INTRODUCTION TO TOPOLOGY OPTIMIZATION

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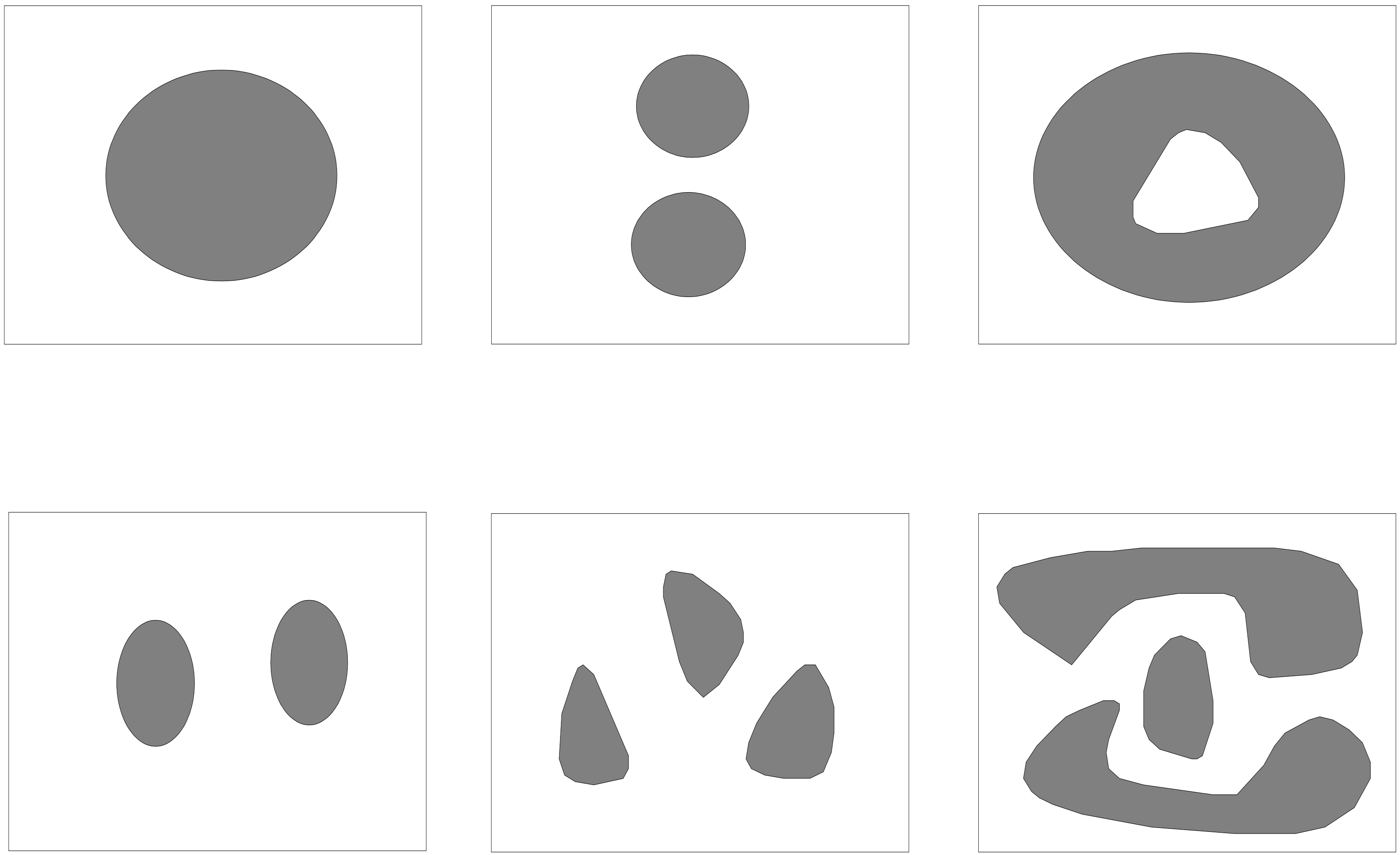
LAY-OUT

- Introduction
- Topology problem formulation: Problem statement
- Compliance minimization
- Homogenization method vs SIMP based
- Filtering techniques
- Sensitivity analysis
- Optimality Criteria for compliance minimization problem
- Solution of optimization problems using structural approximations and dual maximization
- Vibration Problems
- Applications

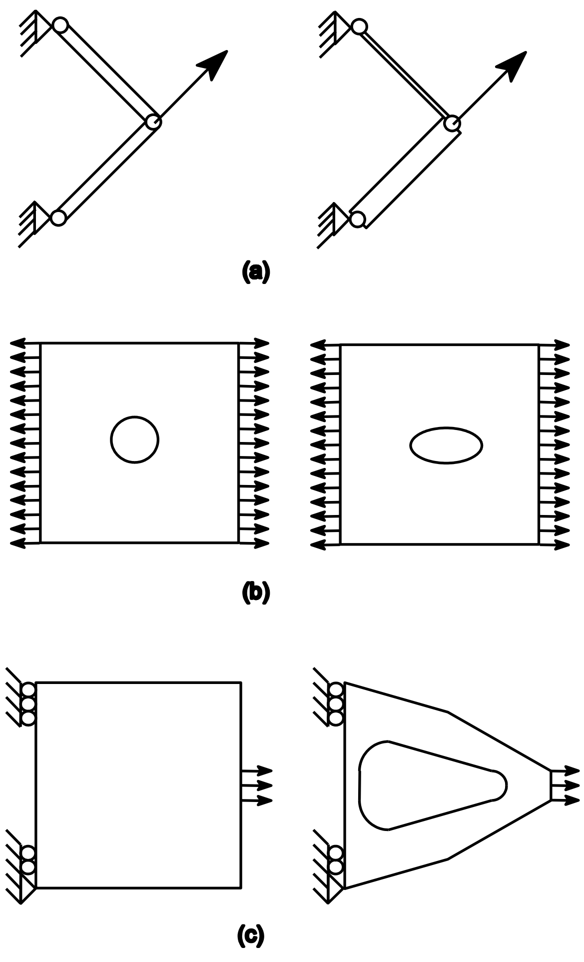
INTRODUCTION

What is topology?



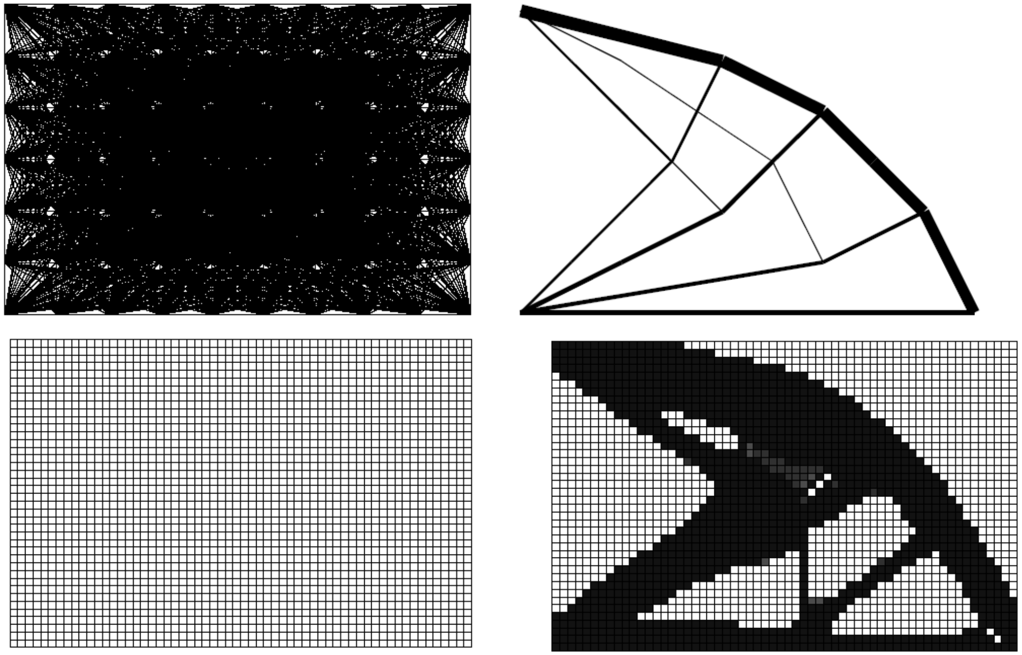
STRUCTURAL & MULTIDISCIPLINARY OPTIMISATION

- TYPES OF VARIABLES
  - a/ Sizing
  - b/ Shape
  - c/ Topology
  - (d/ Material)
- TYPES OF OPTIMISATION
  - structural
  - multidisciplinary
    - structural
    - aerodynamics,
    - thermal,
    - manufacturing



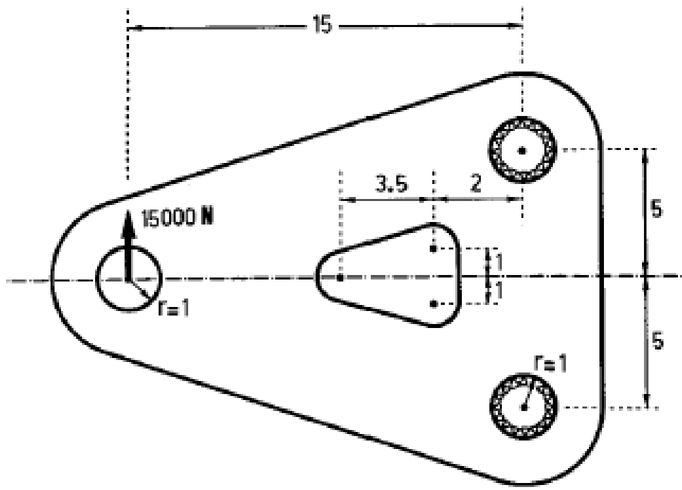
Topology optimization

- One generally distinguishes two approaches of topology optimization:
  - Topology optimization of naturally discrete structures (e.g. trusses)
  - Topology optimization of continuum structures (eventually after FE discretization)

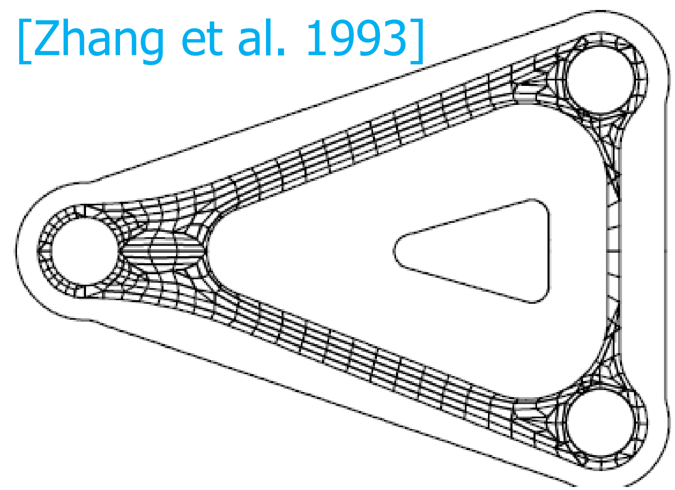


Why topology optimization?

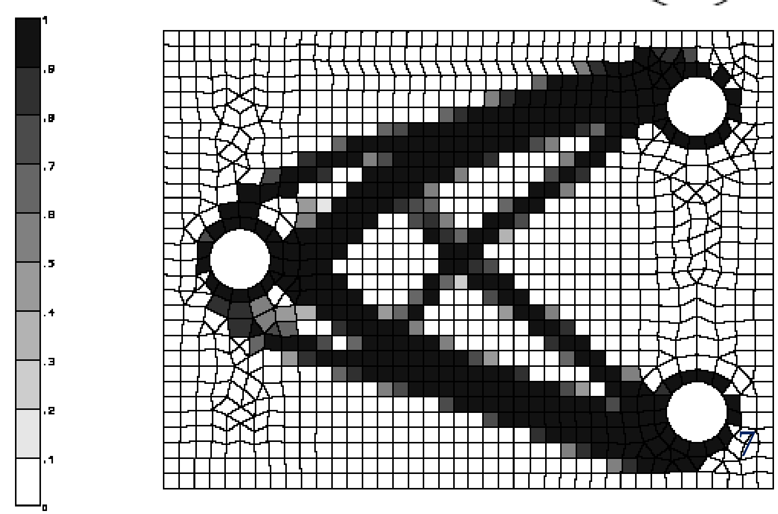
- CAD approach does not allow topology modifications



[Zhang et al. 1993]

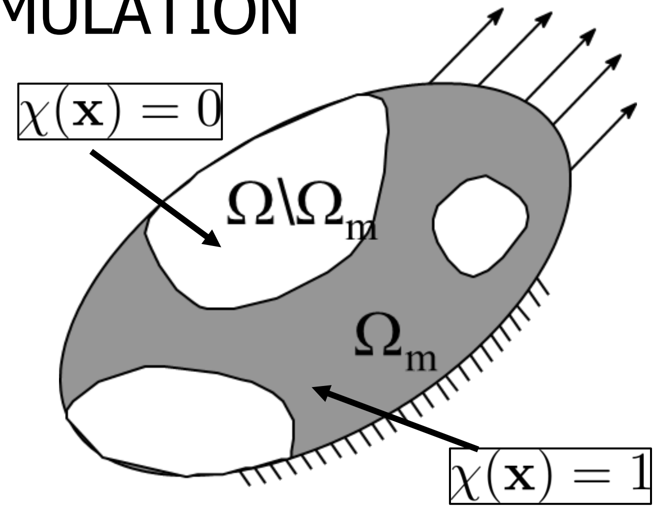


A better morphology by topology optimization
[Duysinx, 1996]



TOPOLOGY PROBLEM FORMULATION

TOPOLOGY OPTIMIZATION FORMULATION



- CAD model description based on boundary description ➔ look for a picture of the optimal structure
- Optimal topology is given by an optimal material distribution problem
- Search for the indicator function of the domain occupied by the material

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m \\ 0 & \text{if } \mathbf{x} \in \Omega \setminus \Omega^m \end{cases}$$

- The physical properties write

$$E_{ijkl}(\mathbf{x}) = \chi(\mathbf{x})\, E^0_{ijkl}$$

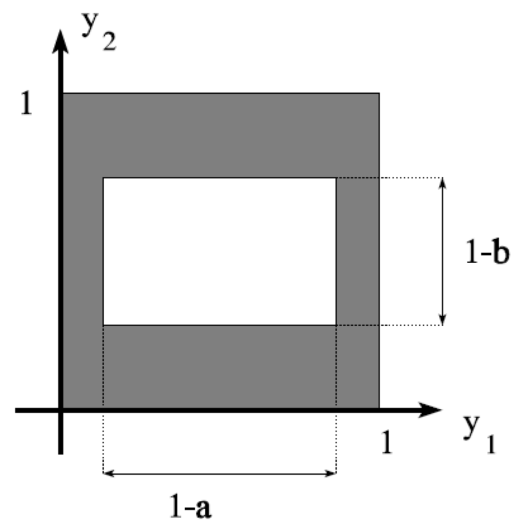
$$\rho(\mathbf{x}) = \chi(\mathbf{x})\, \rho^0$$

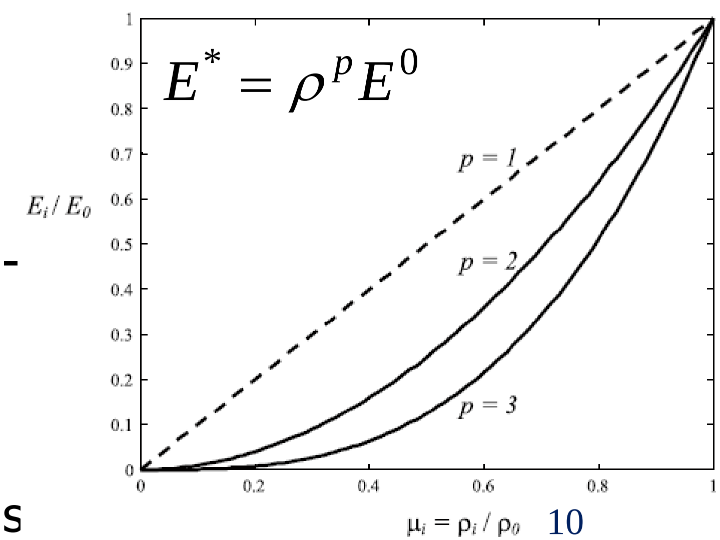
- The problem is intrinsically a binary 0-1 problem ➔ solution is extremely difficult to solve

$$\chi \in \{0, 1\}$$

MATERIAL DENSITY FUNCTION

- Avoid 0/1 problem and replace by a continuous approximation considering a variable density material running from void (0) to solid (1) $\chi \in [0, 1]$
  - Homogenization law of mechanical properties a porous material for any volume fraction (density) of materials
  - Mathematical interpolation and regularization
    - SIMP model $E = x^p\, E^0$
    - RAMP…
- Penalization of intermediate densities to end-up with black and white solutions
- Efficient solution of optimization problem based on sensitivity analysis and gradient based mathematical programming algorithms



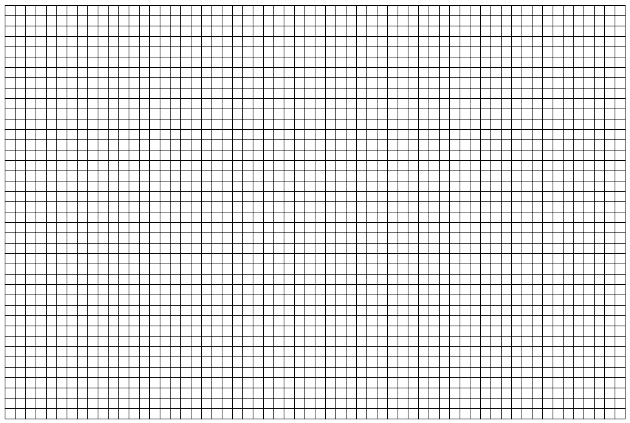


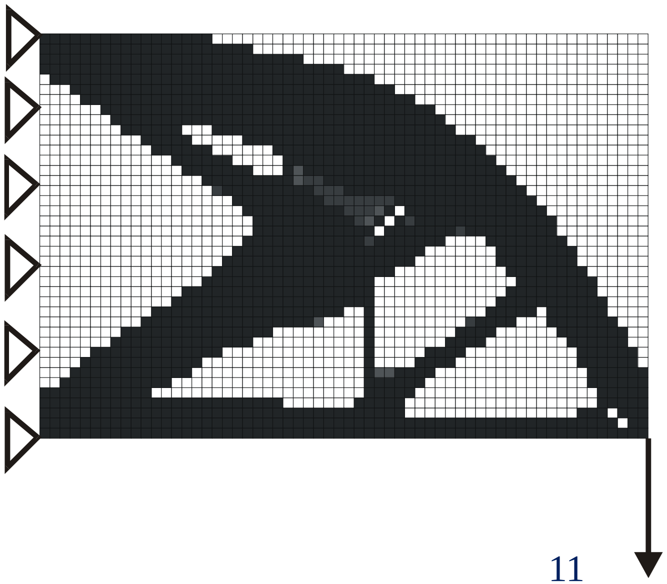
IMPLEMENTION OF MATERIAL DENSITY FUNCTION

- Implementation of material density approach is rather easy:
  1. Fixed mesh
  2. Attach one density design variable x_i to each element
  3. Problem statement is similar to a sizing problem

