SENSITIVITY ANALYSIS

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LAY-OUT

- Introduction
- Finite differences
- Discrete sensitivity analysis
  - Linear static analysis
  - Linear vibration analysis
  - Liner buckling
- Continuum approach of sensitivity analysis

INTRODUCTION

- We want to solve structural optimization problems

$$\min_{\mathbf{x}} \quad g_0(\mathbf{x})$$
$$\text{s.t.:} \quad g_j(\mathbf{x}) \leq \bar{g}_j$$
$$\underline{x}_i \leq x_i \leq \overline{x}_i$$

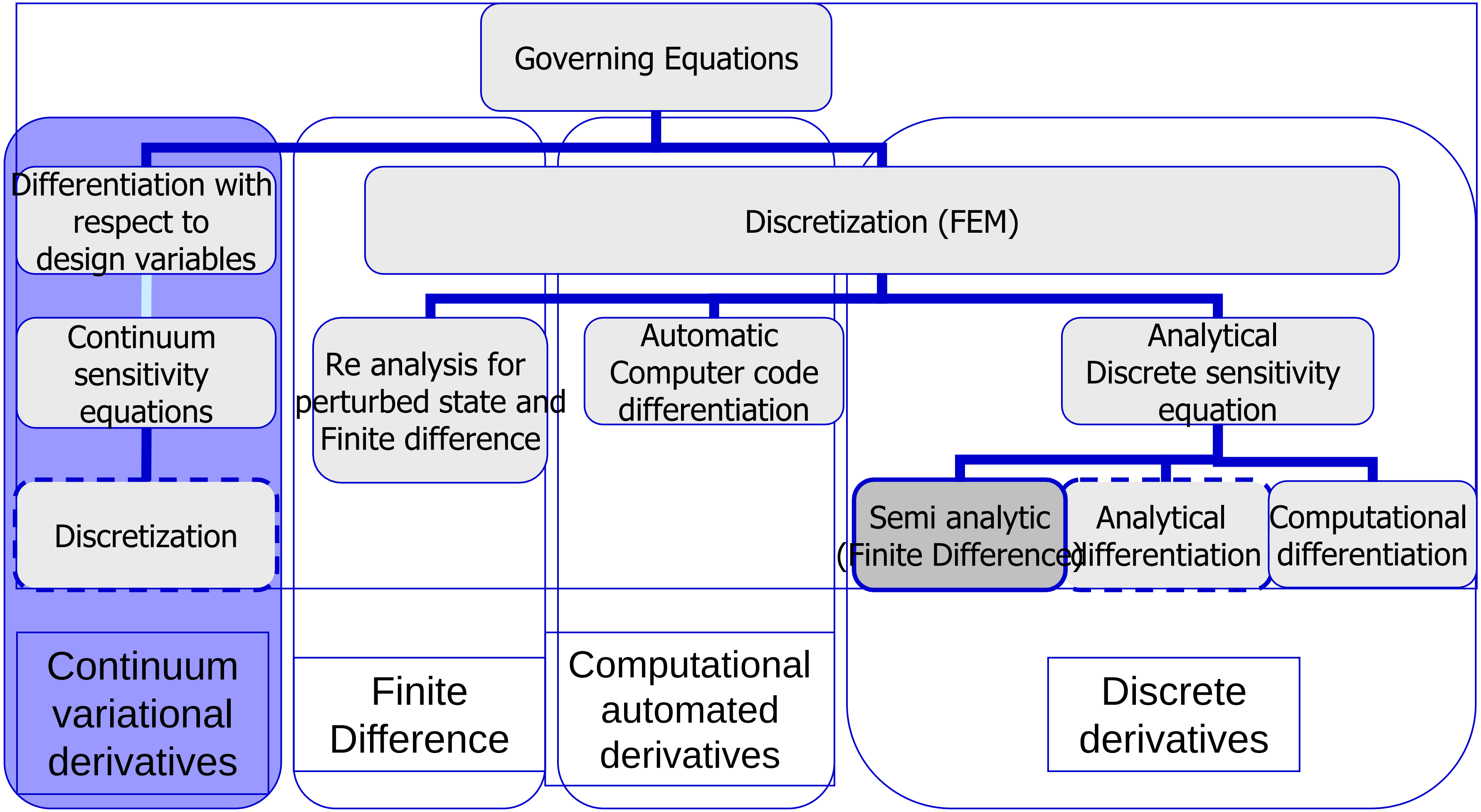
- Solution algorithms & optimality conditions (KKT) require:
  - The values of the objective and constraint functions $g_j(x)$
  - The derivatives of the functions $\frac{\partial g_j}{\partial x_i}$

➔ SENSITIVITY ANALYSIS

INTRODUCTION

- To compute the sensitivities several approaches are possible:
  - Finite differences
  - Differentiate the governing PDE (state) equations and then discretization ➔ Continuum variational derivatives
  - Discretization of the governing PDE equations and then differentiation ➔ Discrete sensitivity analysis
    - Analytic approach
    - Semi Analytic approach
    - Automatic differentiation of computer code

