_2}{1+\lambda_1}\right)^{1/2}$$

$$x_2 = \left(\frac{8\lambda_1}{2+\lambda_2}\right)^{1/2}$$

EXAMPLE: DUAL SOLUTION

- The dual maximization problem then writes

$$
\begin{aligned}
\max_{\lambda} \quad & \ell(\lambda) = \left(\frac{27\lambda_2}{1+\lambda_1}\right)^{1/2} + 4\left(\frac{8\lambda_1}{2+\lambda_2}\right)^{1/2} \\
& \qquad +\lambda_1\left[\left(\frac{27\lambda_2}{1+\lambda_1}\right)^{1/2} + 16\left(\frac{2+\lambda_2}{8\lambda_1}\right)^{1/2} - 8\right] \\
& \qquad +\lambda_2\left[27\left(\frac{1+\lambda_1}{27\lambda_2}\right)^{1/2} + 2\left(\frac{8\lambda_1}{2+\lambda_2}\right)^{1/2} - 17\right] \\
s.t. \quad & \lambda_1,\ \lambda_2\ \geq 0
\end{aligned}
$$

EXAMPLE: DUAL SOLUTION

- It can be easily verified that

$$\frac{\partial \ell}{\partial \lambda_1} = \left(\frac{27\lambda_2}{1+\lambda_1}\right)^{1/2} + 16\left(\frac{2+\lambda_2}{8\lambda_1}\right)^{1/2} - 8$$

$$\frac{\partial \ell}{\partial \lambda_2} = 27\left(\frac{1+\lambda_1}{27\lambda_2}\right)^{1/2} + 2\left(\frac{8\lambda_1}{2+\lambda_2}\right)^{1/2} - 17$$

MMA METHOD

Dual method for MMA sub-problems

- MMA subproblems (after normalization):

$$\min_{\boldsymbol{x}} \quad \sum_{i=1}^{n} \frac{p_{i0}}{U_{ij} - x_i} + \sum_{i=1}^{n} \frac{q_{i0}}{x_i - L_{ij}}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \frac{p_{ij}}{U_{ij} - x_i} + \sum_{i=1}^{n} \frac{q_{ij}}{x_i - L_{ij}} \le d_j \quad j = 1 \dots m$$

$$\underline{x}_i \le x_i \le \overline{x}_i \quad i = 1 \dots n$$

Remark: asymptotes U_{ij} and L_{ij} can depend on both variable index i and constraint index j

- Lagrange function (where we have introduced λ_0=1 for simplicity)

$$L(\boldsymbol{x}, \lambda) = \sum_{j=0}^{m} \lambda_j \left(\sum_{i=1}^{n} \frac{p_{ij}}{U_{ij} - x_i} + \sum_{i=1}^{n} \frac{q_{ij}}{x_i - L_{ij}} - d_j \right)$$

Dual method for MMA sub-problems

- The Lagrangian problem:

$$\min_{\underline{x}_i \le x_i \le \overline{x}_i} \quad L(\boldsymbol{x}, \lambda)$$

- Because of separability, the n-dimensional problem can be split into n 1-dimensional problems:

$$\min_{\underline{x}_i \le x_i \le \overline{x}_i} \quad L_i(x_i, \lambda) = \sum_{j=0}^{m} \frac{\lambda_j \, p_{ij}}{U_{ij} - x_i} + \sum_{j=0}^{m} \frac{\lambda_j \, q_{ij}}{x_i - L_{ij}}$$

Dual method for MMA sub-problems

- For pure MMA problems, asymptotes depend only on variable index i, i.e. $U_{ij} = U_i$ and $L_{ij} = L_i$, so that Lagrangian problem can be solved in closed form from optimality conditions:

$$\frac{\sum_{j=0}^{m} \lambda_j \, p_{ij}}{(U_i - x_i)^2} - \frac{\sum_{j=0}^{m} \lambda_j \, q_{ij}}{(x_i - L_i)^2} = 0$$