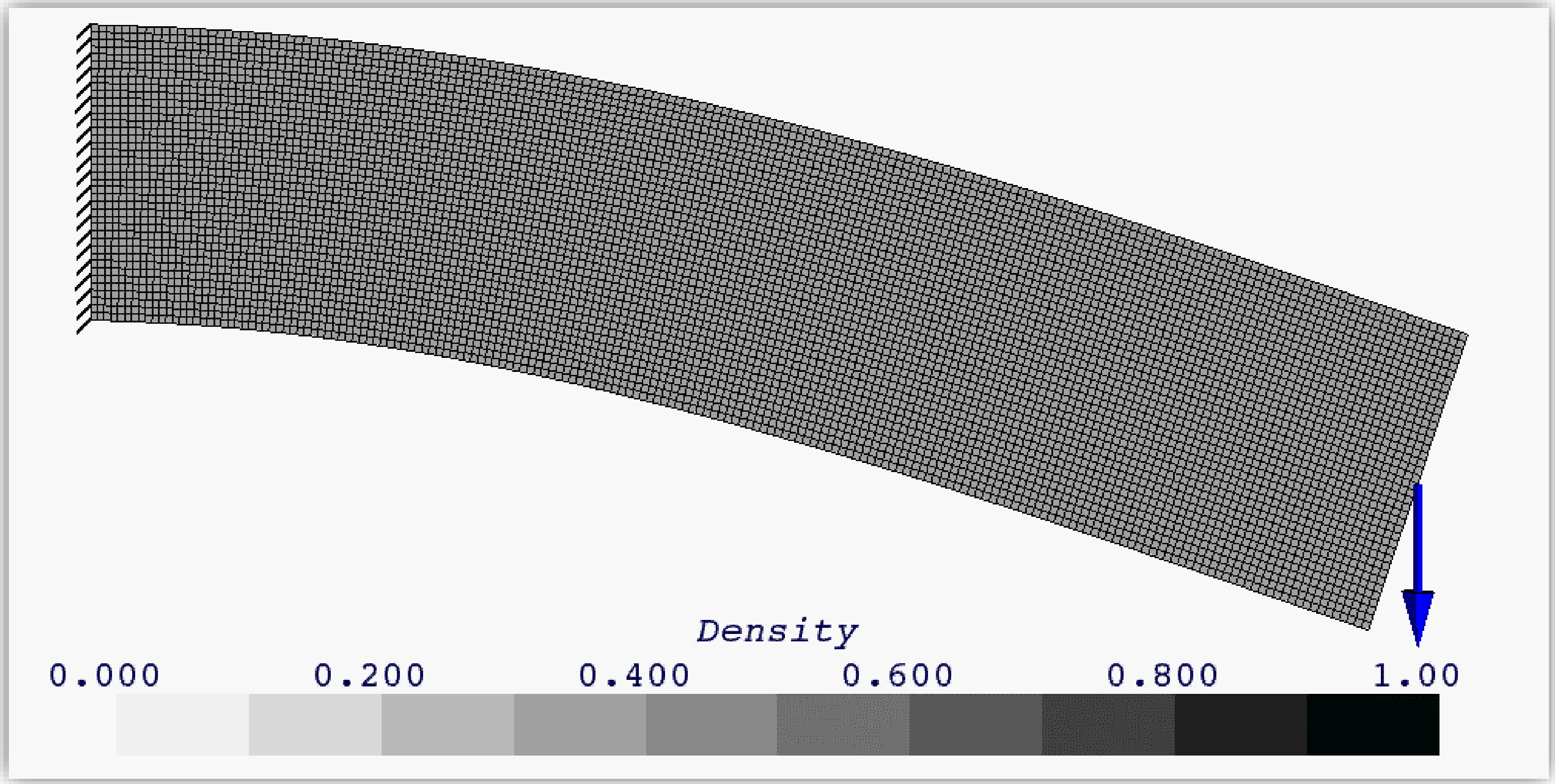
$$\begin{array}{cc} \min\limits_{x \in [0,1]} & \text{Compliance} \\ \text{s.t.} & \text{Volume} \leq \bar{V} \end{array}$$

  4. SIMP law is easy to code
  5. Sensitivity of compliance is cheap
  6. Use an efficient gradient based optimization algorithms as MMA





A FIRST EXAMPLE: GENESIS OF A STRUCTURE



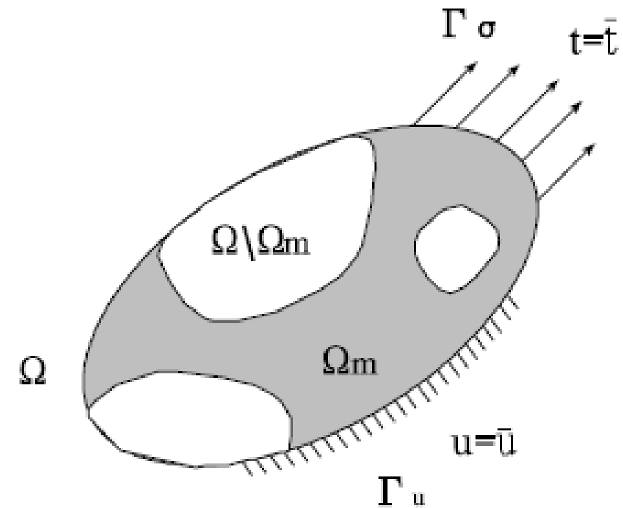
[E. Lemaire, PhD Thesis, Uliege, 2013]

TOPOLOGY OPTIMIZATION AS A COMPLIANCE MINIMIZATION PROBLEM

DESIGN PROBLEM FORMULATION

- The fundamental problem of topology optimization deals with the optimal material distribution within a continuum structure subject to a single static loading.

- In addition, one can assume that the structure is subject to homogeneous boundary conditions on Γ_u.



$$u_i = 0 \quad \text{on} \quad \Gamma_u$$

- The principle of virtual work writes

$$\mathbf{u} \in V_0 \; : \quad a_E(\mathbf{u},\mathbf{v}) = l(\mathbf{v}) \quad \forall\, \mathbf{v} \in V_0$$

$$\delta\mathbf{q}^T\mathbf{K}\mathbf{q} = \delta\mathbf{q}^T\mathbf{g}$$

DESIGN PROBLEM FORMULATION

- A typical topology optimization problem is to find the best subset of the design domain minimizing the volume or alternatively the mass of the structure,

$$V = \int_{\Omega} \rho(\mathbf{x}) \, d\Omega$$

  - while achieving a given level of functional (mechanical) performance.
- Following Kohn (1988), the problem is well posed from a mathematical point of view if the mechanical behaviour is sufficiently smooth. Typically one can consider :
  - Compliance (energy norm)
  - A certain norm of the displacement over the domain
  - A limitation of the maximum stress

DESIGN PROBLEM FORMULATION

- Compliance performance: The mechanical work of the external loads

$$l(\boldsymbol{u}) = \int_{\Omega} \boldsymbol{f}^T \boldsymbol{u} \, d\Omega + \int_{\Gamma} \boldsymbol{t}^T \boldsymbol{u} \, d\Gamma$$

  – Using finite element formulation

$$C(\boldsymbol{q}) = \boldsymbol{g}^T \boldsymbol{q}$$

  – At equilibrium, the compliance is also the strain energy of the structure,

$$\boldsymbol{g}^T \boldsymbol{q} = \boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q}$$

  – One can interpret the compliance as the displacement under the loads. For a single local case, it is the displacement under the load.

$$C(\boldsymbol{q}) = \boldsymbol{g}^T \boldsymbol{q} = f_i \, u_i$$

DESIGN PROBLEM FORMULATION

- The average displacement (according to a selected norm) over the domain or a subdomain Ω_1 excluding some irregular points

$$\int_{\Omega_1} ||\boldsymbol{u}||\, d\Omega$$

- If one considers the quadratic norm and if the finite element discretization is used, one reads

$$\int_{\Omega_1} \boldsymbol{u}^T\, \boldsymbol{u}\, d\Omega \;=\; \int_{\Omega_1} \boldsymbol{q}^T \boldsymbol{N}(\boldsymbol{x})^T\, \boldsymbol{N}(\boldsymbol{x})\, \boldsymbol{q}\, d\Omega \;=\; \boldsymbol{q}^T \boldsymbol{M}\, \boldsymbol{q}$$

- Assuming a lumped approximation of the matrix **M,** one can find the simplified equivalent quadratic norm of the displacement vector

$$\int_{\Omega_1} ||\boldsymbol{u}||\, d\Omega \;\approx\; \boldsymbol{q}^T \boldsymbol{q}$$

DESIGN PROBLEM FORMULATION

- Limitation of a given local stress measure $||\sigma(x)||$ over a sub-domain Ω_2 excluding some neighborhood of singular points related to some geometrical properties of the domain (reentrant corners) or some applied loads

$$\sup_{\boldsymbol{x}\in\Omega_2} ||\sigma(\boldsymbol{x})||$$

  – Stress measure $||\sigma(x)||$ → Von Mises, Tresca, Tsai-Hill…

DESIGN PROBLEM FORMULATION

- The choice of the compliance is generally the main choice by designers.
  - The sensitivity of compliance is easy to calculate. Being self adjoined, compliance is self adjoined, and it does not require the solution of any additional load case.
  - Conversely local stress constraints call for an important amount of additional CPU to compute the local sensitivities.
  - One can find analytical results providing the optimal bounds of composites mixture of materials for a given external strain field. The problem is known as the *G-closure.*

DESIGN PROBLEM FORMULATION

- Finally the statement of the basic topology problem writes:

$$
\begin{array}{ll}
\min\limits_{\underline{x}_i \leq x_i \leq \overline{x}_i} & V(\mathbf{x}) \\
\text{s.t.:} & C(\mathbf{q}) = \mathbf{g}^T \mathbf{q} \leq \bar{C} \\
 & \mathbf{K}\,\mathbf{q} = \mathbf{g}
\end{array}
$$

- Alternatively it is equivalent for a given bounds on the volume and the compliance to solve the minimum compliance subject to volume constraint

$$
\begin{array}{ll}
\min\limits_{\underline{x}_i \leq x_i \leq \overline{x}_i} & C(\mathbf{q}) = \mathbf{g}^T \mathbf{q} \\
\text{s.t.:} & V(\mathbf{x}) \leq \bar{V} \\
 & \mathbf{K}\,\mathbf{q} = \mathbf{g}
\end{array}
$$

DESIGN PROBLEM FORMULATION

- Minimize compliance

s.t.

- Given volume
- (bounded perimeter)
- (other constraints)

- Maximize eigenfrequencies

s.t.

- Given volume
- (bounded perimeter)
- (other constraints)

- Minimize the maximum of the local failure criteria

s.t.

- Given volume
- (bounded perimeter)
- (other constraints)

PROBLEM FORMULATION

- For several load cases, average compliance

$$\begin{array}{cc} \min\limits_{\underline{x}_i \le x_i \le \overline{x}_i} & \sum_k w_k\, \mathbf{g}^{(k)\,T} \mathbf{q}^{(k)} \\ \text{s.t.:} & V(\mathbf{x}) \le \bar{V} \\ & \mathbf{K}\,\mathbf{q}^{(k)} = \mathbf{g}^{(k)} \end{array}$$

- Or better a worst-case approach

$$\begin{array}{cc} \min\limits_{\underline{x}_i \le x_i \le \overline{x}_i} & \max_k \mathbf{g}^{(k)\,T} \mathbf{q}^{(k)} \\ \text{s.t.:} & V(\mathbf{x}) \le \bar{V} \\ & \mathbf{K}\,\mathbf{q}^{(k)} = \mathbf{g}^{(k)} \end{array}$$

  - Where k is load case index, K is the stiffness matrix of FE approximated problem, g_k, and q_k are the load case and generalized displacement vectors for load case k
  - and $\rho(x)$ is the local density and V is the volume

MULTIPLE LOAD CASE FORMULATION

- Min-max formulation

$$\min_{\underline{x}_i \leq x_i \leq \overline{x}_i} \quad \max_k \mathbf{g}^{(k)\,T}\mathbf{q}^{(k)}$$

$$\text{s.t.:} \quad V(\mathbf{x}) \leq \bar{V}$$

$$\mathbf{K}\,\mathbf{q}^{(k)} = \mathbf{g}^{(k)}$$

- Is equivalent to

$$\min_{\underline{x}_i \leq x_i \leq \overline{x}_i} \quad \beta$$

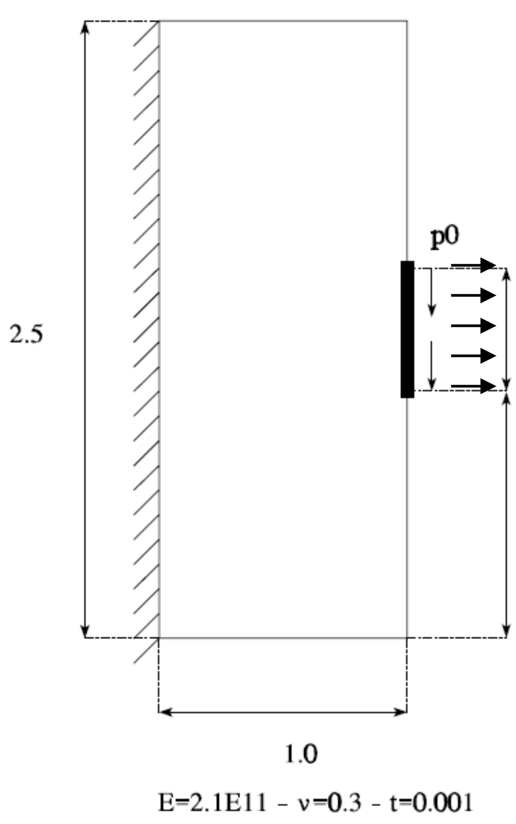
$$\text{s.t.:} \quad V(\mathbf{x}) \leq \bar{V}$$

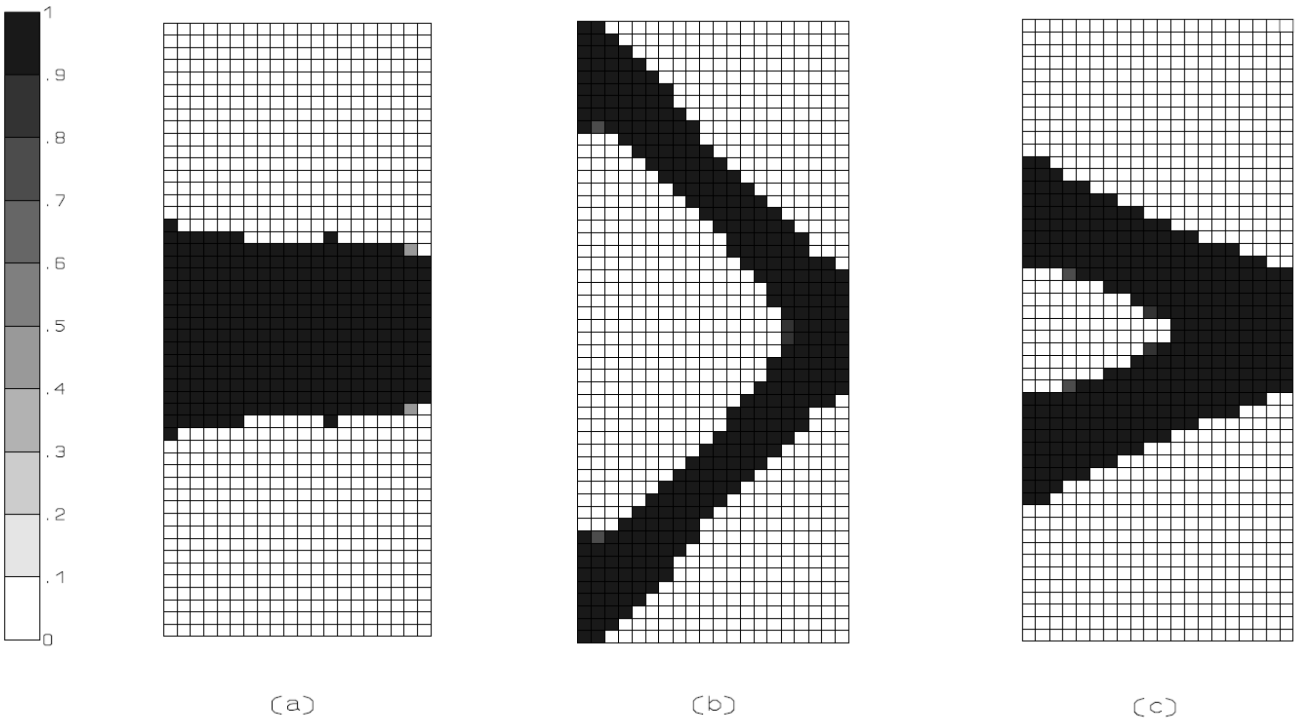
$$\mathbf{g}^{(k)\,T}\mathbf{q}^{(k)} \leq \beta$$

$$\mathbf{K}\,\mathbf{q}^{(k)} = \mathbf{g}^{(k)}$$

MULTIPLE LOAD CASE

- Importance of treating separately the different load case!





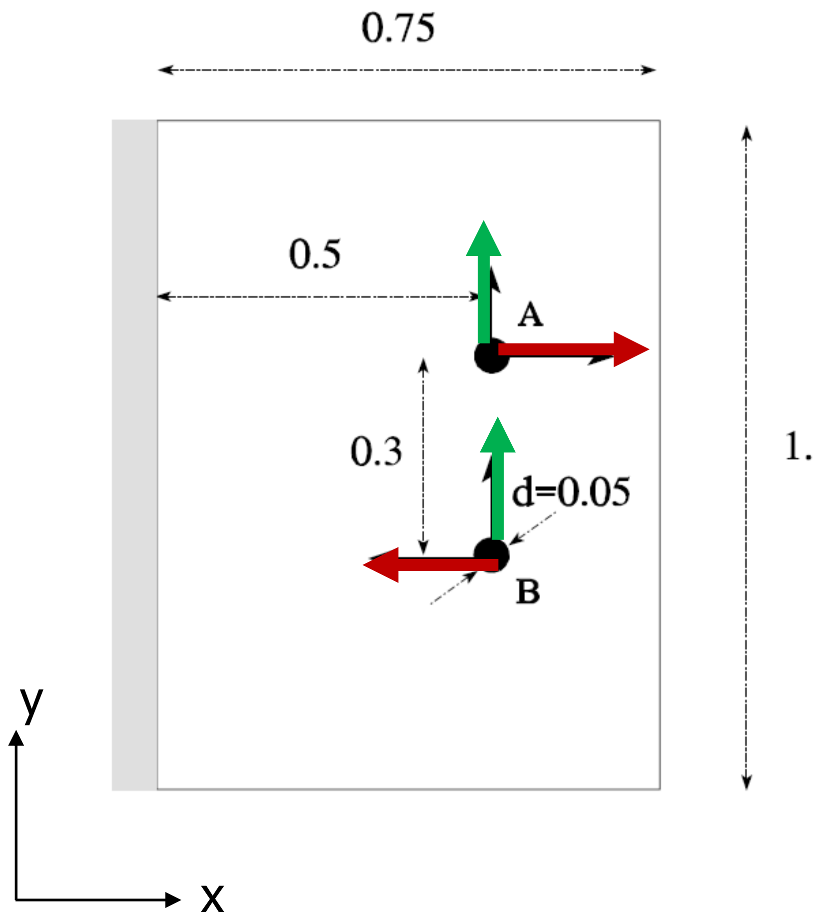
Traction alone

Shear alone

Traction + Shear

MULTIPLE LOAD CASE

- Importance of treating separately the different load case!



Load case 1

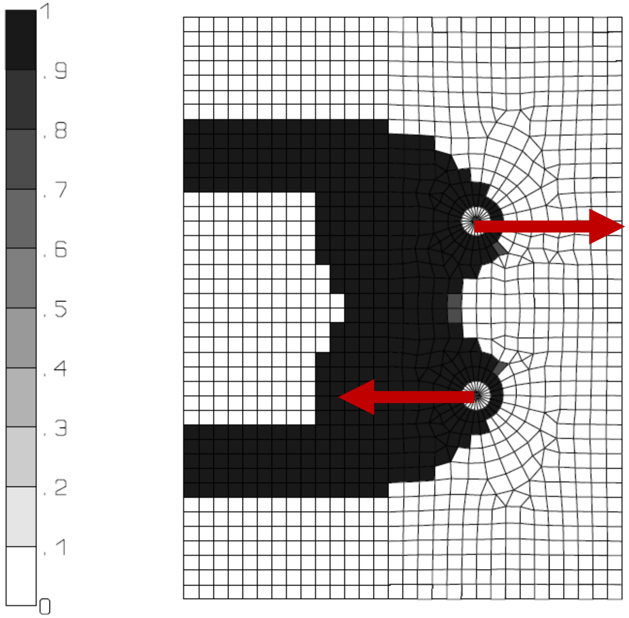
$$F_x^A = 1.E5\ N$$
$$F_x^B = -1.E5\ N$$

Load case 2

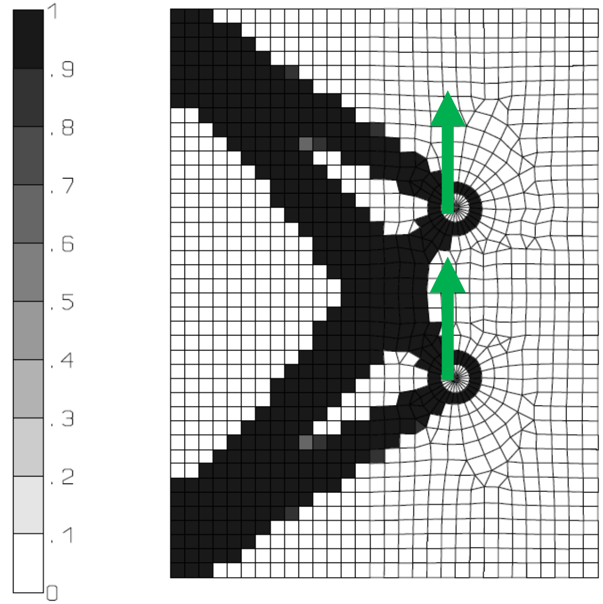
$$F_y^A = 0.5\ E5\ N$$
$$F_y^B = 0.5\ E5\ N$$

MULTIPLE LOAD CASE

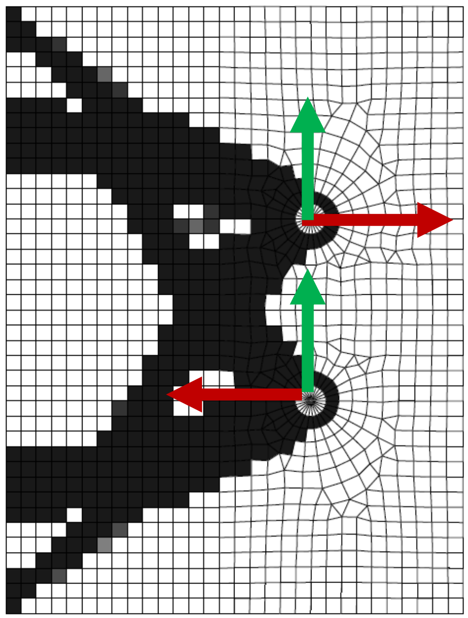
- Importance of treating separately the different load case!