INTRODUCTION



FINITE DIFFERENCE APPROACH

FINITE DIFFERENCE SENSITIVITY

- The simplest approach to compute sensitivities is the first order forward finite difference approximation

$$\frac{dg(\mathbf{x})}{dx_i} \simeq \frac{\Delta g}{\Delta x_i} = \frac{g(\mathbf{x}+\Delta\mathbf{x}^{(i)}) - g(\mathbf{x})}{\Delta x_i}$$

- With the perturbation of the variable x_i:

$$\Delta\mathbf{x}^{(i)} = (0\ldots\Delta x_i\ldots 0)^T$$

- An alternative scheme is the second order central finite difference approximation

$$\frac{dg(\mathbf{x})}{dx_i} \simeq \frac{\Delta g}{\Delta x_i} = \frac{g(\mathbf{x}+\Delta\mathbf{x}^{(i)}) - g(\mathbf{x}-\Delta\mathbf{x}^{(i)})}{2\,\Delta x_i}$$

FINITE DIFFERENCIE SENSITIVITY

- First order (forward and central) difference schemes are the most used in structural and multidisciplinary optimization because of the higher cost associated to higher-order schemes.
- Cost of finite difference schemes
  - Forward difference: n+1 (n=number of design variables)
  - Central difference: 2n+1

- The key to finite difference approximation scheme is the selection of the perturbation step Δx_i
- There are two sources of errors:
  - The truncation error results from the neglected terms in the Taylor expansion
  - The condition error is the difference between the numerical evaluation of the function and its exact value

FINITE DIFFERENCIE SENSITIVITY

- Truncation error can be evaluated by writing the Taylor expansion of the function g:

$$g(\mathbf{x} + \Delta\mathbf{x}^{(i)}) = g(\mathbf{x}) + \frac{dg}{dx_i}\,\Delta x_i + \frac{\Delta x_i^2}{2}\,\frac{d^2}{dx_i^2} g(x + \zeta\Delta x_i) \quad 0 \leq \zeta \leq 1$$

- It comes that the truncation error for the forward difference scheme is:

$$e_T(\Delta x_i) = \frac{\Delta x_i}{2}\,\frac{d^2}{dx_i^2} g(x + \zeta\Delta x_i) \quad 0 \leq \zeta \leq 1$$

- Similarly for the central finite difference scheme, one gets:

$$e_T(\Delta x_i) = \frac{\Delta x_i^2}{6}\,\frac{d^3}{dx_i^3} g(x + \zeta\Delta x_i) \quad 0 \leq \zeta \leq 1$$

FINITE DIFFERENCIE SENSITIVITY

- The condition error results from the numerical error in the evaluation of the function (compared to its exact value).

- One contribution of the condition error is the round-off error in calculating the original and the perturbated values of g.
  - The round-off error is small except if Δx is very small.
  - However when the value of g is computed by lengthy and/or ill-conditioned numerical process, the round-off error can become substantial.
  - The condition error can also grow if calculated by an iterative process that is terminated prematurely.

FINITE DIFFERENCIE SENSITIVITY

- If we have a bound ε_g on the absolute error in the computation of the function g, we can estimate the condition error.

- A very conservative bound on the condition error is:

$$\epsilon_C(\Delta x_i) = \frac{2}{\Delta x_i}\epsilon_g$$

- Dilemma of the selection of the finite difference of the step-size:
  - The truncation error increases with Δx
  - The condition error growths with $1/\Delta x$
- Reduction of Δx reduces the truncation error while increasing the condition error and vice-versa!

FINITE DIFFERENCIE SENSITIVITY

- The total error is the sum of the condition and truncation errors

$$e \simeq \frac{\Delta x_i}{2} |s_b| + \frac{2}{\Delta x_i}\epsilon_g$$

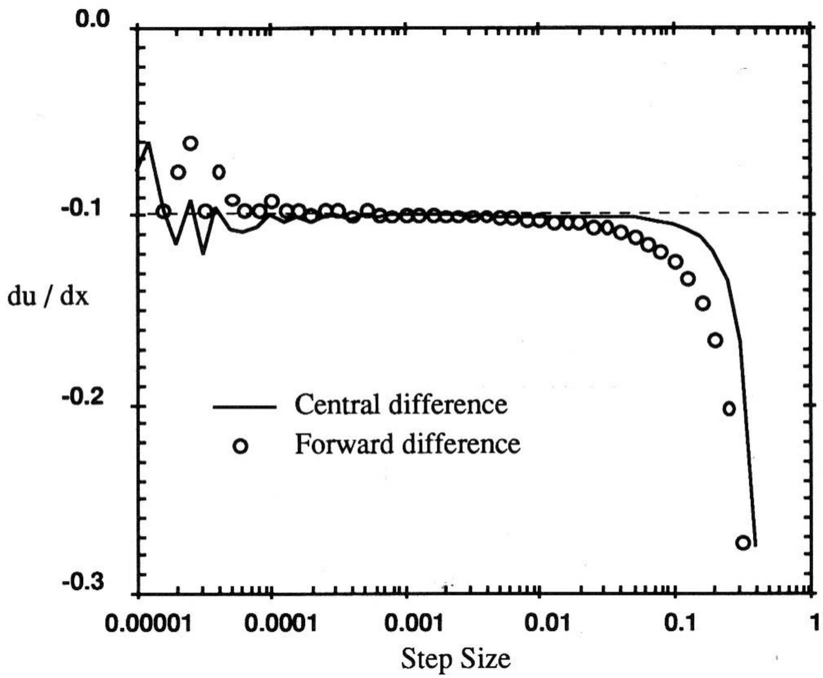
  – Where $|s_b|$ is a bound upon the second order derivatives in the interval $[x, x+\Delta x]$