- Primal-dual relationships for pure MMA:

$$x_i^\star(\lambda) = \frac{U_i + \eta L_i}{\eta + 1} \quad \text{with} \quad \eta = \sqrt{\frac{\sum_{j=0}^{m} \lambda_j \, p_{ij}}{\sum_{j=0}^{m} \lambda_j \, q_{ij}}}$$

Dual method for MMA sub-problems

- For generalized MMA family, each constraint has its own set of asymptotes, and it is not possible anymore to find the closed form solution of Lagrangian problem. Solution is obtained by resorting to an iterative Newton scheme applied to optimality conditions

$$x_i(\lambda^+) = x_i(\lambda) - \frac{\partial L_i / \partial x_i}{\partial^2 L_i / \partial x_i^2}$$

with

$$\frac{\partial L_i}{\partial x_i} = \sum_{j=0}^{m} \frac{\lambda_j \, p_{ij}}{(U_{ij} - x_i)^2} - \sum_{j=0}^{m} \frac{\lambda_j \, q_{ij}}{(x_i - L_{ij})^2}$$

$$\frac{\partial^2 L_i}{\partial x_i^2} = 2 \sum_{j=0}^{m} \frac{\lambda_j \, p_{ij}}{(U_{ij} - x_i)^3} - 2 \sum_{j=0}^{m} \frac{\lambda_j \, q_{ij}}{(x_i - L_{ij})^3}$$

Dual method for MMA sub-problems

- Primal-dual relations with treatment of side-constraints:

$$x_i(\lambda) = x_i^\star \quad \text{if} \quad \underline{x}_i \leq x_i^\star \leq \overline{x}_i,$$
$$x_i(\lambda) = \underline{x}_i \quad \text{if} \quad x_i^\star \leq \underline{x}_i,$$
$$x_i(\lambda) = \overline{x}_i \quad \text{if} \quad x_i^\star \geq \overline{x}_i.$$

- Dual function, gradient of dual function:
  - calculate x = x(λ)
  - compute f(x(λ)) and g(x(λ))
  - insert the calculated values in L(x,λ) = l(λ)
  - gradient is just given by g(x(λ)).
- If sub-problem is too complicated, sub-problem solution is itself decomposed into a sequence of quadratic separable sub-sub-problems that can be efficiently solved by dual method.

Annex 1: GCMMA code in Matlab by Svanberg

GCMMA code in Matlab by Svanberg

- For academic purpose, Svanberg wrote a Matlab version of its GCMMA solver: i.e. Globally convergent approximations + solver (IP method)

- GCMMA : mmasub function in Matlab

```
function [xmma,ymma,zmma,lam,xsi,eta,mu,zet,s,low,upp] = ...
mmasub(m,n,iter,xval,xmin,xmax,xold1,xold2, ...
f0val,df0dx,df0dx2,fval,dfdx,dfdx2,low,upp,a0,a,c,d);
```

- Solver : subsolv function (called from mmasub)

```
function [xmma,ymma,zmma,lamma,xsimma,etamma,mumma,zetmma,
smma] = ...
subsolv(m,n,epsimin,low,upp,alfa,beta,p0,q0,P,Q,a0,a,b,c,d);
```

GCMMA code in Matlab by Svanberg

- Solve canonical problems of the form :

$$
\begin{array}{lll}
\min & f_0(\mathbf{x}) + z + \sum_{j=1}^{m}(c_j y_j + \frac{1}{2} d_j y_j^2) & \\
\text{s.t.:} & f_j(\mathbf{x}) - a_j z - y_j \le 0 & j = 1 \ldots m \\
 & \underline{x}_i \;\le\; x_i \;\le \overline{x}_i & i = 1 \ldots n \\
 & y_j \ge 0 & j = 1 \ldots m \\
 & z \ge 0 &
\end{array}
$$

- With
  - f_0, f_1… f_m real functions, continuous and differentiable
  - a_i, c_i, d_i real non negative numbers, with $a_i+d_i>0$