Load case 1 only



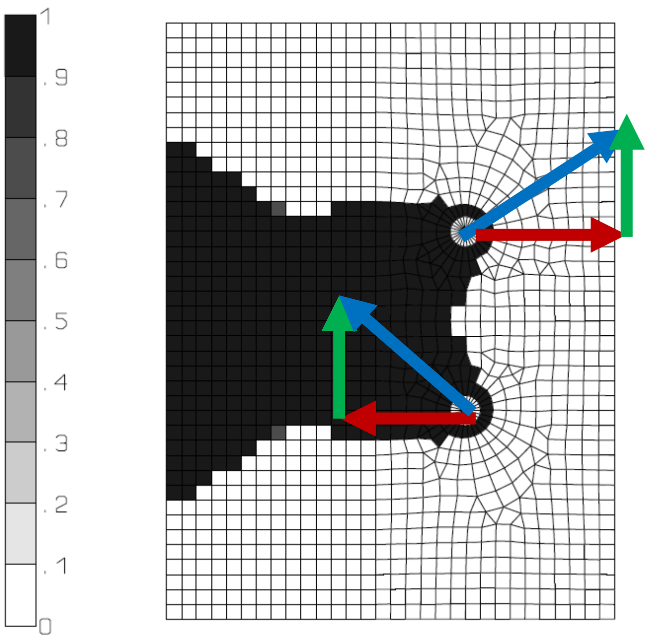
Load case 2 only



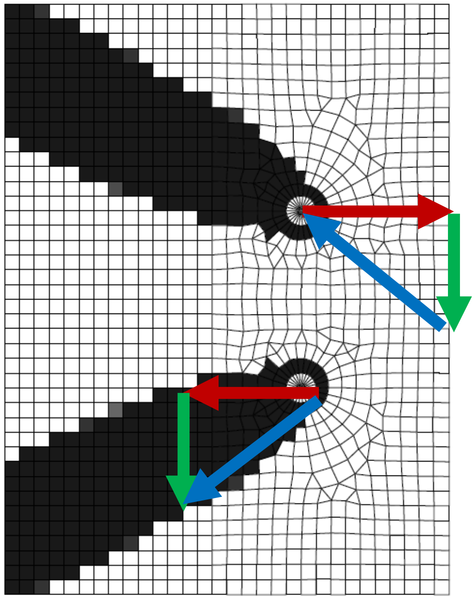
Load case 1 and 2

MULTIPLE LOAD CASE

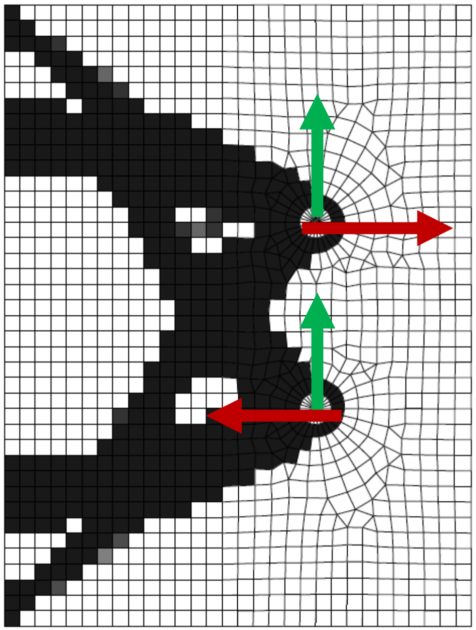
- Importance of treating separately the different load case!



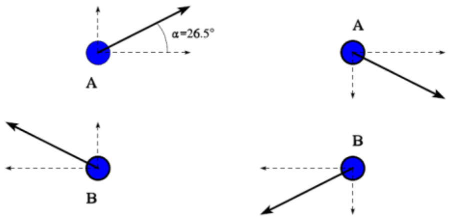
Load case F1+F2



Load case F1-F2



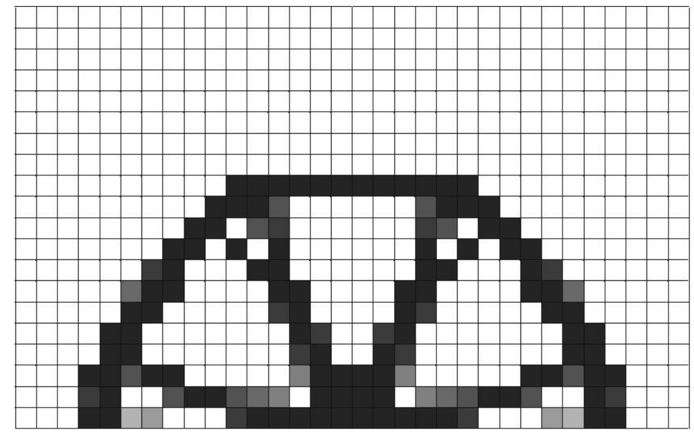
Max Load case F1 and F2

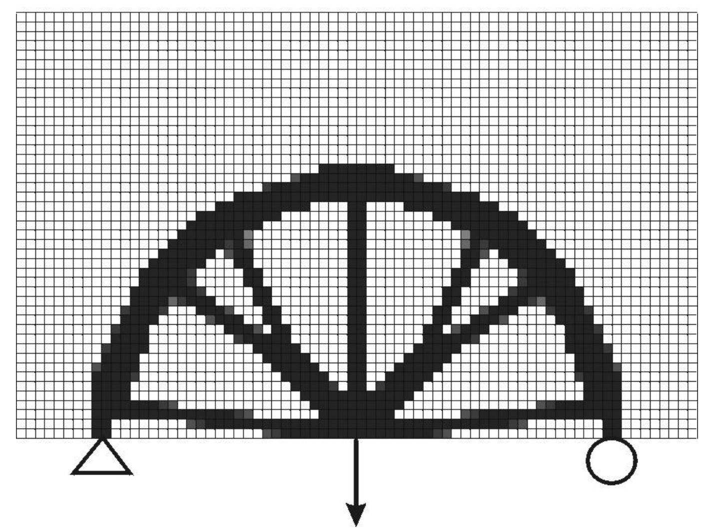


TOPOLOGY OPTIMIZATION USING HOMOGENIZATION VS SIMP BASED TOPOLOGY OPTIMIZATION

TOPOLOGY OPTIMIZATION: FORMULATION

- Well-posed ness of problem?
  Discretised problem is ill-posed
  - Mesh-dependent solutions
  - Recreate microstructures
  - Nonexistence and uniqueness of a solution
- Homogenisation Method:
  → Extend the design space to all porous composites of variable density
- Filter method / Perimeter method / Slope constraints:
  → Restrict the design space by eliminating chattering designs from the design space

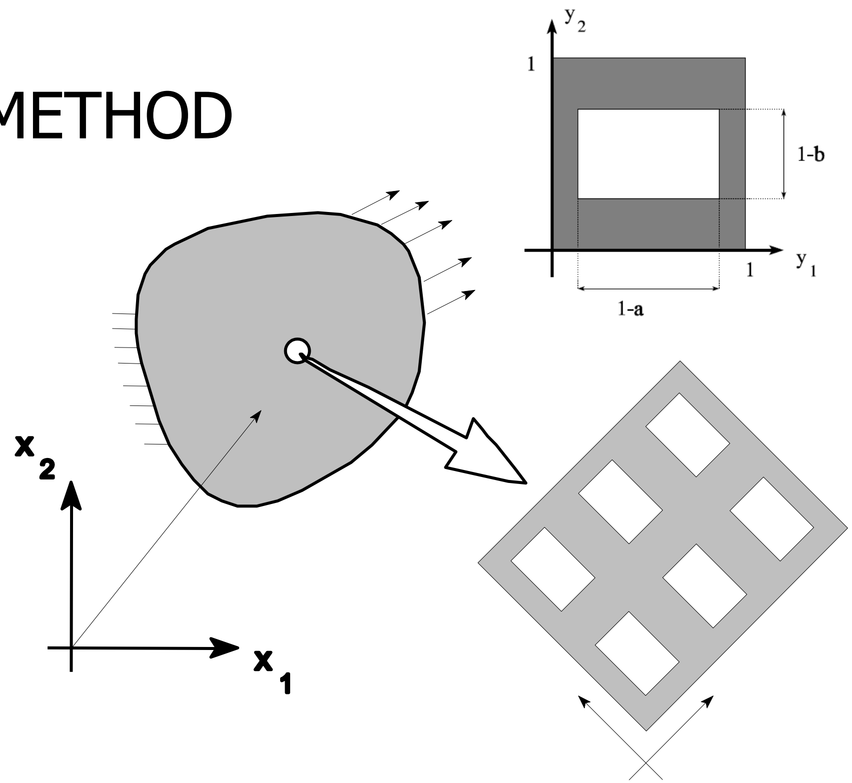




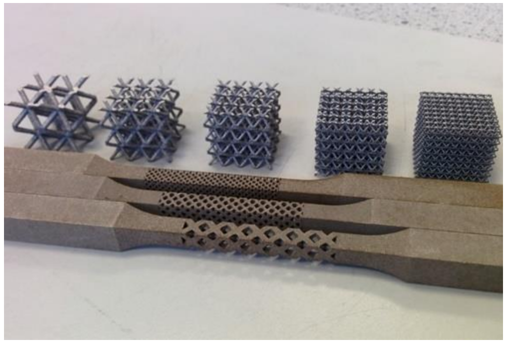
HOMOGENIZATION METHOD

- Select one family of microstructures whose geometry is fully parameterized in terms of a set of design variables [Bendsoe and Kikuchi, 1988]
  - G closure : optimal microstructure (full relaxation)
  - Suboptimal microstructures (partial relaxation)

- Use homogenization theory to compute effectives properties: in terms of microstructural geometrical parameters: $E_{ijk} = E^h_{ijkl}(a,b,\ldots)$
- Difficult to interpret and fabricate the optimal material distribution as it is



- Revival interest with emergence of cellular structures e.g. lattice structures made by additive manufacturing



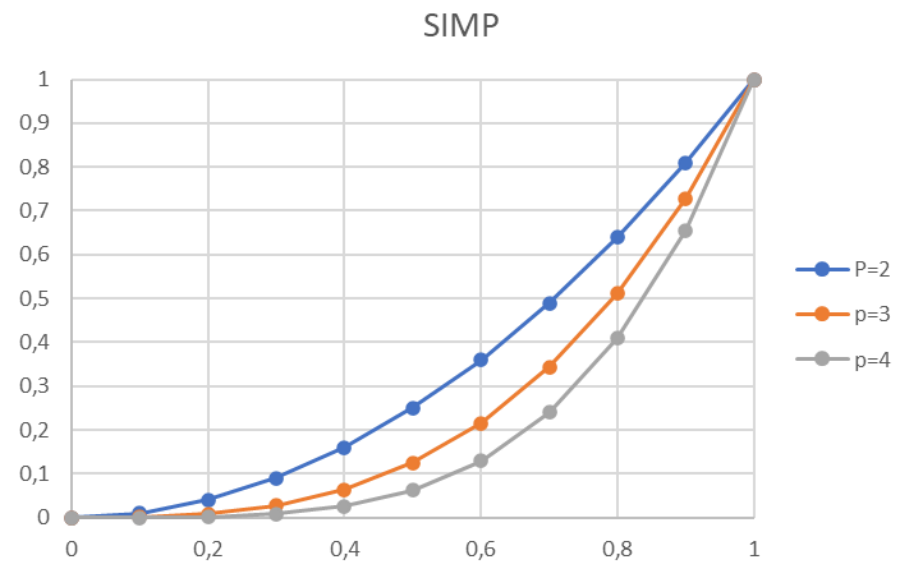
POWER LAW MODEL (SIMP)

- Simplified model of a microstructured material with a penalisation SIMP of intermediate densities [Bendsoe, 1989]
- Stiffness properties:

$$E = x^p \, E^0$$

$$\rho = x \, \rho^0$$

$$0 \leq x \leq 1 \quad p > 1$$



- Can be related to actual micro geometries [Bendsoe and Sigmund, 1999]
- Strength properties:

$$\sigma_l = x^p \, \sigma_l^0$$

- Modified SIMP should be preferred to avoid singularities

$$E(x) = E_{min} + x^p(E^0 - E_{min})$$

- 90% of current topology optimization runs

ALTERNATIVE PARAMETRIZATION TO SIMP

- Alternatively RAMP parameterization [Stolpe & Svanberg, 2001] enables controlling the slope at zero density

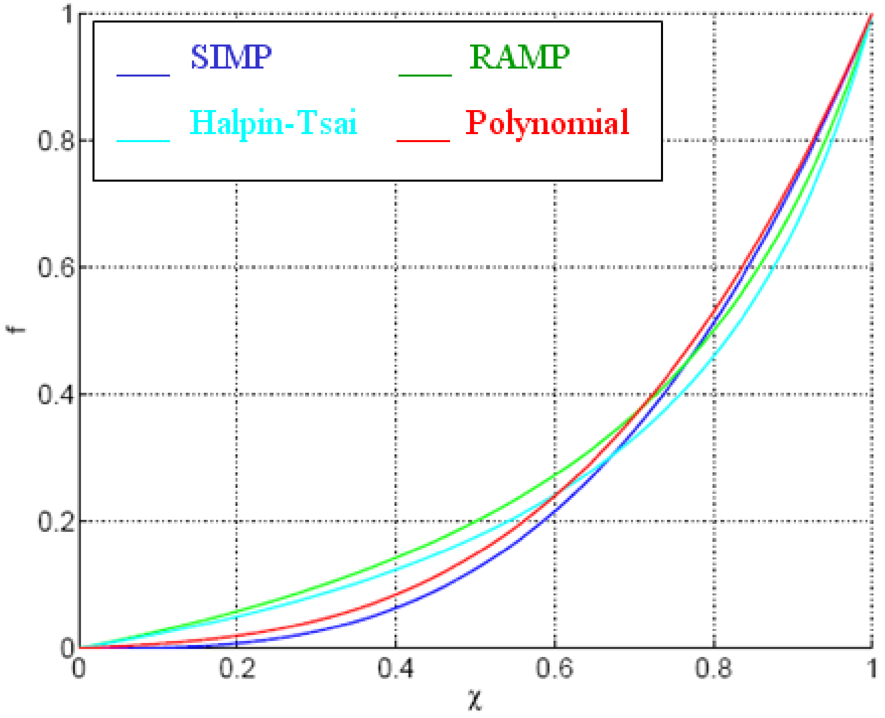
$$E(x) = \frac{x}{1+p(1-x)}\, E^0$$

- Halpin Tsai (1969)

$$E(x) = \frac{r\, x}{(1+r)-x}\, E^0$$

- Polynomial penalization [Zhu, 2009]:

$$E(x) = \left(\frac{\alpha-1}{\alpha} x^p + \frac{1}{\alpha} x \right) E^0$$



POWER LAW MODEL (SIMP)

- Prescribing immediately a high penalization may introduce some numerical difficulties:
  - Optimization problem becomes difficult to solve because of the sharp variation of material properties close to x=1
  - Optimization problem includes a lot of local optima and solution procedure may be trapped in one of these.

- To mitigate these problems, one resorts to the so-called continuation procedure in which p is gradually increased from a small initial value till the desired high penalization.
- Typically:

$$p^{(0)} = 1.6$$

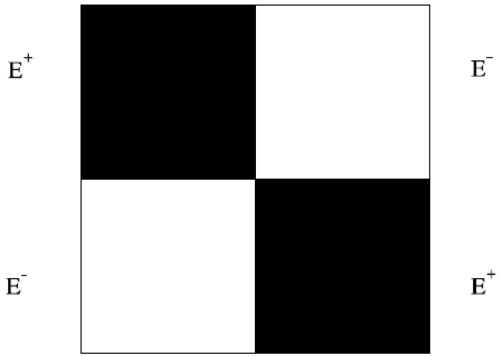
$$p^{(k+1)} = p^{(k)} + \Delta p \quad k > K \text{ or } KKT \leq \epsilon$$

  - after a given number of iterations or when a convergence criteria is OK

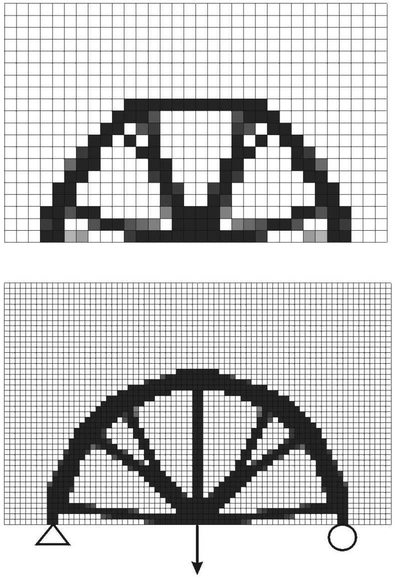
FILTERING TECHNIQUES AND MESH INDEPENDENCY STRATEGIES

Two numerical difficulties

- Checkerboard patterns: numerical instabilities related to the inconsistency between the displacement and density fields.
  - Appearance of alternate black-white patterns
  - Checkerboard patterns replaces intermediate densities

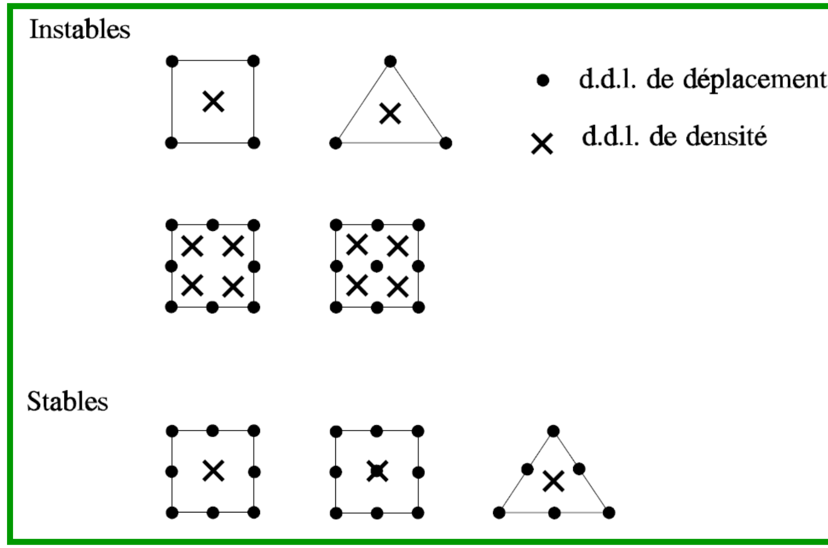


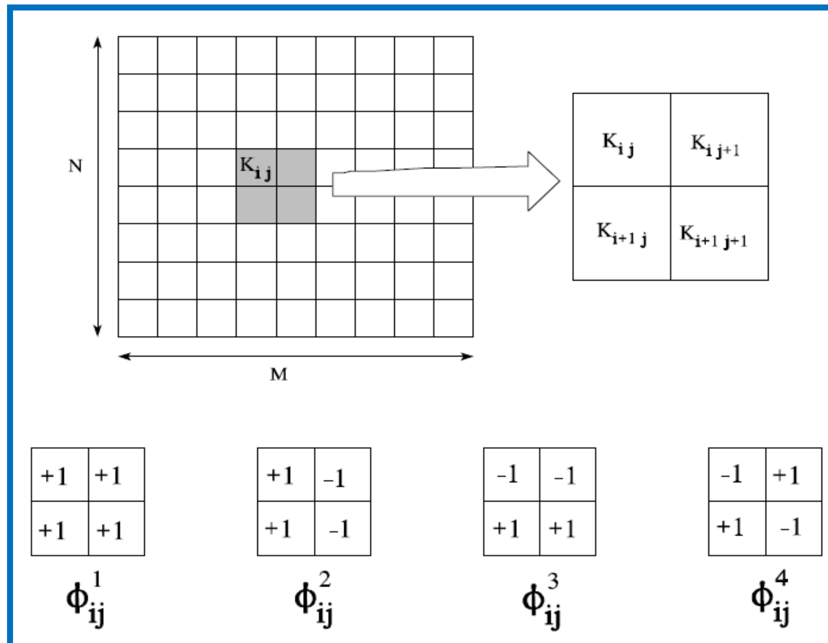
- Mesh dependency: the solution depends on the computing mesh.
  - New members appear when refining the mesh
  - The number of holes and features is modified when changing the mesh.
  - Stability (and meaning) of solutions?