$$\left| \frac{d^2}{dx_i^2} g(x + \zeta \Delta x_i) \right| \leq |s_b| \quad 0 \leq \zeta \leq 1$$

- If we know the values of $|s_b|$ and ε_g , then we can compute the optimum step-size that minimizes the total error:

$$\Delta x_i^{\text{opt}} = 2\sqrt{\epsilon_g / |s_b|}$$

FINITE DIFFERENCIE SENSITIVITY



Effect of step size on derivative by Haftka and Adelman (1993)

- The figure shows a typical dependence of the finite difference derivative on the step-size.
- For small step-sizes, the round-off error is random in nature
- For large step-sizes, the error varies smoothly. The central difference scheme gives a small advantage by producing accurate derivatives in a slightly larger range of the step-size.

FINITE DIFFERENCE FOR ITERATIVELY SOLVED PROBLEMS

FINITE DIFFERENCE FOR ITERATIVELY SOLVED PROBLEMS

- Condition errors can become important when iterative methods are used to evaluate the response functions.

- Let us consider:

$$f(\mathbf{x}, \mathbf{u}) = 0$$

- The iterative solution process starts from an initial guess u and terminates when the iterant u_Δ is within a given tolerance ε of the exact u.
- Using a forward difference scheme, we perturb one variable by $\Delta x^{(i)}$ and solve again the problem

$$f(\mathbf{x} + \Delta\mathbf{x}^{(i)}, \mathbf{u}_\Delta) = 0$$

FINITE DIFFERENCE FOR ITERATIVELY SOLVED PROBLEMS

- The iterative solution yields an approximation $\tilde{u}_\Delta$ then derivatives du/dx is approximated by

$$\frac{du}{dx_i} \simeq \frac{\tilde{\mathbf{u}}_\Delta - \tilde{\mathbf{u}}}{\Delta x_i}$$

- To start the iterative process for obtaining u_Δ, two initial guesses are obvious:
  - Start from the same guess as for the non perturbated problem. There is a good chance the condition errors will be the same and cancels if the iterative process is monotonic
  - Start from the solution of the iterative ũ. The initial guess is very good and the convergence may be fast, but the condition error is likely to have changed and they do not cancel

FINITE DIFFERENCE FOR ITERATIVELY SOLVED PROBLEMS

- To circumvent the problem, Haftka (1985) suggests a strategy allowing to start the iteration for u_Δ from ũ without excessive condition errors.
- Let's pretend that ũ is the exact solution instead of being the approximation solution. ũ is the exact solution of the (slightly) modified problem:

$$f(\mathbf{x}, \mathbf{u}) - f(\mathbf{x}, \tilde{\mathbf{u}}) = 0$$

- We find the derivative of du/dx by obtaining u_Δ as the solution:

$$f(\mathbf{x} + \Delta\mathbf{x}^{(i)}, \mathbf{u}_\Delta) - f(\mathbf{x}, \tilde{\mathbf{u}}) = 0$$

- Because ũ is the exact solution for $\Delta x=0$, the iterative process reflects only the influence of Δx and we get a good approximation for making the finite differences.

SENSITIVITY OF DISCRETE SYSTEMS

STATIC ANALYSIS

- Study of the derivatives of the structure under linear static analysis when discretized by finite elements.
- The study is carried out for a single load case, but it can be easily extended to multiple load cases.

- Equilibrium equation of the discretized structure:

$$\boxed{\mathbf{K}\,\mathbf{q} = \mathbf{g}}$$

  - **q** generalized displacement of the structure
  - **K** stiffness matrix of the structure discretized into F.E.
  - **g** generalized load vector consistent with the F.E. discretization

STATIC ANALYSIS

- Let **x** be the vector of design variables in number n.

- The differentiation of the equilibrium equation yields the sensitivity of the generalized displacements:

$$\mathbf{K}\frac{\partial \mathbf{q}}{\partial x_i} = \{\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q}\}$$

- The right-hand side term is called pseudo load vector

$$\tilde{\mathbf{g}}_i = \frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q}$$

- Physical interpretation of the pseudo load (Irons): load that is necessary to re-establish the equilibrium when perturbating the design.