GCMMA code in Matlab by Svanberg

- Classic problems of non-linear programming

$$
\begin{array}{lll}
\min & f_0(\mathbf{x}) & \\
\text{s.t.:} & f_j(\mathbf{x}) \le 0 & j = 1 \ldots m \\
 & \underline{x}_i \;\le\; x_i \;\le \overline{x}_i & i = 1 \ldots n
\end{array}
$$

- Parameter choice:

$$
\begin{array}{ll}
a_j = 0 & j = 1 \ldots m \\
d_j = 0 & j = 1 \ldots m \\
c_j = \text{big} & j = 1 \ldots m
\end{array}
$$

GCMMA code in Matlab by Svanberg

- Least square problem

$$
\begin{array}{lll}
\min & \sum_{k=1}^{p} (h_k(\mathbf{x}))^2 & \\
\text{s.t.:} & g_j(\mathbf{x}) \leq 0 & j = 1 \dots q \\
 & \underline{x}_i \ \leq \ x_i \ \leq \overline{x}_i & i = 1 \dots n
\end{array}
$$

- Play with the quadratic term in y_j and the coefficient d_j
- Parameter choice:

$$
\begin{array}{rlrl}
m = 2p + q & & a_j = 0 & j = 1 \dots m \\
f_0(\mathbf{x}) = 0 & & d_k = 2 & k = 1 \dots 2p \\
f_k(\mathbf{x}) = h_k(\mathbf{x}) & k = 1 \dots p & d_{2p+j} = 0 & j = 1 \dots q \\
f_{p+k}(\mathbf{x}) = -h_k(\mathbf{x}) & k = 1 \dots p & c_k = 0 & k = 1 \dots 2p \\
f_{2p+j}(\mathbf{x}) = g_j(\mathbf{x}) & j = 1 \dots q & c_{2p+j} = \text{big} & j = 1 \dots q
\end{array}
$$

GCMMA code in Matlab by Svanberg

- L_1 minimization problem

$$
\begin{array}{lll}
\min & \sum_{k=1}^{p} |h_k(\mathbf{x})| & \\
\text{s.t.:} & g_j(\mathbf{x}) \leq 0 & j = 1 \dots q \\
 & \underline{x}_i \leq x_i \leq \overline{x}_i & i = 1 \dots n
\end{array}
$$

- Play with the linear terms in y_j and the coefficient c_j
- Parameter choice:

$$
\begin{array}{rlrl}
m = 2p + q & & & \\
f_0(\mathbf{x}) = 0 & & a_j = 0 & j = 1 \dots m \\
f_k(\mathbf{x}) = h_k(\mathbf{x}) & k = 1 \dots p & d_j = 0 & j = 1 \dots m \\
f_{p+k}(\mathbf{x}) = -h_k(\mathbf{x}) & k = 1 \dots p & c_k = 1 & k = 1 \dots 2p \\
f_{2p+j}(\mathbf{x}) = g_j(\mathbf{x}) & j = 1 \dots q & c_{2p+j} = \text{big} & j = 1 \dots q
\end{array}
$$

GCMMA code in Matlab by Svanberg

- L_∞ minimization problems or min max problems

$$
\begin{array}{lll}
\min & \max_{k=1\ldots p} |h_k(\mathbf{x})| & \\
\text{s.t.:} & g_j(\mathbf{x}) \le 0 & j = 1\ldots q \\
 & \underline{x}_i \;\le\; x_i \;\le \overline{x}_i & i = 1\ldots n
\end{array}
$$

- Play with the linear terms in z and the coefficient a_j
- Parameter choice:

$$
\begin{array}{llll}
m = 2p + q & & & \\
f_0(\mathbf{x}) = 0 & & a_k = 1 & k = 1\ldots 2p \\
f_k(\mathbf{x}) = h_k(\mathbf{x}) & k = 1\ldots p & a_{2p+j} = 0 & j = 1\ldots q \\
f_{p+k}(\mathbf{x}) = -h_k(\mathbf{x}) & k = 1\ldots p & d_j = 0 & j = 1\ldots m \\
f_{2p+j}(\mathbf{x}) = g_j(\mathbf{x}) & j = 1\ldots q & c_j = \text{big} & j = 1\ldots m
\end{array}
$$