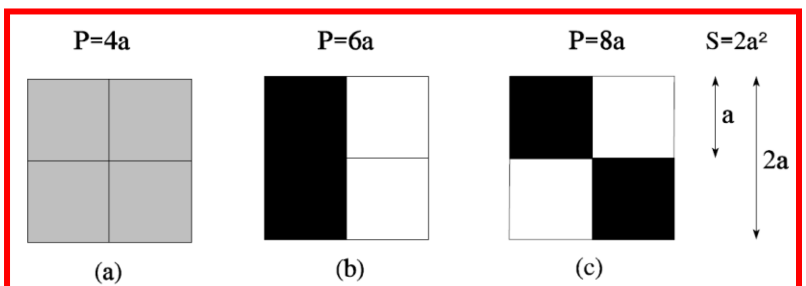


Checkerboard patterns

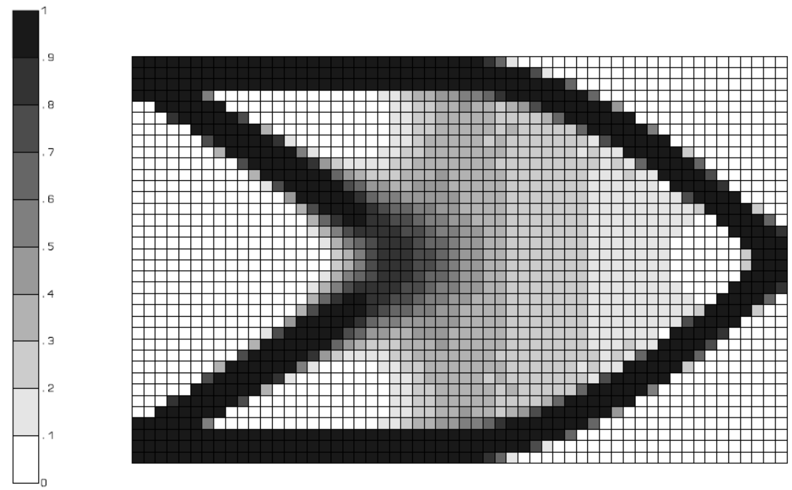
- Babuska Brezzi conditions of discretization schemes
- Checkerboard free numerical schemes
  - High order FE elements
  - Filtering density field solutions ➔ lower order density fields
  - Perimeter constraint



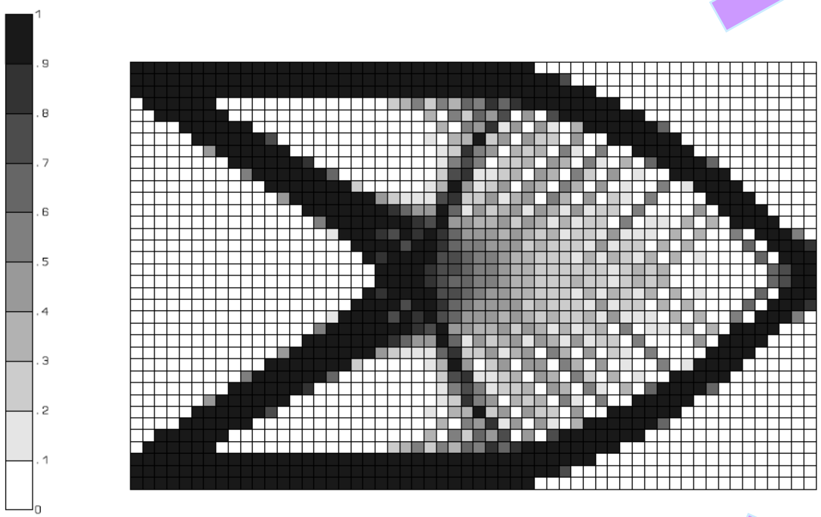




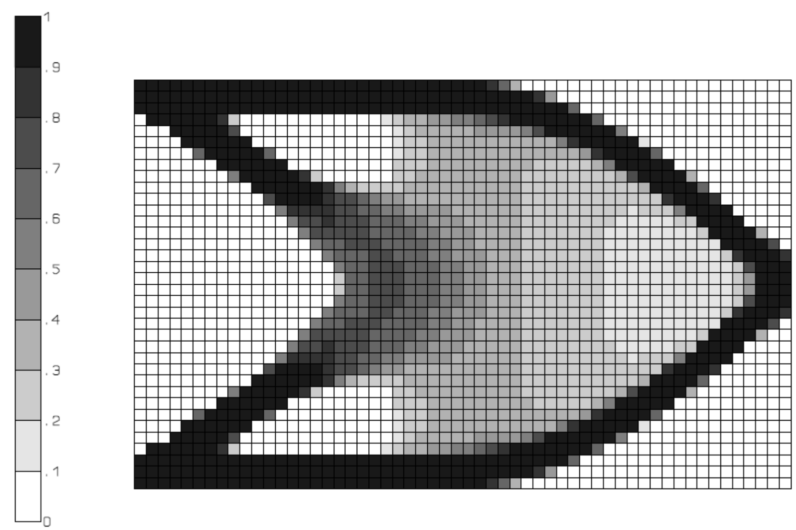
Checkerboard patterns



FE u: degree 2 / Density : constant

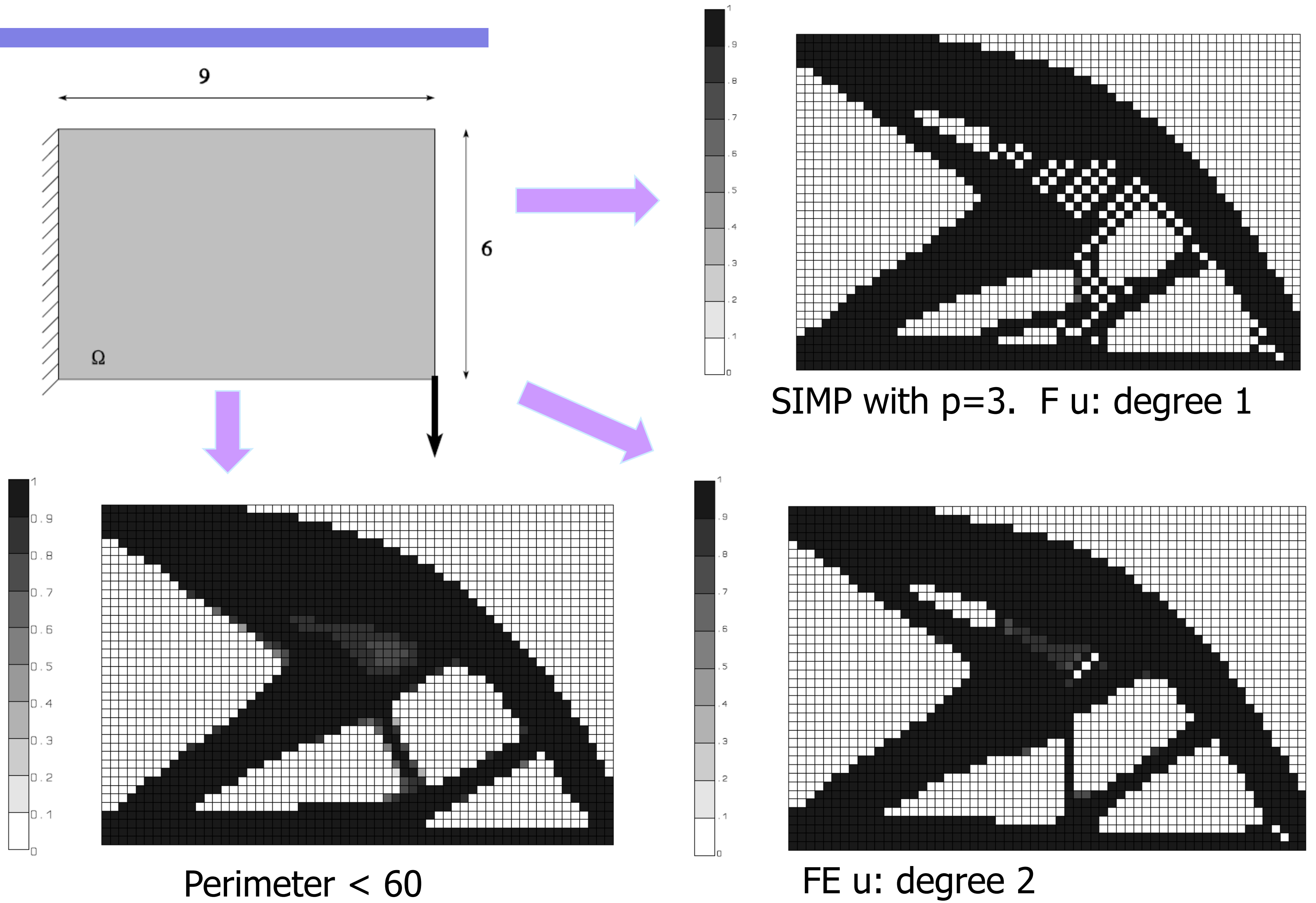


Solution with checkerboards
SIMP with p=2
FE u: degree 1 / Density : constant

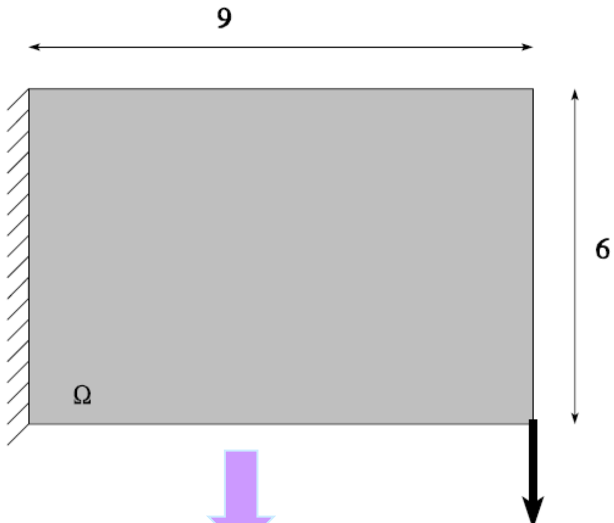


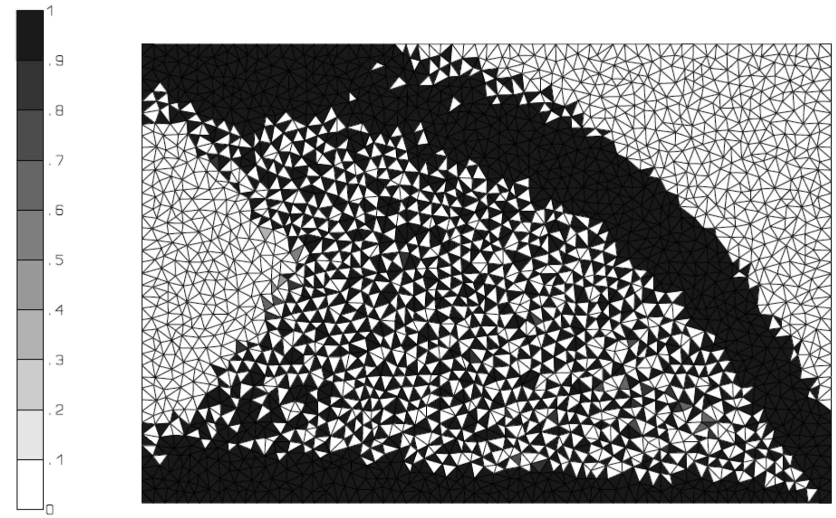
With perimeter constraint

Checkerboard patterns

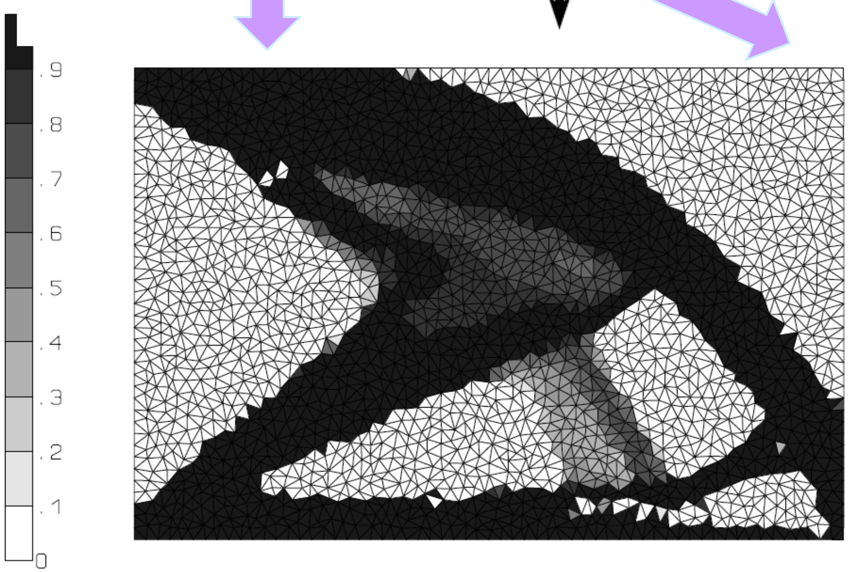


Checkerboard patterns

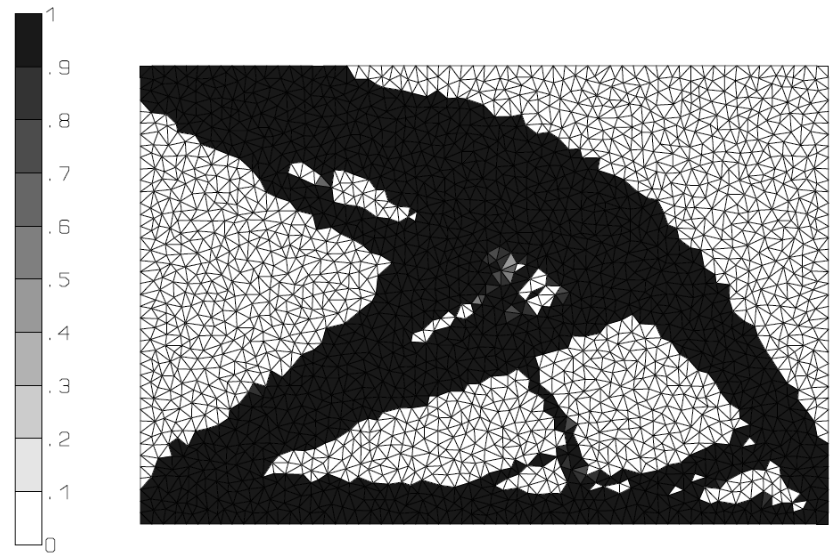




SIMP with p=3. FE u: degree 1



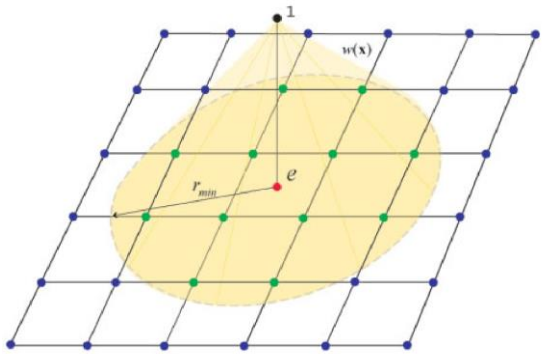
Perimeter < 60



FE u: degree 2

Mesh dependency

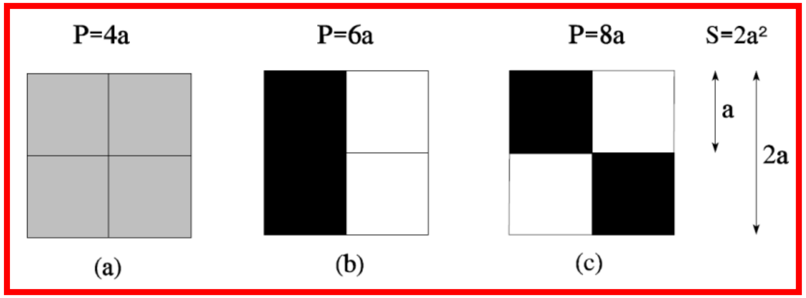
- Mesh independent solution: insure mesh independent filtering of lower size details
  - Low pass filter [Sigmund (1998)]



$$\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X)\, v_i\, x_i}{\sum_i w_i(X_i)\, v_i}$$

  - Perimeter constraint [Ambrosio & Butazzo (1993)]

$$\begin{aligned} \min_{0 \le \rho(\boldsymbol{x}) \le 1} \quad & \boldsymbol{g}^T \boldsymbol{q} \\ \text{s.t.} \quad & P(\rho(\boldsymbol{x})) \le \bar{P} \\ & V = \int_\Omega \rho(\boldsymbol{x})\, d\Omega \le \bar{V} \\ & g_j(\boldsymbol{x}) \le \bar{g}_j \qquad j = 1 \ldots m \end{aligned}$$



PERIMETER METHOD

- Continuous version of perimeter measure

$$P(\rho) = \int_{\Omega} ||\nabla \rho|| d\Omega + \int_{\Gamma_j} ||[\rho]||_j \; d\Gamma$$

  – With the gradient of the density field and the jump $[]_j$ of the density across discontinuity surfaces j

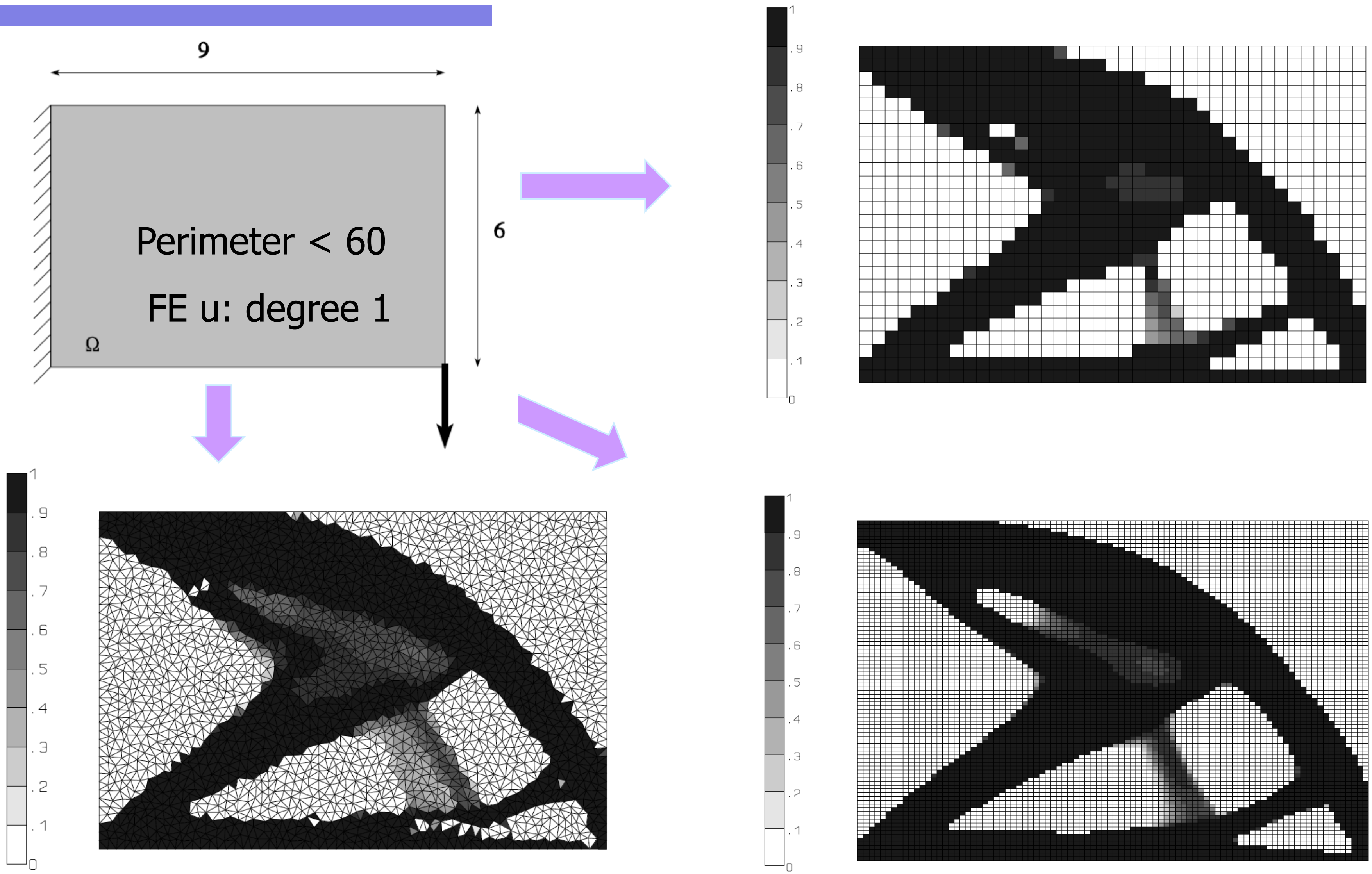
- The continuous approximation of the modulus of the gradient

$$P(\rho) \simeq \int_{\Omega} \sqrt{\nabla \rho^T \nabla \rho + \frac{\varepsilon^2}{h^2}} - \frac{\varepsilon}{h} \; d\Omega + \int_{\Gamma_j} \sqrt{[\rho]_j^2 + \varepsilon^2} - \varepsilon \; d\Gamma$$

x

$||\nabla x||$

Mesh independency

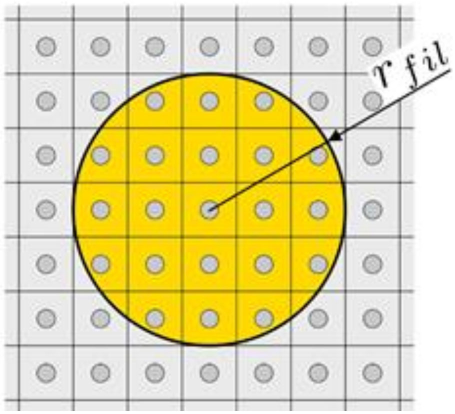


FILTERING MATERIAL DENSITIES

- To avoid mesh dependency and numerical instabilities like checkerboards patterns, one approach consists in restricting the design space of solutions by forbidding high frequency variations of the density field.