STATIC ANALYSIS

- Differentiating one more time, one gets the second order derivatives of the generalized displacements:

$$\mathbf{K}\frac{\partial^2 \mathbf{q}}{\partial x_i \partial x_j} = \{\frac{\partial^2 \mathbf{g}}{\partial x_i \partial x_j} - \frac{\partial^2 \mathbf{K}}{\partial x_i \partial x_j}\mathbf{q} - \frac{\partial \mathbf{K}}{\partial x_i}\frac{\partial \mathbf{q}}{\partial x_j} - \frac{\partial \mathbf{K}}{\partial x_j}\frac{\partial \mathbf{q}}{\partial x_i}\}$$

- Computational effort:
  - The sensitivity of the generalized displacements requires the solution of n additional load cases for the first order sensitivities and n(n+1)/2 for the second order derivatives.

STATIC ANALYSIS

- Derivative of a response function R:

$$R(\mathbf{x}, \mathbf{q}) \leq 0$$

- Deriving the equation of the response function

$$\begin{aligned} \frac{dR(\mathbf{x}, \mathbf{q})}{dx_i} &= \frac{\partial R}{\partial x_i} + \frac{\partial R}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial x_i} \\ &= \frac{\partial R}{\partial x_i} + \mathbf{b}^T \frac{\partial \mathbf{q}}{\partial x_i} \end{aligned}$$

- With $b_i = \dfrac{\partial R}{\partial q_i}$

STATIC ANALYSIS

- Direct approach: consists in evaluating the derivatives of the generalized displacement first

$$\mathbf{K}\frac{\partial \mathbf{q}}{\partial x_i} = \{\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q}\}$$

  and then substituting into the expression of the derivative of the constraint

$$\frac{dR(\mathbf{x},\mathbf{q})}{dx_i} = \frac{\partial R}{\partial x_i} + \mathbf{b}^T\,\mathbf{K}^{-1}\,(\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q})$$

- The direct approach requires solving ‘n’ additional load cases (pseudo loads)

STATIC ANALYSIS

- The virtual load approach is based on the observation that the problem is self adjoin so that $K^{-1} = K^{-T}$ so that one can evaluate the matrix product first

$$\mathbf{b}^T\,\mathbf{K}^{-1}$$

- This is equivalent to solving the adjoin state equation:

$$\mathbf{K}\,\boldsymbol{\Lambda} = \mathbf{b} = \frac{\partial R}{\partial \mathbf{q}}$$

- And then substitute into the expression of the derivative of the constraint

$$\frac{dR(\mathbf{x},\mathbf{q})}{dx_i} = \frac{\partial R}{\partial x_i} + \boldsymbol{\Lambda}^T\,(\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q})$$

- Computational cost: The adjoin method requires one additional load case per constraint

STATIC ANALYSIS

- For the second derivatives of the response function R, one gets:

$$\frac{d^2 R}{dx_i dx_j} = \frac{\partial^2 R}{\partial x_i \partial x_j} + \frac{\partial \mathbf{q}^T}{\partial x_j} \mathbf{B} \frac{\partial \mathbf{q}}{\partial x_i} + \mathbf{b}^T \frac{\partial^2 \mathbf{q}}{\partial x_i \partial x_j}$$

- The matrix R collects the partial second order derivatives of the response with respect to the generalized displacements q.

$$\mathbf{B} = \left[\frac{\partial^2 R}{\partial \mathbf{q} \partial \mathbf{q}} \right]$$

- The direct approach requires computing the first and the second order derivatives of the generalized displacements, that is solving n + n(n+1)/2 additional load cases

STATIC ANALYSIS

- Haftka (1982) showed that it is often more economical to solve the ‘n’ pseudo loads and the ‘m’ virtual loads.

$$\mathbf{K}\frac{\partial \mathbf{q}}{\partial x_i} = \{\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q}\} \qquad \mathbf{K}\,\mathbf{\Lambda} = \mathbf{b} = \frac{\partial R}{\partial \mathbf{q}}$$

- It comes

$$\frac{d^2 R}{dx_i dx_j} = \frac{\partial^2 R}{\partial x_i \partial x_j} + \frac{\partial \mathbf{q}^T}{\partial x_j}\,\mathbf{B}\,\frac{\partial \mathbf{q}}{\partial x_i} + \mathbf{\Lambda}^T \left(\frac{\partial^2 \mathbf{g}}{\partial x_i \partial x_j} - \frac{\partial^2 \mathbf{K}}{\partial x_i \partial x_j}\,\mathbf{q} - \frac{\partial \mathbf{K}}{\partial x_i}\frac{\partial \mathbf{q}}{\partial x_j} - \frac{\partial \mathbf{K}}{\partial x_j}\frac{\partial \mathbf{q}}{\partial x_i} \right)$$

ANALYTICAL APPROACH

- A central issue is the calculation of the derivatives of the stiffness matrix and of the load vector.