- Basic density filtering by Bruns and Tortorelli (2001), proven by Bourdin (2001)

$$\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X)\, v_i\, x_i}{\sum_i w_i(X_i)\, v_i}$$



$$N_e = \{ i \mid ||\mathbf{X}_i - \mathbf{X}_e|| \leq R \}$$

$$w_i(\mathbf{X}_i) = \begin{cases} R - \; ||\mathbf{X}_i - \mathbf{X}_e|| & i \in N_e \\ 0 \quad i \notin N_e \end{cases}$$

FILTERING MATERIAL DENSITIES

- Other weighting functions
  - Gaussian

$$w(\mathbf{X}_i) = \exp\left[-\frac{1}{2}\left(\frac{||\mathbf{X}_i - \mathbf{X}_e||}{\sigma_d}\right)^2\right]$$

  - Constant

$$w(\mathbf{X}_i) = 1$$

- Density filter is equivalent to solving a Helmotz equation [Lazarov & Sigmund (2010)]

$$-R_{\text{min}}\,\nabla^2\bar{\delta} \;+\; \bar{\delta} \;=\; \delta$$

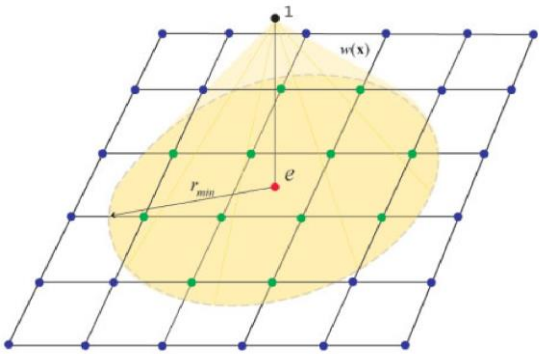
- With the following Neuman boundary conditions

$$\frac{\partial\bar{\delta}}{\partial n} \;=\; 0$$

FILTERING MATERIAL DENSITIES

- Historically Ole Sigmund (1994, 1997) introduced a filter of the sensitivities

$$\frac{\partial \tilde{f}}{\partial x_e} = \frac{\sum_{i \in N_e} w(\mathbf{X}_i) x_i \frac{\partial f}{\partial x_i}}{x_e \sum_{i \in N_e} w(\mathbf{X}_i)}$$



- with

$$w(\mathbf{X}_i) = R - ||\mathbf{X}_i - \mathbf{X}_e||$$

- For non uniform meshes, Sigmund proposed to use

$$\frac{\partial \tilde{f}}{\partial x_e} = \frac{\sum_{i \in N_e} w(\mathbf{X}_i) \frac{x_i}{v_i} \frac{\partial f}{\partial x_i}}{\frac{x_e}{v_e} \sum_{i \in N_e} w(\mathbf{X}_i)}$$

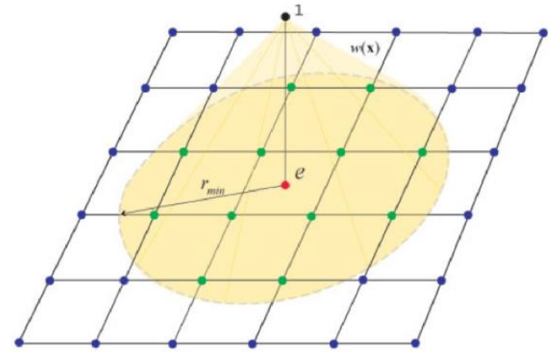
- The smoothed sensitivities correspond to the sensitivities of a smoothed version of the objective function (as well as the constraints)

FILTERING MATERIAL DENSITIES

x

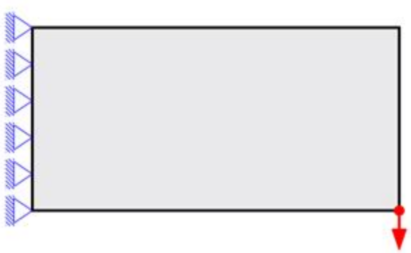
$\tilde{x}(r_{fil} = 8)$

$\tilde{x}(r_{fil} = 15)$



$$E_i = x_i^p E_0$$

a)



(b)

(d)

$$E_i = \tilde{x}_i^p E_0$$

(e)

$$E_i = \tilde{x}_i^p E_0$$

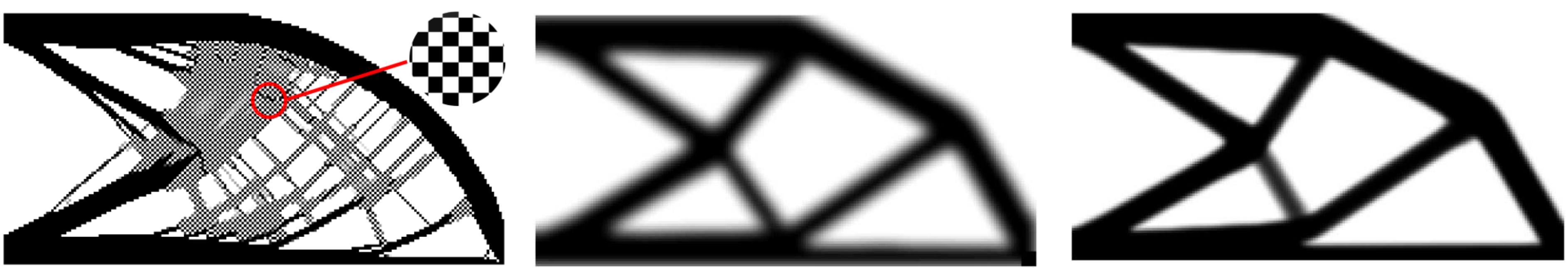
- **~~Mesh dependent~~**
- **~~Checkerboard~~**
- **Non-Discrete Solution (intermediate densities!)**

HEAVISIDE FILTER

- To obtain 0/1 solutions , Guest et al. (2014) modifies the density filter with a Heaviside function such that if $x_e>0$, the Heaviside gives a physical value of the density equal to ‘1’ and if the $x_e=0$, the Heaviside gives a density ‘0’

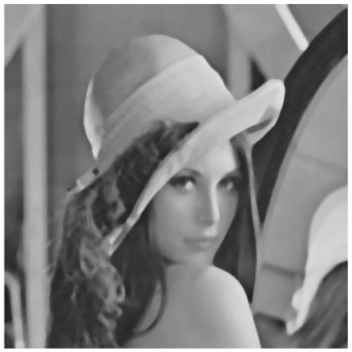
$$\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X)\, v_i\, x_i}{\sum_i w_i(X_i)\, v_i}$$

x $\tilde{x}$ $\hat{x}$



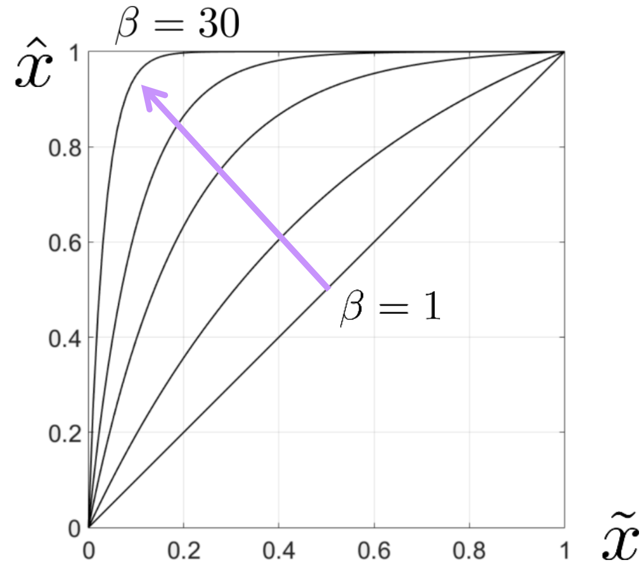
$$\hat{x}_i = \begin{cases} 1 & \text{if } \tilde{x}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

HEAVISIDE FILTER



x

$\hat{x}$



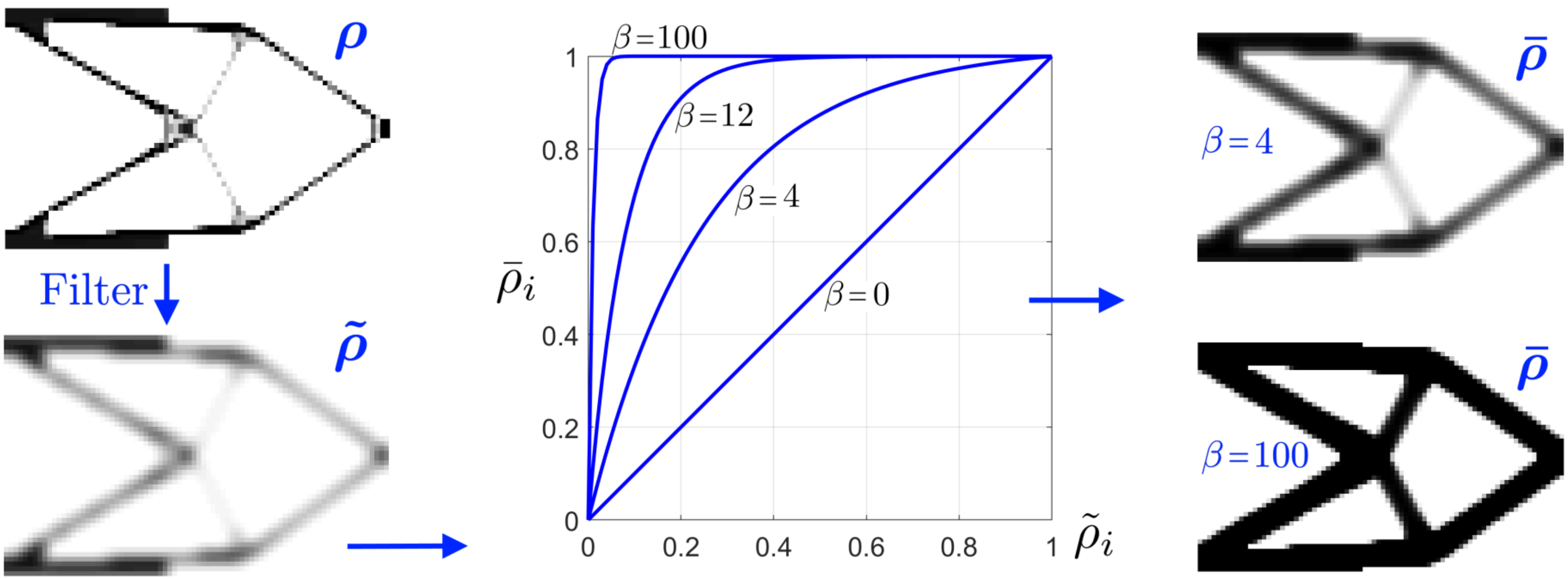
- Heaviside smooth approximation

$$\hat{x}_i = \begin{cases} 1 & \text{if } \tilde{x}_i > 0 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad \hat{x}_i = 1 - e^{-\beta \tilde{x}_i} + \tilde{x}_i \, e^{-\beta}$$

  - For β→ 0, the filter gives the original filter
  - For β→ infinity, the function reproduces the max operator, that is the density becomes 1 if there is any element in the neighborhood that is nonzero.

HEAVISIDE FILTER

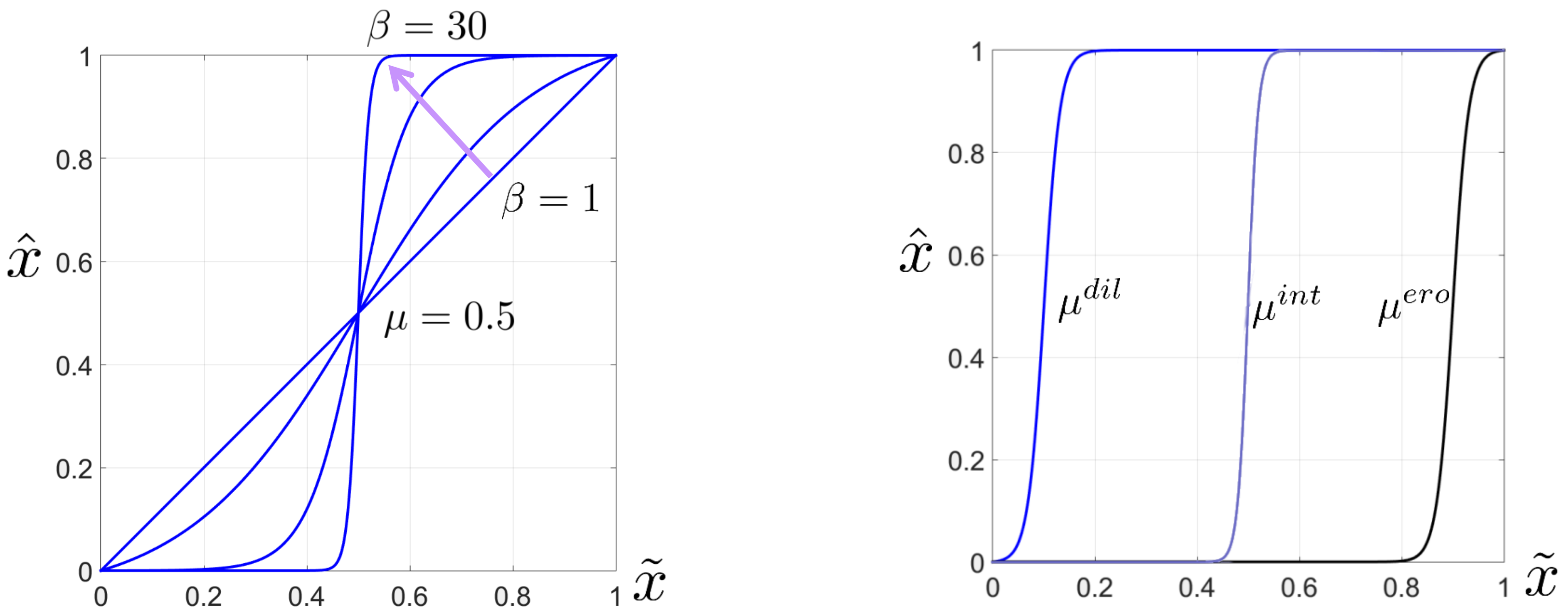
- Heaviside smooth approximation $\hat{x}_i = 1 - e^{-\beta \tilde{x}_i} + \tilde{x}_i \, e^{-\beta}$



- **~~Mesh dependent~~**
- **~~Checkerboard~~**
- **~~Non-Discrete Solution~~**
- **Need of continuation / Large number of iterations (>100)**

HEAVISIDE FILTER

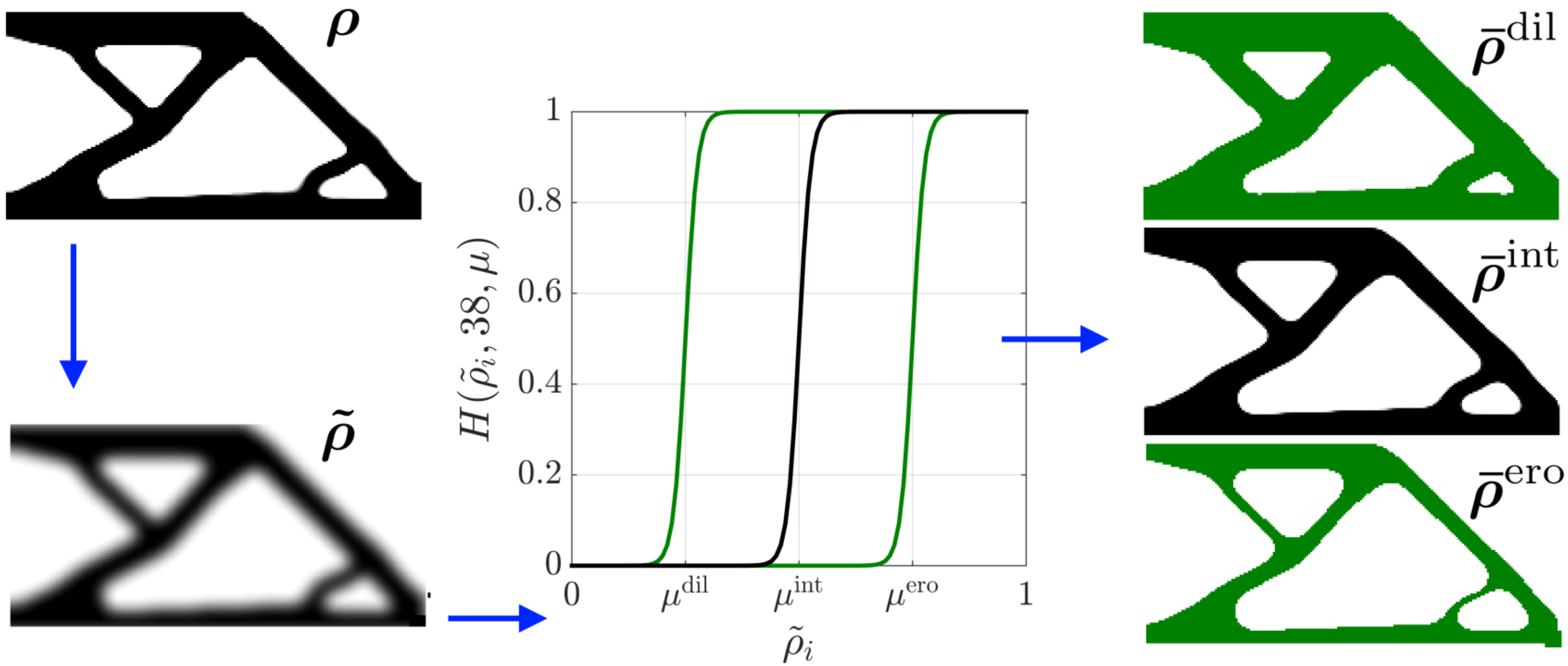
- Heaviside function can be extended [Wang, Lazarov, Sigmund, 2011] to control minimum and maximum length scale



$$\hat{x}_e = \frac{\tanh(\beta\,\eta) \;+\; \tanh(\beta\,(\tilde{x}_e - \eta))}{\tanh(\beta\,\eta) + \tanh(\beta\,(1.0 - \eta))}$$

HEAVISIDE FILTER

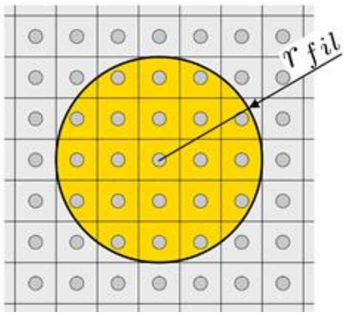
- Heaviside function enables a control of manufacturing tolerant designs ➔ robust design



$$\hat{x}_e = \frac{\tanh(\beta\,\eta) \;+\; \tanh(\beta\,(\tilde{x}_e - \eta))}{\tanh(\beta\,\eta) + \tanh(\beta\,(1.0 - \eta))}$$

THE THREE FIELD APPROACH

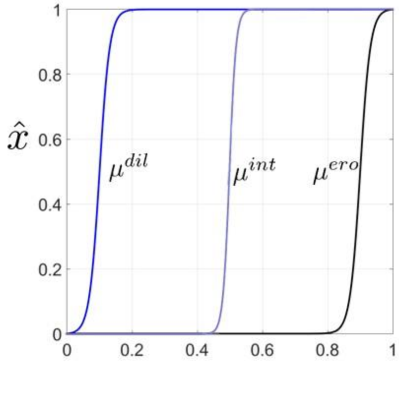
- Combining density filtering and Heaviside filter give rise to the so called three field topology optimization scheme proposed by Wang et al. (2011), one uses a design field, a filtered field and a physical field whose relations are defined though the following filter and thresholding processes
  - Filtering



$$\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X)\, v_i\, x_i}{\sum_i w_i(X_i)\, v_i}$$

  - Heaviside

$$\hat{x}_e = \frac{\tanh(\beta\,\eta) + \tanh(\beta\,(\tilde{x}_e - \eta))}{\tanh(\beta\,\eta) + \tanh(\beta\,(1.0 - \eta))}$$



SOLVING TOPOLOGY OPTIMIZATION PROBLEM USING OPTIMALITY CRITERIA

OPTIMALITY CRITERIA

- One considers the fundamental problem of compliance minimization subject to a volume constraint

$$
\begin{array}{cl}
\min\limits_{0 \leq \mathbf{x} \leq 1} & C(\mathbf{x}) \\
\text{s.t.:} & V(\mathbf{x}) \ \leq \ \bar{V}
\end{array}
$$

- Compliance is an implicit, nonlinear function of the density variables. To do so one can either
  – Use the virtual work principle

  Or

  – Make a first order Taylor expansion with respect to the inverse (reciprocal) variables.

OPTIMALITY CRITERIA

- Compliance is the strain energy at equilibrium

$$C = u \times f = \mathbf{q}^T \mathbf{g} = \mathbf{q}^T \mathbf{K} \mathbf{q}$$

- Decompose into FE element contributions

$$C = \mathbf{q}^T \mathbf{K} \mathbf{q} = \sum_i \mathbf{q}_i^T \mathbf{K}_i \mathbf{q}_i$$

- Stiffness matrix dependency on density variables

$$\mathbf{K} = \sum_i x_i^p \bar{\mathbf{K}}_i$$

- And

$$C = \sum_i \mathbf{q}_i^T \mathbf{K}_i \mathbf{q}_i = \sum_i x_i^p \mathbf{q}_i^T \bar{\mathbf{K}}_i \mathbf{q}_i$$

OPTIMALITY CRITERIA

- For isostatic structures, for which the load vector remains constant, we have also that the element displacement vector of an inverse function of the design variable:

$$\mathbf{K}_i \div x_i^p$$

$$\mathbf{q}_i \div \frac{1}{x_i^p}$$

- Therefore the following flexibility coefficients remain constant for isostatic structures.

$$c_i = x_i^p \, \mathbf{q}_i^T \mathbf{K}_i \, \mathbf{q}_i = x_i^{2p} \, \mathbf{q}_i^T \bar{\mathbf{K}}_i \, \mathbf{q}_i$$

- For hyperstatic structures, it does not remain constant, but it is reasonable to assume that generally it does change too much from one iteration to another

OPTIMALITY CRITERIA

- Therefore one can define:

$$c_i \;=\; \left(\mathbf{q}^T\,\mathbf{K}_i\,\mathbf{q}\right)\; x_i^p$$

- c_i is constant for a statically determinate structure. We get an explicit expression which is a first order approximation of the compliance in the neighborhood of the current design point:

$$\boxed{C \;=\; \sum_i \frac{c_i}{x_i^p}}$$

OPTIMALITY CRITERIA

- It can be noticed from now that the same expression can be obtained if we perform a first order Taylor expansion of the compliance with respect to the intermediate variables

$$y_i = \frac{1}{x_i^p}$$

- The Taylor expansion of the compliance writes

$$\begin{aligned}
C &\simeq C(\mathbf{x}_0) + \sum_i \frac{\partial C}{\partial y_i}(\mathbf{x}_0)\,(y_i - y_i^0) \\
&\simeq C_0 + \sum_i \frac{\partial C}{\partial x_i}\frac{\partial x_i}{\partial y_i}\left(\frac{1}{x_i^p} - \frac{1}{x_i^{0\;p}}\right) \\
&\simeq C_0 + \sum_i -\frac{x_i^{0\;p+1}}{p}\frac{\partial C}{\partial x_i}\left(\frac{1}{x_i^p} - \frac{1}{x_i^{0\;p}}\right) \\
&\simeq \left(C_0 - \sum_i -\frac{x_i^{0\;p+1}}{p}\frac{\partial C}{\partial x_i}\frac{1}{x_i^{0\;p}}\right) + \sum_i -\frac{x_i^{0\;p+1}}{p}\frac{\partial C}{\partial x_i}\frac{1}{x_i^p}
\end{aligned}$$

OPTIMALITY CRITERIA

- One can show that

$$C_0 - \sum_i -\frac{x_i^{0\;p+1}}{p} \frac{\partial C}{\partial x_i} \frac{1}{x_i^{0\;p}} = 0$$

- Let's define the coefficient

$$c_i = -\frac{x_i^{0\;p+1}}{p} \frac{\partial C}{\partial x_i}$$

- so that we get exactly the same explicit expression of the compliance

$$C = \sum_i \frac{c_i}{x_i^p}$$

OPTIMALITY CRITERIA

- We can show that this last expression is exactly the same as the first one because, the coefficients c_i are those that we defined previously using the engineering approach based on the decomposition of the strain energy

$$\begin{aligned}
\frac{\partial C}{\partial x_i} &= -\,\mathbf{q}_i^T \,\frac{\partial \mathbf{K}_i}{\partial x_i}\mathbf{q}_i \\
&= -\,\mathbf{q}_i^T \,p\, x_i^{p+1}\bar{\mathbf{K}}_i\,\mathbf{q}_i \\
&= -\,\mathbf{q}_i^T \,\frac{p}{x_i}\mathbf{K}_i\,\mathbf{q}_i
\end{aligned}$$

- We have

$$\begin{aligned}
c_i &= -\frac{x_i^{0\;p+1}}{p}\,\frac{\partial C}{\partial x_i} \\
&= -\frac{x_i^{0\;p+1}}{p}\,(-1)\,\mathbf{q}_i^T\,\frac{p}{x_i^0}\mathbf{K}_i\,\mathbf{q}_i \\
&= +\,x_i^{0\;p}\,\mathbf{q}_i^T\,\mathbf{K}_i\,\mathbf{q}_i \;=\; +\,x_i^{0\;2p}\,\mathbf{q}_i^T\,\bar{\mathbf{K}}_i\,\mathbf{q}_i
\end{aligned}$$

OPTIMALITY CRITERIA

- The fundamental compliance minimization problem

$$
\begin{array}{cl}
\min\limits_{0 \le \mathbf{x} \le 1} & C(\mathbf{x}) \\
\text{s.t.:} & V(\mathbf{x}) \;\le\; \bar{V}
\end{array}
$$

- Using approximation concepts, we get an explicit (sub)-problem that can be solved more efficiently may be at the price of repeating iteratively the process of generating the subproblems:

$$
\begin{array}{cl}
\min\limits_{0 \le \mathbf{x} \le 1} & C \;=\; \sum_{i=1}^{n} \frac{c_i}{x_i^p} \\
\text{s.t.:} & V(\mathbf{x}) = \sum_{i=1}^{n} v_i \, x_i \;\le\; \bar{V}
\end{array}
$$

OPTIMALITY CRITERIA

- Let introduce the Lagrange multiplier λ and let's shape the Lagrange function

$$L(x_i, \lambda) = \sum_{i=1}^{n} \frac{c_i}{x_i^p} + \lambda \left(\sum_{i=1}^{n} v_i \, x_i - \bar{V} \right)$$

- Stationary conditions (from KKT conditions)

$$\frac{\partial}{\partial x_i} L(x_i, \lambda) = 0$$

- It gives

$$c_i \, \frac{-p}{x_i^{p+1}} + \lambda \, v_i = 0$$

OPTIMALITY CRITERIA

- The stationarity conditions gives:

$$\boxed{x_i^{p+1} = \frac{p\, c_i}{\lambda\, v_i}}$$

- If $c_i>0$, the x_i variable is called as active.

- If $c_i<0$, the Langrangian function is monotonic increasing, so that the minimum is

$$x_i = \underline{x}_i = 0$$

- The variable is said passive.

OPTIMALITY CRITERIA

- To identify the Lagrange variable, one substitutes the $x_i(\lambda)$ by its value into the volume constraint that is satisfied as an equality constraint:

$$\sum_{i=1}^{n} v_i \, x_i(\lambda) \;=\; \bar{V}$$

- It comes

$$\begin{aligned} V \;&=\; \sum_{i=1}^{n} v_i \, x_i(\lambda) \;=\; \bar{V} \\ &=\; \sum_{\{i|c_i \leq 0\}} v_i \, \underline{x}_i \;+\; \sum_{\{i|c_i > 0\}} v_i \, x_i(\lambda) \;=\; \bar{V} \end{aligned}$$

- And

$$\overline{V} \;-\; V_0 \;=\; \sum_{\{i|c_i > 0\}} c_i \, x_i(\lambda) \;=\; \sum_{\{i|c_i > 0\}} v_i \; \sqrt[p+1]{\frac{p\, c_i}{\lambda \, v_i}}$$

- with

$$V_0 \;=\; \sum_{\{i|c_i \leq 0\}} c_i \, \underline{x}_i$$

OPTIMALITY CRITERIA

- We have

$$\overline{V} - V_0 = \sum_{\{i|c_i>0\}} c_i \, x_i(\lambda) = \sum_{\{i|c_i>0\}} v_i \sqrt[p+1]{\frac{p \, c_i}{\lambda \, v_i}}$$

- It can be solved analytically or numerically

$$\frac{1}{\sqrt[p+1]{\lambda}} \sum_{\{i|c_i>0\}} v_i \sqrt[p+1]{\frac{p \, c_i}{v_i}} = \bar{V} - V_0$$

$$\sqrt[p+1]{\lambda} = \frac{\left(\sum_{k=1}^{\tilde{n}} v_i \sqrt[p+1]{\frac{p \, c_i}{v_i}}\right)}{\bar{V} - V_0} \qquad \lambda = \frac{\left(\sum_{k=1}^{\tilde{n}} v_i \sqrt[p+1]{\frac{p \, c_i}{v_i}}\right)^{p+1}}{(\bar{V} - V_0)^{p+1}}$$

- Finally the optimized value is reused into the primal variables

$$x_i = \sqrt[p+1]{\frac{p \, c_i}{\lambda \, v_i}}$$

OPTIMALITY CRITERIA

- For statically determinate case, the c_i remain constant and one structural analysis and reach the optimum. For statically indeterminate case c_i is not constant, one has to use an iterative scheme.

$$
\begin{aligned}
x_i^{p+1} &= \frac{p\,c_i}{\lambda\,v_i} \\
&= x_i^{p+1}\,\frac{p\,c_i}{\lambda\,v_i\,x_i^{p+1}}
\end{aligned}
$$

- And the iteration scheme

$$
\boxed{x_i^{(k+1)} = x_i^{(k)}\,\sqrt[p+1]{\frac{p\,c_i}{\lambda\,v_i\,x_i^{(k)\,p+1}}}}
$$

OPTIMALITY CRITERIA

- If we remember that

$$\frac{p\,c_i}{x_i^{p+1}} = \frac{p}{x_i^{p+1}}\,\frac{\partial C}{\partial x_i}\,\frac{-x_i^{p+1}}{p} = -\frac{\partial C}{\partial x_i} = \mathbf{q}_i^T\,\frac{\partial \mathbf{K}_i}{\partial x_i}\,\mathbf{q}_i$$

- It follows

$$\boxed{x_i^{(k+1)} = x_i^{(k)}\,\sqrt[p+1]{\frac{\mathbf{q}_i^T\,\frac{\partial \mathbf{K}_i}{\partial x_i}\,\mathbf{q}_i}{\lambda\,v_i}}}$$

- The quantity has the meaning of strain energy per unit volume

$$\varepsilon_i^{(k)} = \frac{\mathbf{q}_i^T\,\frac{\partial \mathbf{K}_i}{\partial x_i}\,\mathbf{q}_i}{\lambda\,v_i}$$

OPTIMALITY CRITERIA

- For active variables (i.e. $c_i>0$),

$$x_i^{(k+1)} = x_i^{(k)} \sqrt[p+1]{\varepsilon_i^{(k)}/\lambda}$$

- For passive variables (i.e. $c_i<0$), $\varepsilon_i^{(k)} \leq 0$

$$x_i^{(k+1)} = \underline{x}_i$$

- This reminds the iteration scheme proposed by Bendsoe and Kikuchi (1988)

$$x_i^{(k+1)} = \begin{cases} x_{min}, & \text{if} \quad x_i^{(k)} \left(\varepsilon_i^{(k)}/\lambda\right)^\eta \leq x_{min} \\ x_i^{(k+1)} = x_i^{(k)} \left(\varepsilon_i^{(k)}/\lambda\right)^\eta, & \text{if} \quad x_{min} \leq x_i^{(k)} \left(\varepsilon_i^{(k)}/\lambda\right)^\eta \leq x_{max} \\ x_{max}, & \text{if} \quad x_i^{(k)} \left(\varepsilon_i^{(k)}/\lambda\right)^\eta \geq x_{max} \end{cases}$$

NUMERICAL SOLUTION OF TOPOLOGY PROBLEMS USING GRADIENT BASED MATH PROGRAMMING

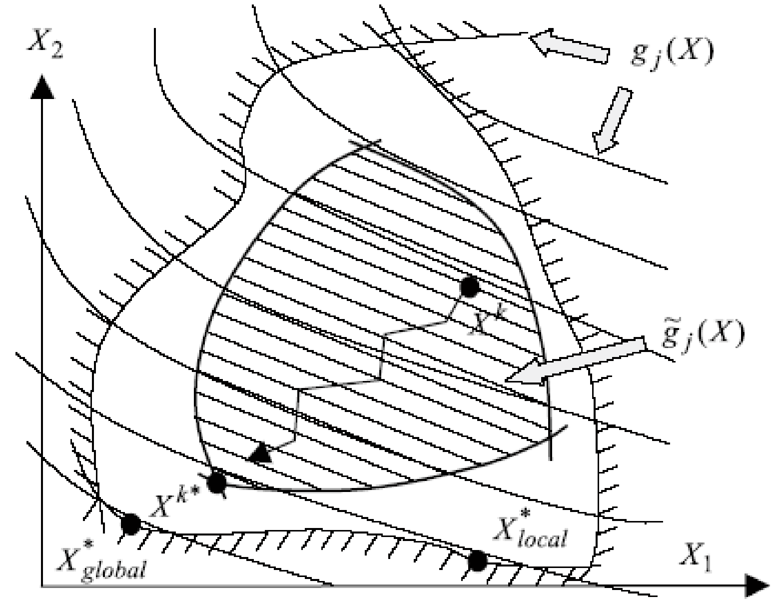
NUMERICAL SOLUTION OF TOPOLOGY OPTIMIZATION PROBLEMS

- Optimal material distribution = very large-scale problem
  - Large number of design variables: 1 000 $\rightarrow$ 100 000
  - Number of restrictions:
    - 1 $\rightarrow$ 10 (for stiffness problems)
    - 1 000 $\rightarrow$ 10 000 (for strength problem with local constraints)

- Solution approach based on the sequential programming approach and mathematical programming
  - Sequence of convex separable problems based on structural approximations
  - Efficient solution of sub problems based on dual maximization

- Major reduction of solution time of optimization problem
- Generalization of problems that can be solved

SEQUENTIAL CONVEX PROGRAMMING APPROACH



Direct solution of the original optimisation problem which is generally non-linear, implicit in the design variables

$$\begin{array}{llll} \min\limits_{\boldsymbol{x}} & g_0(\boldsymbol{x}) & & \\ \text{s.t.} & g_j(\boldsymbol{x}) \leq \bar{g}_j & & j = 1 \ldots m \end{array}$$

is replaced by a sequence of optimisation sub-problems

$$\begin{array}{llll} \min\limits_{\boldsymbol{x}} & \tilde{g}_0(\boldsymbol{x}) & & \\ \text{s.t.} & \tilde{g}_j(\boldsymbol{x}) \leq \bar{g}_j & & j = 1 \ldots m \end{array}$$

by using approximations of the responses and using powerful mathematical programming algorithms

SEQUENTIAL CONVEX PROGRAMMING APPROACH

- Two basic concepts:
  - Structural approximations replace the implicit problem by an explicit optimisation sub-problem using convex, separable, conservative approximations; e.g. CONLIN, MMA
  - Solution of the convex sub-problems: efficient solution using dual methods algorithms or SQP method.
- Advantages of SCP:
  - Optimised design reached in a reduced number of iterations: typically 100 F.E. analyses in topology optimization
  - Efficiency, robustness, generality, and flexibility, small computation time
  - Large scale problems in terms of number of design constraints and variables

STRUCTURAL APPROXIMATIONS

- Convex Linearisation (CONLIN)

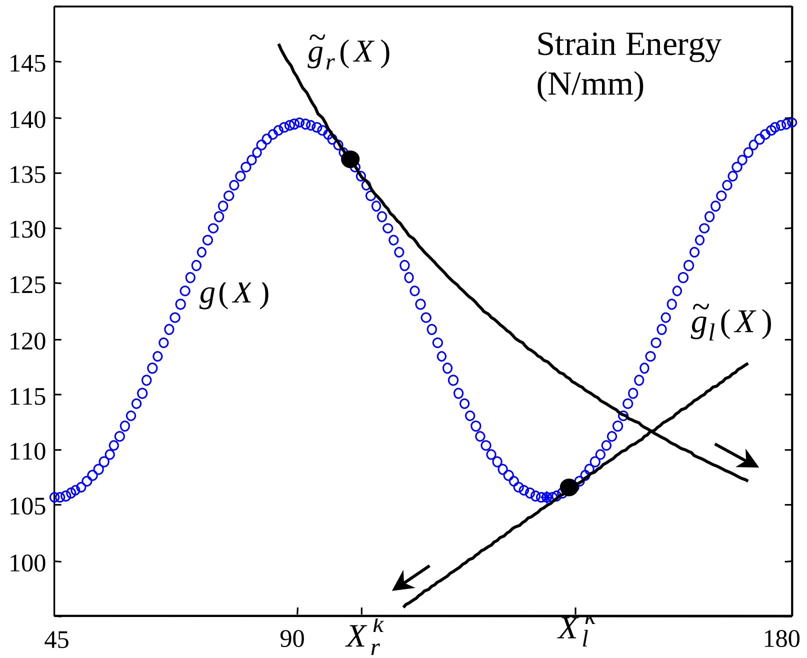
$$\tilde{g}_j(\boldsymbol{x}) = g_j(\boldsymbol{x}^{\mathbf{o}}) + \sum_{+} \frac{\partial g_j}{\partial x_i}(x_i - x_i^0) - \sum_{-} (x_i^0)^2 \frac{\partial g_j}{\partial x_i}(\frac{1}{x_i} - \frac{1}{x_i^0})$$

- Method of Moving Asymptotes (MMA)

$$\tilde{g}_j(\boldsymbol{x}) = r_j^0 + \sum_{i=1}^{n} \frac{p_{ij}}{U_i - x_i} + \sum_{i=1}^{n} \frac{q_{ij}}{x_i - L_i}$$

$$\begin{aligned} p_{ij} &= \max\{0, (U_i - x_i^0)^2 \frac{\partial g_j}{\partial x_i}\} \\ q_{ij} &= \max\{0, -(x_i^0 - L_i)^2 \frac{\partial g_j}{\partial x_i}\} \end{aligned}$$

CONLIN approximation



Approximation of the strain energy in a two plies symmetric laminate subject to shear load and torsion (Bruyneel and Fleury, 2000)

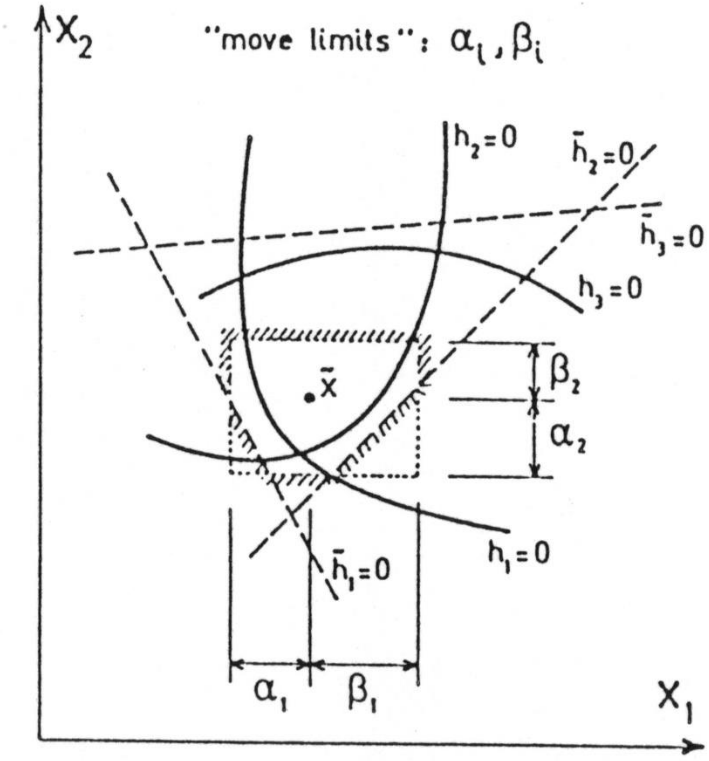
MOVE LIMITS STRATEGY

- Introduce a box constraint around the current design point to limit the variation domain of the design variables

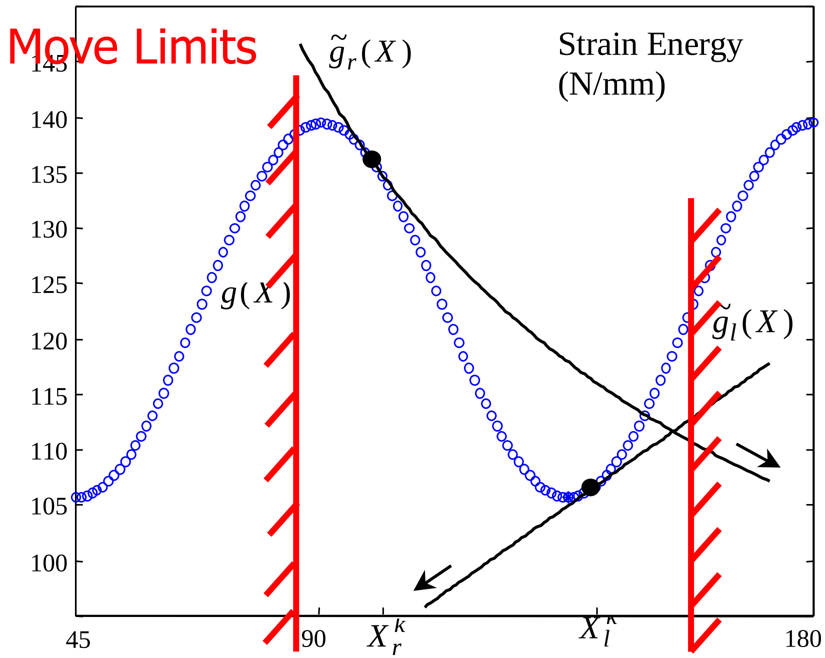
$$\hat{x}_i - \alpha_i \;\leq\; x_i \;\leq\; \hat{x}_i + \beta_i$$