- In some cases the structure of the stiffness matrix makes it easy to have the sensitivity of the matrix with respect to the design variable

- For thin walled structures (bars, membranes):

$$\mathbf{K}_e = x_e \, \overline{\mathbf{K}}_e$$

- It comes

$$\frac{\partial \mathbf{K}_e}{\partial x_e} = \overline{\mathbf{K}}_e$$

ANALYTICAL APPROACH

- For bending elements, one can write:

$$\mathbf{K}_e = t_e^3 \, \overline{\mathbf{K}}_e$$

- It comes

$$\frac{\partial \mathbf{K}_e}{\partial t_e} = 3 \, t_e^2 \, \overline{\mathbf{K}}_e$$

- In topology optimization using SIMP model:

$$E = \mu^p \, E^0$$

- The stiffness matrix

$$\mathbf{K}_e = \mu_e^p \, \overline{\mathbf{K}}_e$$

- And its derivatives

$$\frac{\partial \mathbf{K}_e}{\partial \mu_e} = p \, \mu_e^{p-1} \, \overline{\mathbf{K}}_e$$

SEMI ANALYTICAL APPROACH

- However in many cases, it is impossible to exhibit a closed form structure in terms of the design variables, thus one generally resorts to a finite difference to evaluate the derivatives of the stiffness matrix and of the load vectors.

$$\frac{\partial \mathbf{K}}{\partial x_i} \simeq \frac{\mathbf{K}(\mathbf{x}+\Delta\mathbf{x}^{(i)}) - \mathbf{K}(\mathbf{x})}{\Delta x_i}$$

$$\frac{\partial \mathbf{g}}{\partial x_i} \simeq \frac{\mathbf{g}(\mathbf{x}+\Delta\mathbf{x}^{(i)}) - \mathbf{g}(\mathbf{x})}{\Delta x_i}$$

- For the second derivatives, one has also

$$\frac{\partial^2 \mathbf{K}}{\partial x_i^2} \simeq \frac{\mathbf{K}(\mathbf{x}+\Delta\mathbf{x}^{(i)}) + \mathbf{K}(\mathbf{x}-\Delta\mathbf{x}^{(i)}) - 2\,\mathbf{K}(\mathbf{x})}{2\,\Delta x_i^2}$$

SENSITIVITY OF DISPLACEMENTS

- Obviously for displacement constraints, one has the following simple form:

$$u = \mathbf{b}^T \mathbf{q}$$

With $$\mathbf{b}^T = (0 \ldots 1 \ldots 0)$$

- The vector **b** being constant, its derivative is zero and the derivative of the displacement constraint is:

$$\frac{du}{dx_i} = \mathbf{b}^T \frac{\partial \mathbf{q}}{\partial x_i}$$

- The second order derivatives writes also

$$\frac{d^2 u}{dx_i^2} = \mathbf{b}^T \frac{\partial^2 \mathbf{q}}{\partial x_i^2}$$

SENSITIVITY OF DISPLACEMENTS

- The sensitivity writes

$$\frac{du}{dx_i} = \mathbf{b}^T \mathbf{K}^{-1} \left(\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q}\right)$$

- It can be evaluated using the direct approach and computing the derivatives of the generalized displacements

$$\mathbf{K} \frac{\partial \mathbf{q}}{\partial x_i} = \left\{\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q}\right\}$$

- or by using the adjoin approach and solving for the virtual load

$$\mathbf{K} \, \mathbf{\Lambda} = \mathbf{b}$$

SENSITIVITY OF COMPLIANCE

- The compliance is defined as the work of the applied load.

$$C = \mathbf{g}^T \mathbf{q}$$

- It is equal to the twice the deformation energy

$$C = \mathbf{g}^T \mathbf{q} = \mathbf{q}^T \mathbf{K} \mathbf{q}$$

- The derivative of the compliance constraint gives:

$$\frac{dC}{dx_i} = \frac{\partial \mathbf{g}^T}{\partial x_i} \mathbf{q} + \mathbf{g}^T \frac{\partial \mathbf{q}}{\partial x_i}$$

- Introducing the value of the derivatives of the generalized displacements:

$$\mathbf{K} \frac{\partial \mathbf{q}}{\partial x_i} = \left\{ \frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q} \right\}$$

SENSITIVITY OF COMPLIANCE

- The expression of the sensitivity of the compliance writes

$$\frac{dC}{dx_i} = -\mathbf{q}^T \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q} + 2 \frac{\partial \mathbf{g}^T}{\partial x_i} \mathbf{q}$$

- Generally the load vector derivative is zero (case of no body load), it comes:

$$\frac{\partial \mathbf{g}}{\partial x_i} = 0 \qquad \frac{dC}{dx_i} = -\mathbf{q}^T \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q}$$

- In fact, we could have obtained this result also by using the virtual load approach:

$$\mathbf{b} = \frac{\partial C}{\partial \mathbf{q}} = \mathbf{g} \qquad \mathbf{K}\,\mathbf{\Lambda} = \mathbf{b} = \mathbf{g} \qquad \mathbf{\Lambda} = \mathbf{q}$$

SENSITIVITY OF STRESSES

- The stress in the finite element can be written in a vector form

$$\sigma = \mathbf{T}\mathbf{q}$$

where T is the stress matrix of the element.

- The component k of the stresses writes

$$\sigma_k = \mathbf{t}_k^T \, \mathbf{q}$$

where t^T_k is the row k of the stress matrix T .

- The sensitivity of a component k of the stress with respect to a design variable x is given by:

$$\frac{\partial \sigma_k}{\partial x_i} = \frac{\partial \mathbf{t}_k^T}{\partial x_i}\mathbf{q} + \mathbf{t}_k^T \frac{\partial \mathbf{q}}{\partial x_i}$$

SENSITIVITY OF STRESSES

- Introduction the sensitivity the displacement, the sensitivity writes

$$\frac{\partial \sigma_k}{\partial x_i} = \frac{\partial \mathbf{t}_k^T}{\partial x_i}\mathbf{q} + \mathbf{t}_k^T \mathbf{K}^{-1} \left(\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\mathbf{q} \right)$$

- It is clear that in order to evaluate this expression, one has to compute either the <u>pseudo loads</u>

$$\mathbf{K}^{-1}\left(\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\mathbf{q}\right)$$

- Or the <u>virtual load </u>case

$$\mathbf{K}^{-1}\mathbf{t}_k$$

SENSITIVITY OF STRESSES

- For an equivalent stress criterion, one can compute at first the derivatives of each stress component and then apply the chain rule for the derivative. But when the criterion is quadratic like the von Mises equivalent stress, it is more economical to use another approach.

- For plane problems, the stress in the finite element can be written in a vector form $\sigma = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}^T$
- On can write the equivalent von Mises stress as a follow:

$$\sigma_{VM} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2}$$

as a quadratic form using matrix V

$$\sigma_{VM}^2 = \sigma^T \mathbf{V} \sigma \qquad \mathbf{V} = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

SENSITIVITY OF STRESSES

- The equivalent von Mises stress writes as a follow:

$$\begin{aligned}\sigma_{VM}^2 &= \mathbf{q}^T\ \mathbf{T}^T\mathbf{V}\mathbf{T}\mathbf{q} \\ &= \mathbf{q}^T\mathbf{M}\mathbf{q}\end{aligned}$$

with

$$\mathbf{M} = \mathbf{T}^T\mathbf{V}\mathbf{T}$$

- Differentiating the von Mises expression gives

$$2\,\sigma_{VM}\ \frac{\partial \sigma_{VM}}{\partial x_i} = \mathbf{q}^T\frac{\partial \mathbf{M}}{\partial x_i}\mathbf{q} + 2\ \mathbf{q}^T\mathbf{M}\ \frac{\partial \mathbf{q}}{\partial x_i}$$

SENSITIVITY OF STRESSES

- The sensitivity of the von Mises stress can be written as:

$$\frac{\partial \sigma_{VM}}{\partial x_i} = \frac{1}{\sigma_{VM}} \left(\mathbf{q}^T \frac{\partial \mathbf{M}}{\partial x_i} \mathbf{q} + 2\ \mathbf{q}^T \mathbf{M}\ \mathbf{K}^{-1} \left(\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q} \right) \right)$$

- and it comes that this can be evaluated also by solving pseudo load cases for every design variables or by using a virtual load approach.
- The adjoin loads are for this problem of the form

$$\tilde{\mathbf{g}} = \mathbf{M}\mathbf{q}$$

SENSITIVITY OF EIGENVALUES AND EIGENVECTORS

SENSITIVITY OF EIGENVALUE PROBLEMS

- Eigenvalue problem
  - **K** stiffness matrix, **M** mass matrix
  - **q** the eigenmode vector
  - And λ the eigenfrequency $\lambda = \omega^2$

$$\boxed{(\mathbf{K} - \lambda\,\mathbf{M})\,\mathbf{q} = \mathbf{0}}$$

- The magnitude of the modes is arbitrary, so they are normalized according to a given matrix W (generally the mass matrix M)

$$\boxed{\mathbf{q}^T\,\mathbf{W}\,\mathbf{q} = 1}$$

- At first let's consider the simplified approach: we assume that all eigenvalues are distinct and ordered from the smallest to the largest:

$$0 < \lambda^{(1)} < \lambda^{(2)} < \ldots < \lambda^{(N)}$$

$$0\,, \quad \mathbf{q}^{(1)}\,, \quad \mathbf{q}^{(2)}\,, \quad \ldots \quad \mathbf{q}^{(N)}$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- Let's differentiate the eigenvalue equation

$$\left(\mathbf{K} - \lambda^{(k)}\mathbf{M}\right)\frac{\partial \mathbf{q}^{(k)}}{\partial x_i} = \frac{\partial \lambda^{(k)}}{\partial x_i}\mathbf{M}\mathbf{q}^{(k)} - \left(\frac{\partial \mathbf{K}}{\partial x_i} - \lambda^{(k)}\frac{\partial \mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)}$$