- Of course take the most restrictive constraints with the side constraints

$$\max\{\underline{x}_i, \hat{x}_i - \alpha_i\} \;\leq\; x_i \;\leq\; \min\{\overline{x}_i, \hat{x}_i + \beta_i\}$$



CONLIN approximation +Move Limits



Approximation of the strain energy in a two plies symmetric laminate subject to shear load and torsion (Bruyneel and Fleury, 2000)

DUAL METHODS

- Primal problem

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x})$$
$$\text{s.t.} \quad g_j(\boldsymbol{x}) \leq 0 \qquad j = 1, \dots, m$$

- Lagrange function:

$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \sum_{j=1}^{m} \lambda_j g_j(\boldsymbol{x})$$

- If the problem is convex…

$$\min_{\mathbf{x}} \max_{\lambda \geq 0} L(\mathbf{x}, \lambda) \quad \Longleftrightarrow \quad \max_{\lambda \geq 0} \min_{\mathbf{x}} L(\mathbf{x}, \lambda)$$

DUAL METHODS

- Dual problem

$$\begin{array}{lll} \max\limits_{\lambda_j} & \ell(\lambda) & \\ \text{s.t.} & \lambda_j \;\geq 0 & j = 1, \ldots, m \end{array}$$

  – with

$$\ell(\lambda) \;=\; \min_{\boldsymbol{x} \in X} \quad L(\boldsymbol{x}, \lambda)$$

- Solve Lagrangian problem

$$\boldsymbol{x} \;=\; \boldsymbol{x}(\lambda) \;=\; \arg\min_{\boldsymbol{x}} L(\boldsymbol{x}, \lambda)$$

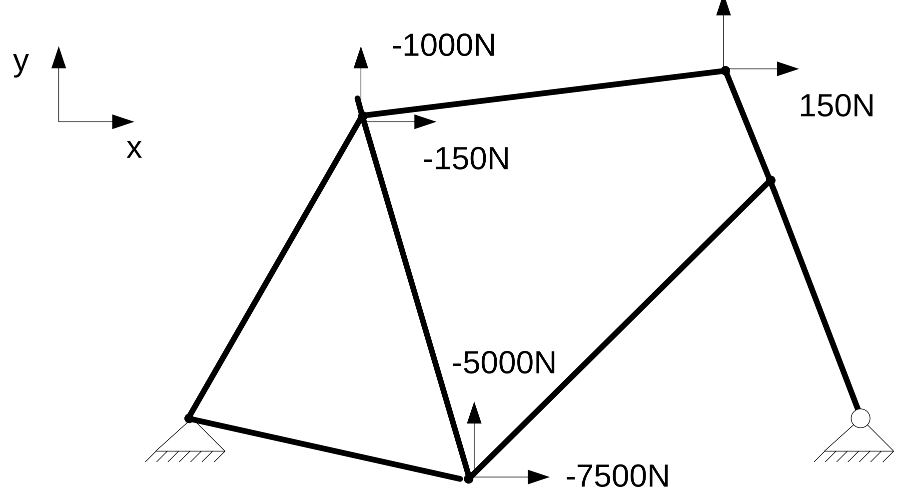
- Lagrangian problem

$$\begin{aligned} \ell(\lambda) &= L(\boldsymbol{x}(\lambda), \lambda) \\ &= f(\boldsymbol{x}(\lambda)) + \sum_{j=1}^{m} \lambda_j \, g_j(\boldsymbol{x}(\lambda)) \end{aligned}$$

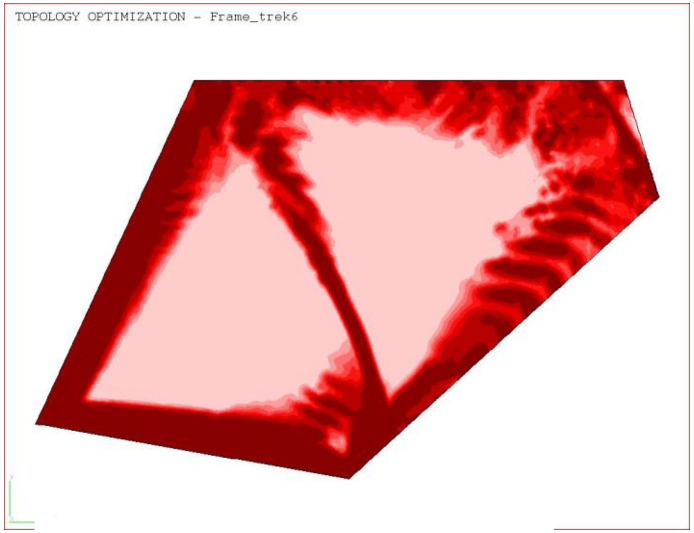
NUMERICAL APPLICATIONS OF COMPLIANCE MINIMIZATION BASED TOPOLOGY OPTIMIZATION

Optimization of a maximum stiffness bicycle frame

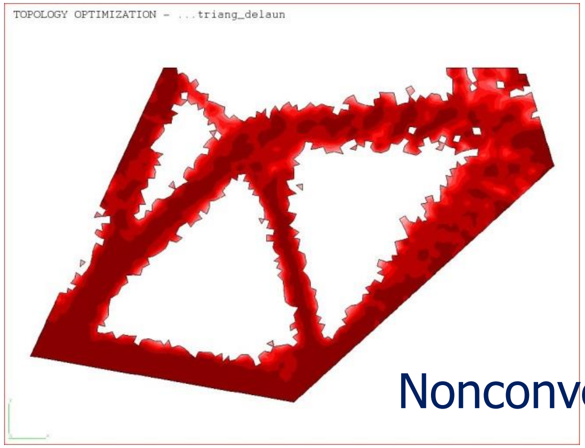
Load cases



Conventional design

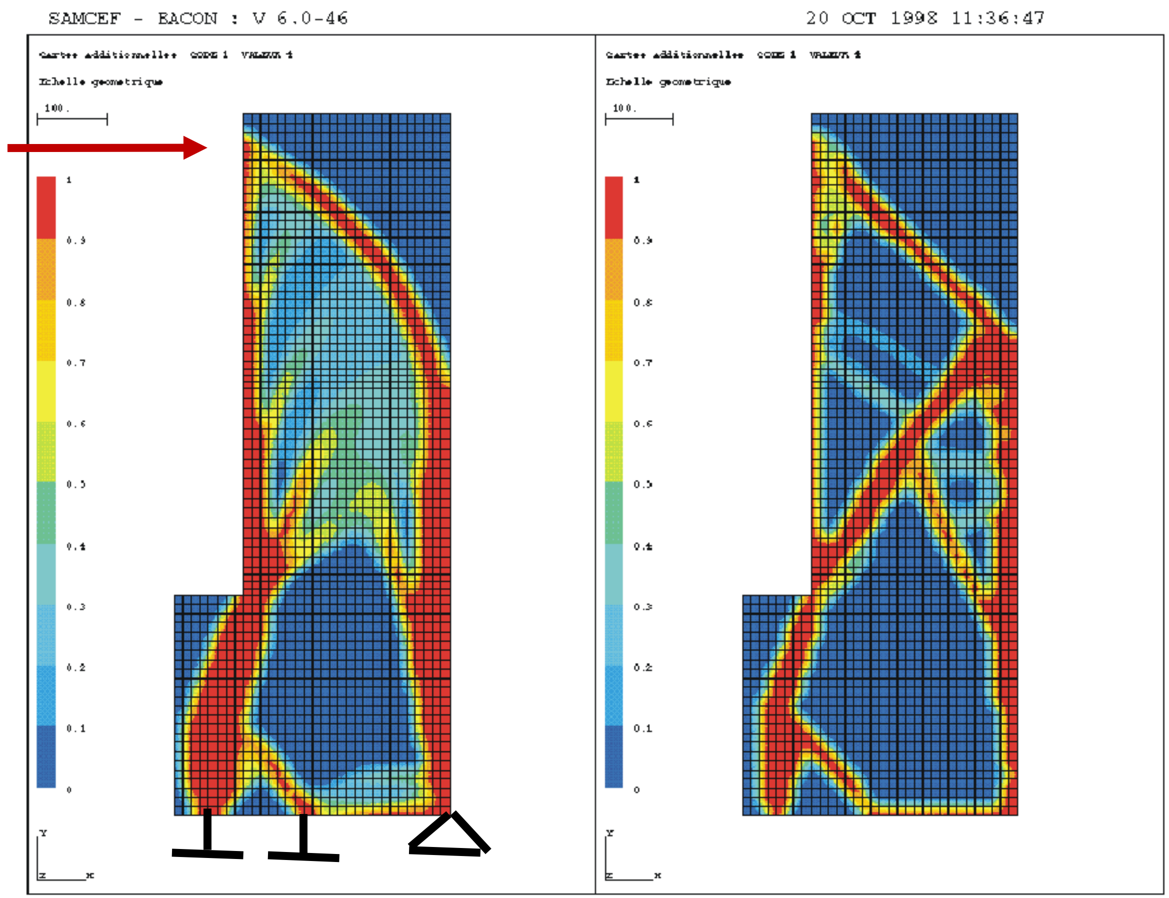


Optimum topology

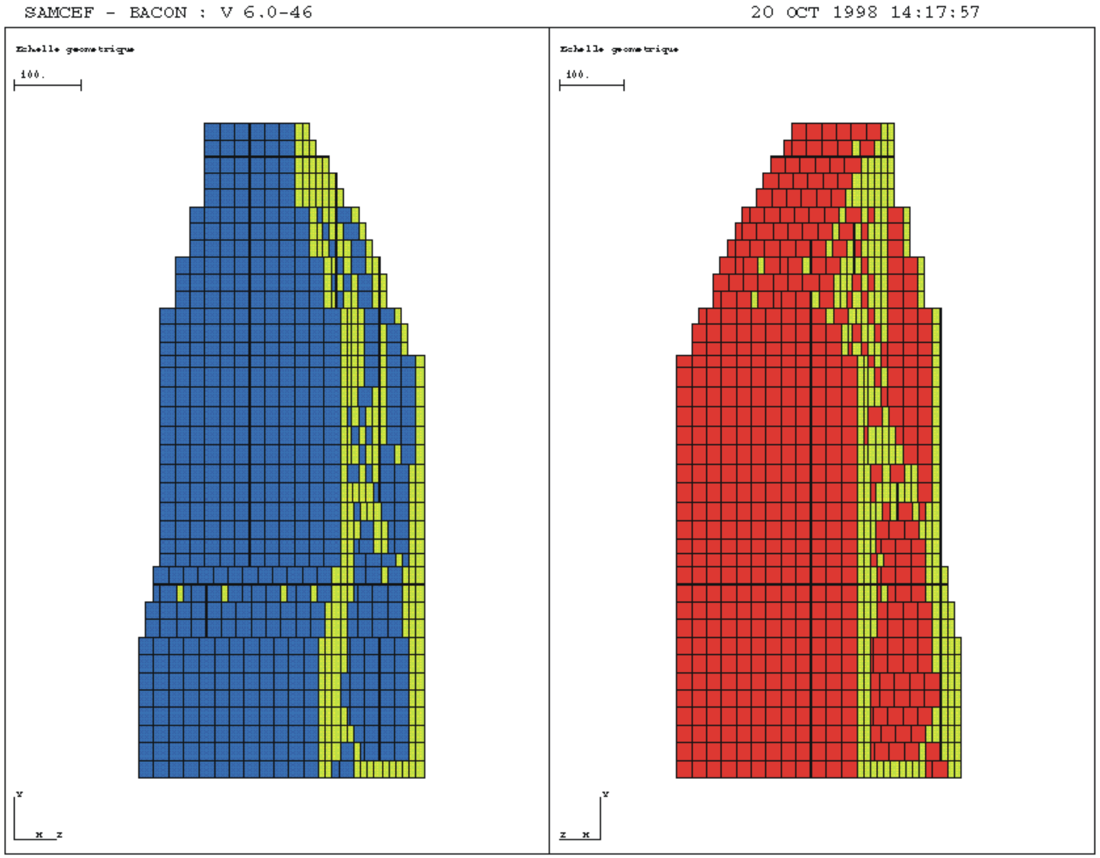


Nonconventional design

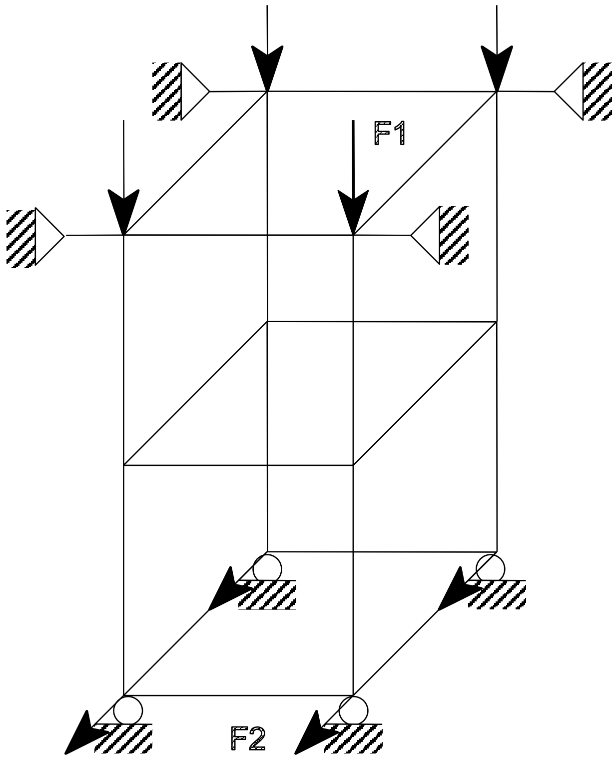
Design of a Crash Barrier Pillar (SOLLAC)

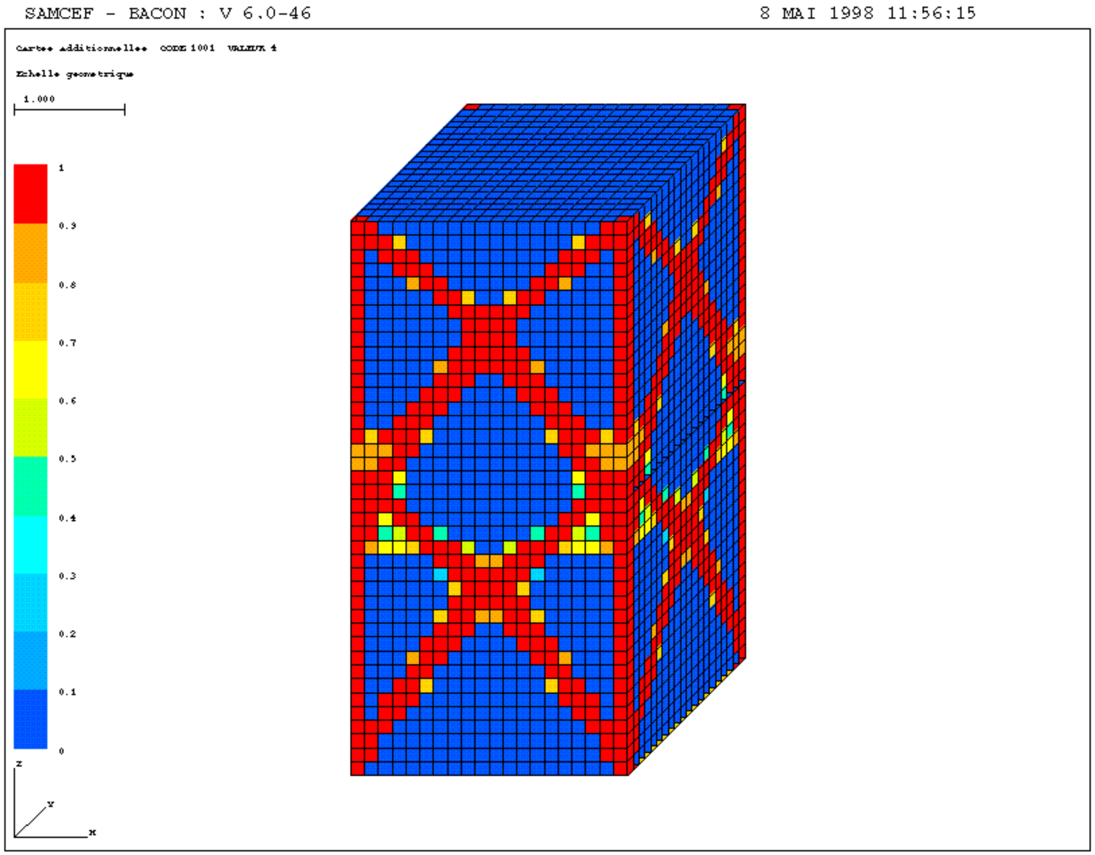


Design of a Crash Barrier Pillar (SOLLAC)

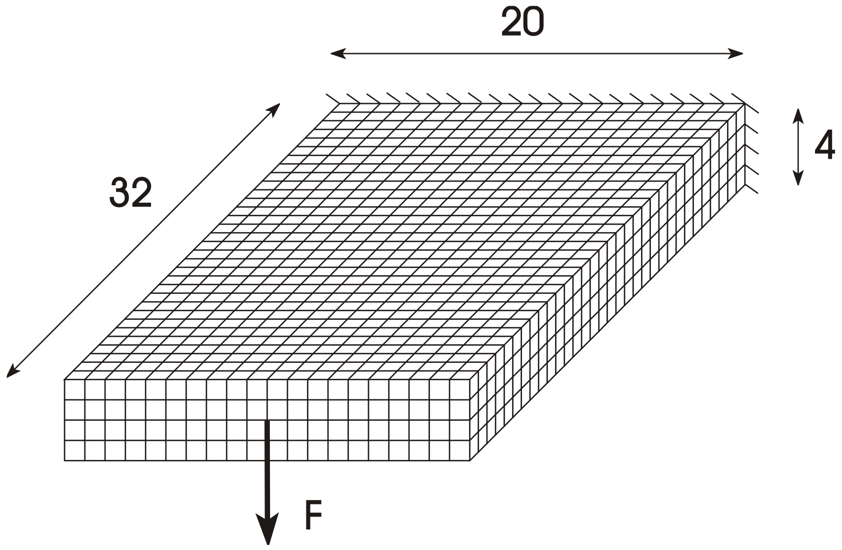


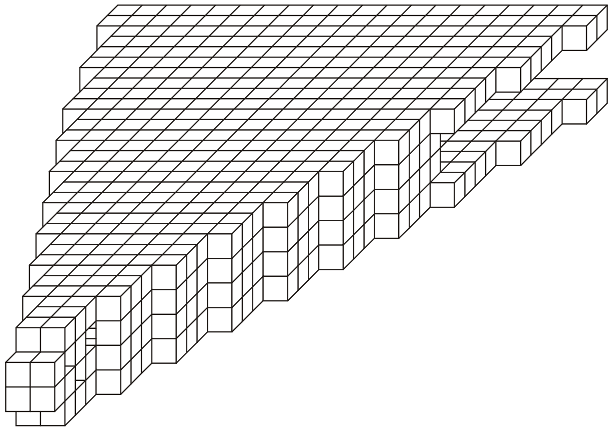
Topology Optimization of a Parasismic Building (DOMECO)