- Differentiating the normalization equation gives

$$\mathbf{q}^{(k)\,T}\mathbf{W}\,\frac{\partial \mathbf{q}^{(k)}}{\partial x_i} = -\frac{1}{2}\mathbf{q}^{(k)T}\frac{\partial \mathbf{W}}{\partial x_i}\mathbf{q}^{(k)}$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- To obtain the derivatives of the eigenvalue $\lambda^{(k)}$, one has to premultiply the first equation by the eigenmode $q^{(k)}$

$$\mathbf{q}^{(k)\,T}\left[\left(\mathbf{K}-\lambda^{(k)}\mathbf{M}\right)\,\frac{\partial \mathbf{q}^{(k)}}{\partial x_i}\;-\;\frac{\partial \lambda^{(k)}}{\partial x_i}\mathbf{M}\mathbf{q}^{(k)}\;+\;\left(\frac{\partial \mathbf{K}}{\partial x_i}-\lambda^{(k)}\frac{\partial \mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)}=\mathbf{0}\right]$$

- Since $q^{(k)}$ is an eigenmode

$$\mathbf{q}^{(k)T}\left(\mathbf{K}-\lambda^{(k)}\mathbf{M}\right)=\mathbf{0}$$

- And one gets

$$-\left(\mathbf{q}^{(k)T}\mathbf{M}\mathbf{q}^{(k)}\right)\,\frac{\partial \lambda^{(k)}}{\partial x_i}\;+\;\mathbf{q}^{(k)\,T}\left(\frac{\partial \mathbf{K}}{\partial x_i}-\lambda^{(k)}\frac{\partial \mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)}=\mathbf{0}$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- We finally obtain the final expression of the sensitivity of the eigen values:

$$\boxed{\frac{\partial \lambda^{(k)}}{\partial x_i} = \frac{1}{m^{(k)}}\,\mathbf{q}^{(k)T}\left(\frac{\partial \mathbf{K}}{\partial x_i} - \lambda^{(k)}\frac{\partial \mathbf{M}}{\partial x_i}\right)\,\mathbf{q}^{(k)}}$$

- With the scaling factor

$$m^{(k)} = \mathbf{q}^{(k)T}\mathbf{M}\mathbf{q}^{(k)}$$

- Sensitivity does not need any additional solution. Low CPU cost to obtain the eigenfrequency sensitivities

$$\boxed{\frac{\partial \omega^2_{(k)}}{\partial x_i} = \frac{1}{m^{(k)}}\left(\mathbf{q}^{(k)T}\frac{\partial \mathbf{K}}{\partial x_i}\mathbf{q}^{(k)} - \omega^2_{(k)}\,\mathbf{q}^{(k)T}\frac{\partial \mathbf{M}}{\partial x_i}\mathbf{q}^{(k)}\right)}$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- Calculating the sensitivity of the eigenvectors is more complicated.

- If we want to determine the derivatives of the eigenvectors, we have to solve simultaneously the two equations because the matrix $\mathbf{K} - \lambda^{(k)}\mathbf{M}$ is singular, and it is impossible to invert it.

$$\begin{cases} \left(\mathbf{K} - \lambda^{(k)}\mathbf{M}\right) \dfrac{\partial \mathbf{q}^{(k)}}{\partial x_i} = \dfrac{\partial \lambda^{(k)}}{\partial x_i}\mathbf{M}\mathbf{q}^{(k)} - \left(\dfrac{\partial \mathbf{K}}{\partial x_i} - \lambda^{(k)}\dfrac{\partial \mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)} \\ \mathbf{q}^{(k)\,T}\mathbf{W}\dfrac{\partial \mathbf{q}^{(k)}}{\partial x_i} = -\frac{1}{2}\mathbf{q}^{(k)T}\dfrac{\partial \mathbf{W}}{\partial x_i}\mathbf{q}^{(k)} \end{cases}$$

- The system of equations writes under matrix form:

$$\begin{bmatrix} \mathbf{K} - \lambda^{(k)}\mathbf{M} & -\mathbf{M}\mathbf{q}^{(k)} \\ -\mathbf{q}^{(k)T}\mathbf{W} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \mathbf{q}^{(k)}}{\partial x_i} \\ \dfrac{\partial \lambda^{(k)}}{\partial x_i} \end{Bmatrix} = \begin{Bmatrix} -\left(\dfrac{\partial \mathbf{K}}{\partial x_i} - \lambda^{(k)}\dfrac{\partial \mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)} \\ -\frac{1}{2}\mathbf{q}^{(k)T}\dfrac{\partial \mathbf{W}}{\partial x_i}\mathbf{q}^{(k)} \end{Bmatrix}$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- To solve this system, there are several methods. The most popular one is the temporary fixation strategy proposed by Nelson (1976).

- Expand the derivative of the eigenmode in the basis of eigenvectors:

$$\frac{\partial \mathbf{q}^{(k)}}{\partial x_i} = \sum_{l=1}^{N} c_{lk} \mathbf{q}^{(l)} = \sum_{l=1, l \neq k}^{N} c_{lk} \mathbf{q}^{(l)} + c_{kk} \mathbf{q}^{(k)}$$

$$= \mathbf{V}_k + c_{kk} \mathbf{q}^{(k)}$$

  - V_k is orthogonal to the eigenmode $q^{(k)}$;

SENSITIVITY OF EIGENVALUE PROBLEMS

- V_k is the solution of a reduced version of the eigenvalue equation obtained by deleting the kth row and column from

$$\mathbf{K} - \lambda^{(k)}\mathbf{M}$$

and by setting to zero the kth component of V_k

$$\left(\mathbf{K} - \lambda^{(k)}\mathbf{M}\right)^{*}\mathbf{V}_k = \frac{\partial \lambda^{(k)}}{\partial x_i}\mathbf{M}\mathbf{q}^{(k)} - \left(\frac{\partial \mathbf{K}}{\partial x_i} - \lambda^{(k)}\frac{\partial \mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)}$$

- V_k has been made orthogonal to $q^{(k)}$ in the norm of matrix M

$$\mathbf{V}_k^{\perp} = \mathbf{V}_k - \frac{\mathbf{q}^{(k)T}\mathbf{M}\mathbf{V}_k}{m^{(k)}}\mathbf{q}^{(k)} \quad \text{with} \quad m^{(k)} = \mathbf{q}^{(k)T}\mathbf{M}\mathbf{q}^{(k)}$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- The multiplier c_{kk} is evaluated by substituting into the derivative of the normalization equation with respect to the mass matrix M=W

$$\begin{aligned}
\mathbf{q}^{(k)\,T}\mathbf{M}\,\frac{\partial \mathbf{q}^{(k)}}{\partial x_i} &= -\frac{1}{2}\mathbf{q}^{(k)T}\frac{\partial \mathbf{M}}{\partial x_i}\mathbf{q}^{(k)} \\
\mathbf{q}^{(k)\,T}\mathbf{M}\,(\mathbf{V}_k^{\perp} + c_{kk}\ \mathbf{q}^{(k)}) &= -\frac{1}{2}\mathbf{q}^{(k)T}\frac{\partial \mathbf{M}}{\partial x_i}\mathbf{q}^{(k)} \\
\mathbf{0} + c_{kk}\,\mathbf{q}^{(k)T}\mathbf{M}\mathbf{q}^{(k)} &= -\frac{1}{2}\mathbf{q}^{(k)T}\frac{\partial \mathbf{M}}{\partial x_i}\mathbf{q}^{(k)}
\end{aligned}$$

- As the modes are orthogonal, it comes

$$c_{kk} = \frac{-1}{2m^{(k)}}\,\mathbf{q}^{(k)T}\frac{\partial \mathbf{M}}{\partial x_i}\mathbf{q}^{(k)}$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- **Remark 1:** In the framework of the theory of small perturbations, the modifications of the eigen frequencies and of the mode shapes are decoupled.

- **Remark 2:** in case of multiple eigenfrequencies, the eigenfrequencies are non smooth and the derivative does not exist anymore (subdifferential).

  Calculating the derivatives requires a projection into the basis of eigenvectors of the multiple eigenfrequencies.

SENSITIVITY OF STABILITY PROBLEMS

- Stability equation

$$\left(\mathbf{K} - \lambda^{(j)}\,\mathbf{S}\right)\,\mathbf{q}^{(j)} = 0$$

- The matrix **S** is the stability matrix resulting from the geometrical and prestressing terms

$$\mathbf{S} = \mathbf{S}(\sigma^0(\mathbf{x}))$$

- Let's differentiate the stability equation

$$\left(\mathbf{K} - \lambda^{(j)}\mathbf{S}\right)\,\frac{\partial \mathbf{q}^{(j)}}{\partial x_i} = \frac{\partial \lambda^{(j)}}{\partial x_i}\mathbf{S}\mathbf{q}^{(j)} - \left(\frac{\partial \mathbf{K}}{\partial x_i} - \lambda^{(j)}\,\frac{\partial \mathbf{S}}{\partial x_i}\right)\mathbf{q}^{(j)}$$

SENSITIVITY OF STABILITY PROBLEMS

- Derivative of the buckling load factor

$$\frac{\partial \lambda^{(j)}}{\partial x_i} = \frac{1}{\mathbf{q}^{(j)T}\mathbf{S}\mathbf{q}^{(j)}}\, \mathbf{q}^{(j)T} \left(\frac{\partial \mathbf{K}}{\partial x_i} - \lambda^{(j)} \frac{\partial \mathbf{S}}{\partial x_i} \right) \mathbf{q}^{(j)}$$

- The major issue it to calculate the sensitivity of the stability matrix **S** !!!
- Approximation in SAMCEF

$$\frac{\mathbf{S}(\sigma^0(\mathbf{x}))}{\partial x_i} \simeq \frac{\mathbf{S}_{\mathbf{x}+\delta\mathbf{x}}(\sigma^*) - \mathbf{S}_{\mathbf{x}}(\sigma^0(\mathbf{x}))}{\delta x}$$

$$\sigma^* = \sigma^0 + \frac{\partial \sigma}{\partial x}\delta x$$