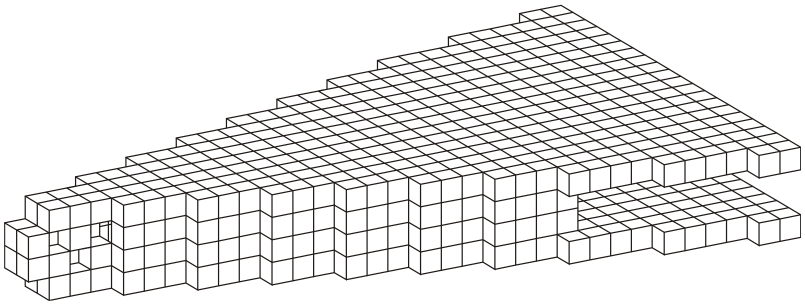
3D cantilever beam problem





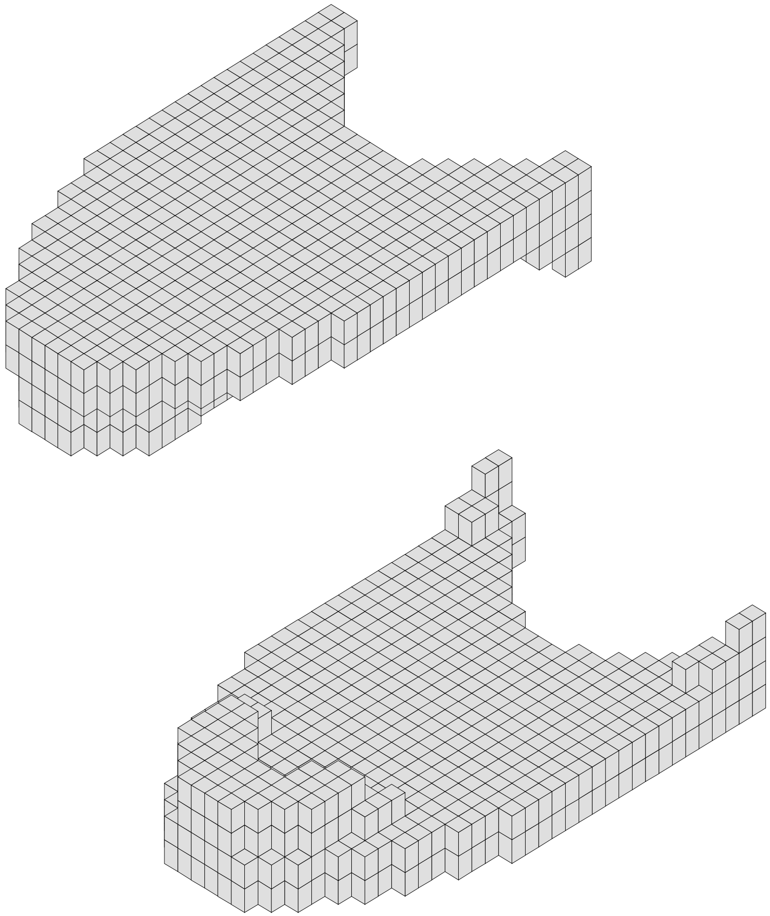
E=100 N/m², ν=0.3

20 x 32 x 4 = 2560 F.E.s

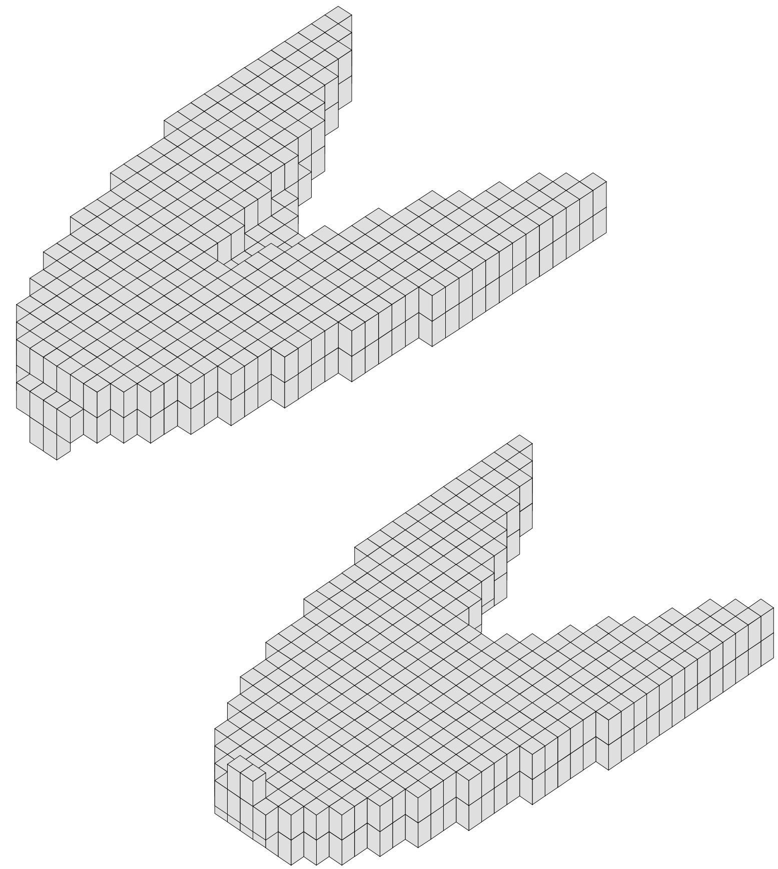


No perimeter constraint

3D cantilever beam problem



Perimeter = 1400

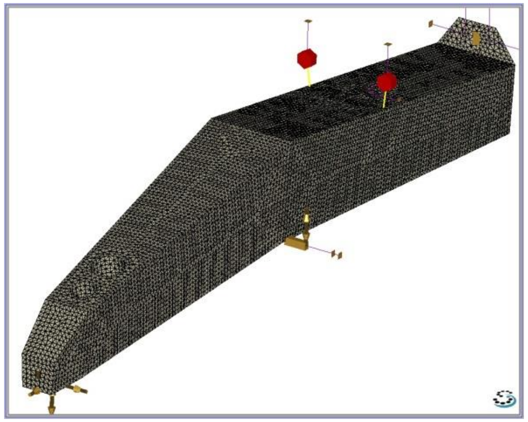


Perimeter = 1000

An industrial application: Airbus engine pylon

- Application
  - carried out by SAMTECH and ordered by AIRBUS
- Engine pylon
  = structure fixing engines to the wing

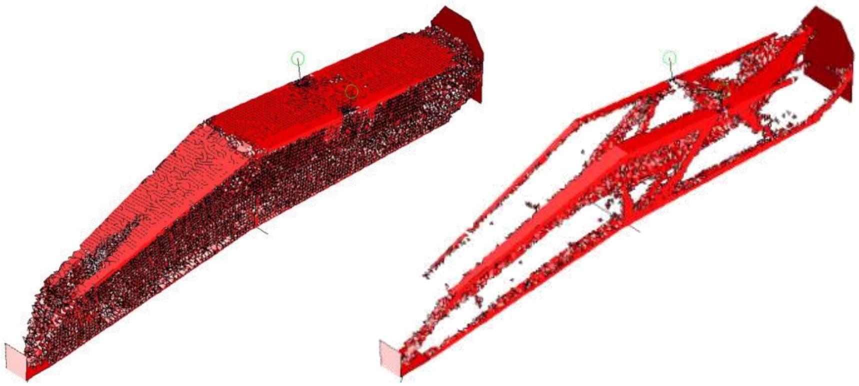
- Initial Model
  - CATIA V5 import → Samcef Model
  - BC's: through shell and beam FE
  - 10 load cases:
    - GUSTS
    - FBO (Fan blade out)
    - WUL (Without undercarriage landing)



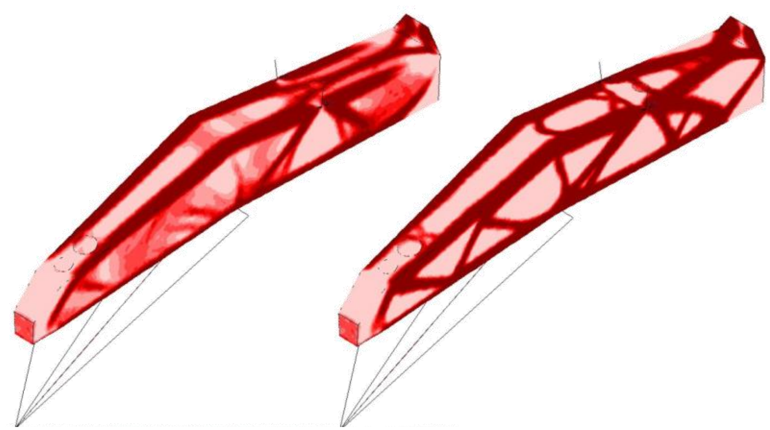
Over 250.000 tetraedral FE

An industrial application: Airbus engine pylon

- Target mass: 10%
- Additional constraints:
  - Engine CoG position
- Optimization parameters
  - Sensitivity filtering: (Sigmund's filter)
  - Symmetry (left right) condition
  - Penalty factor
- CONLIN optimizer: special version for topology optimization

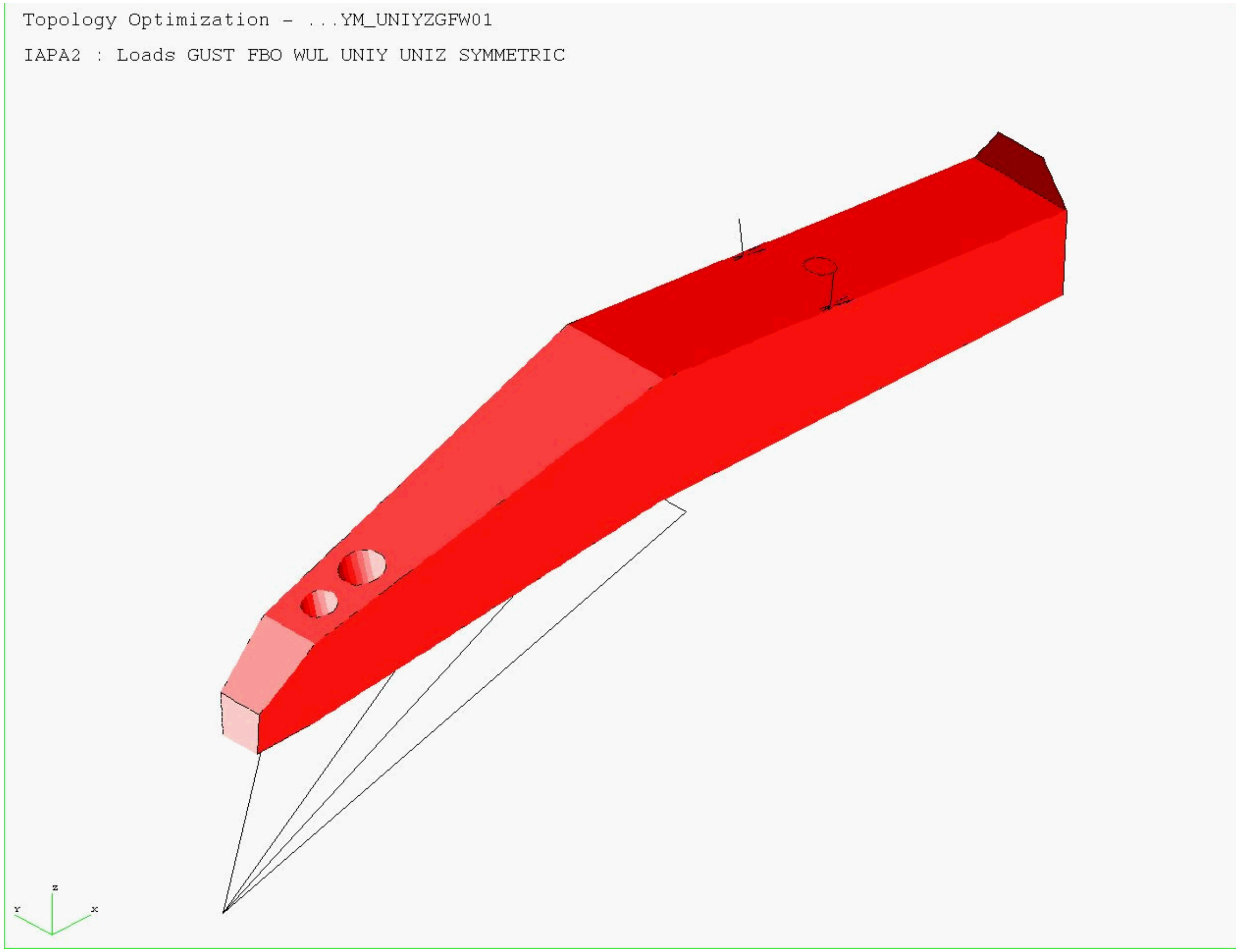


Sensitivities filtering



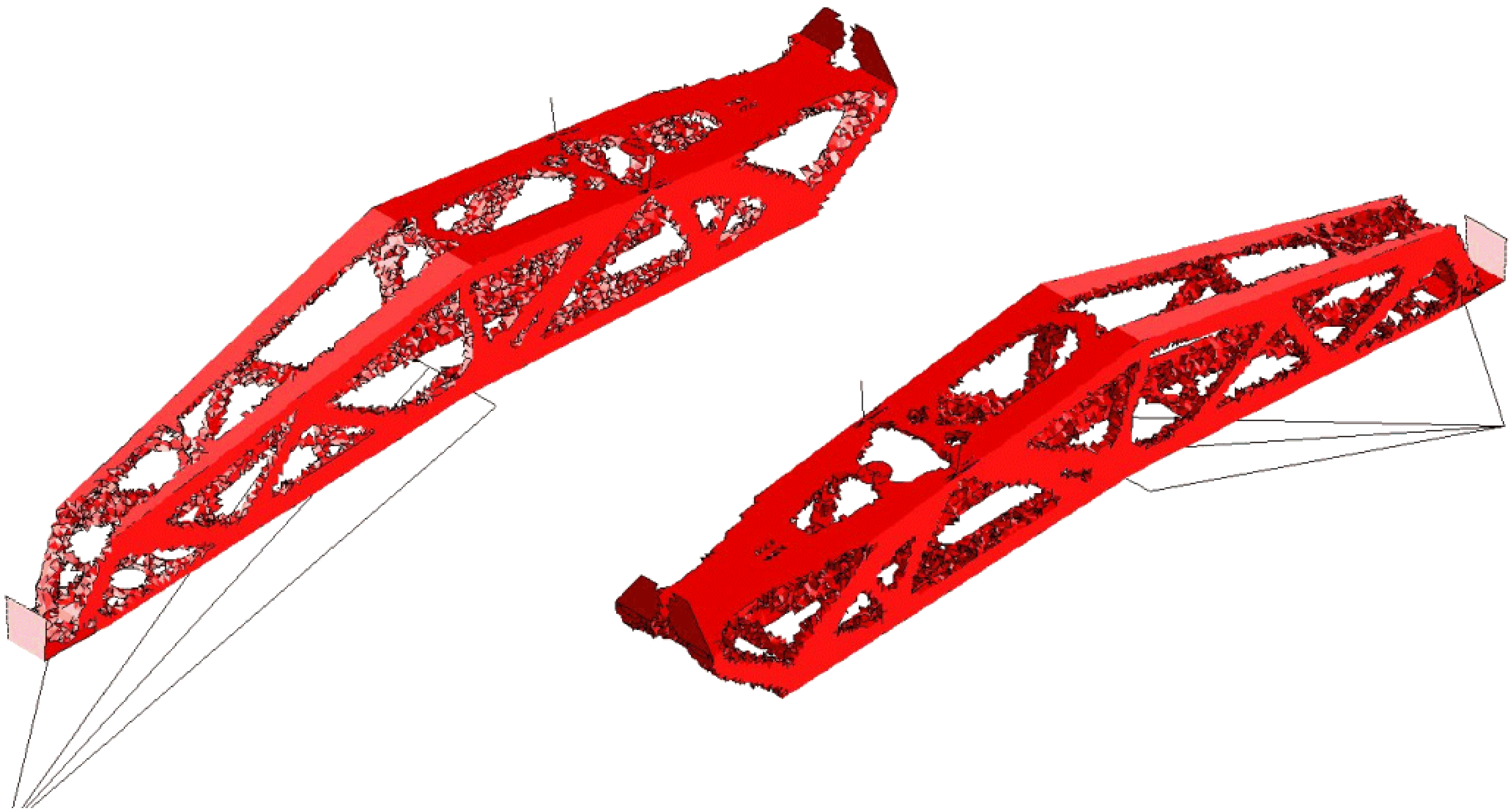
Penalty factor from 2 to 4

Airbus engine pylon



With courtesy by Samtech and Airbus Industries

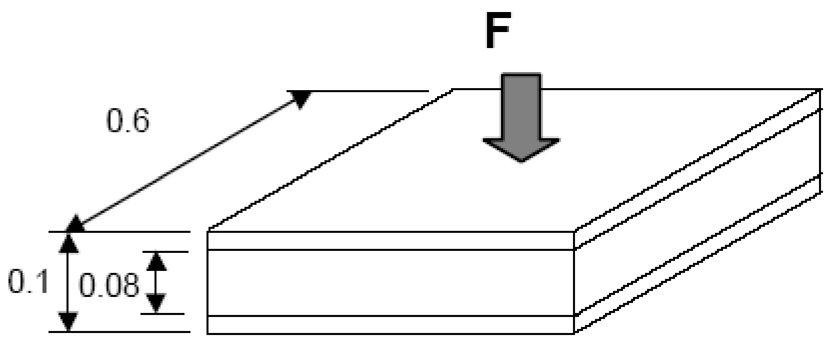
Airbus engine pylon



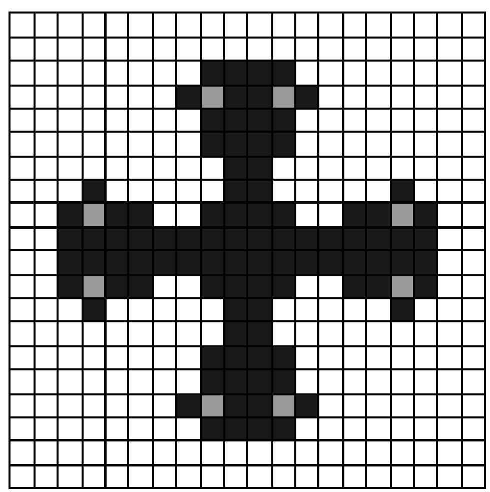
With courtesy by Samtech and Airbus Industries

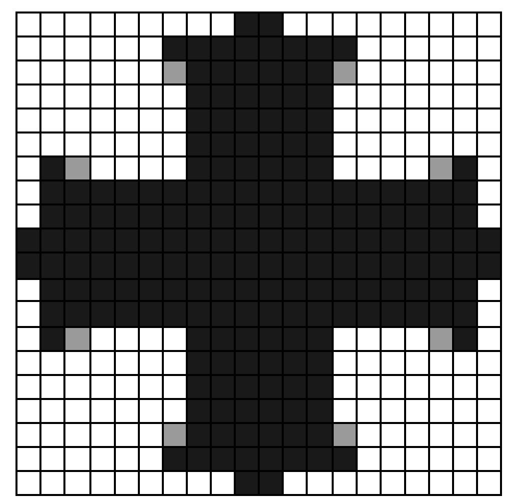
Sandwich panel optimization

- Geometry of the sandwich panel reinforcement problem



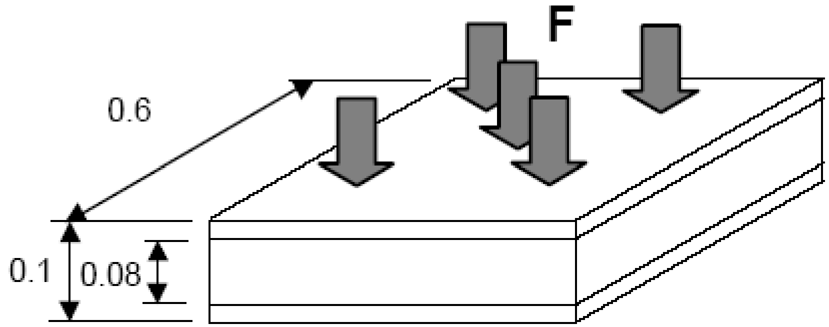
- Optimal topolog



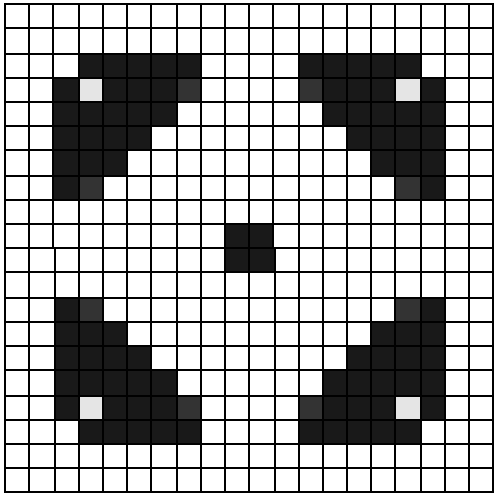


Sandwich panel optimization

- Geometry of the sandwich panel reinforcement problem



- Optimal topology



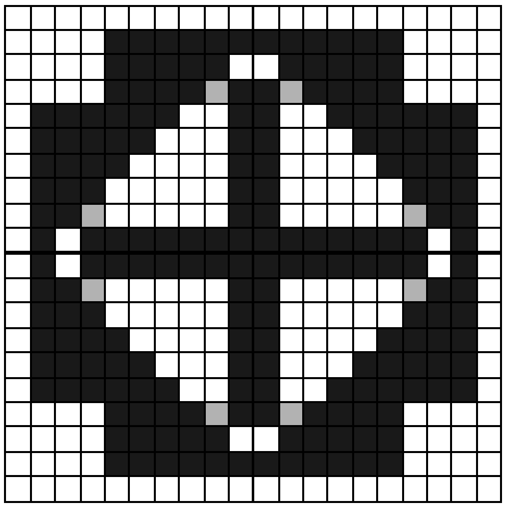
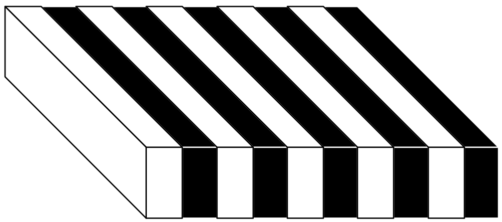
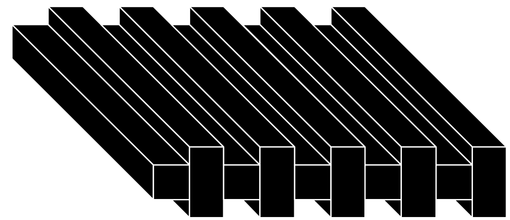


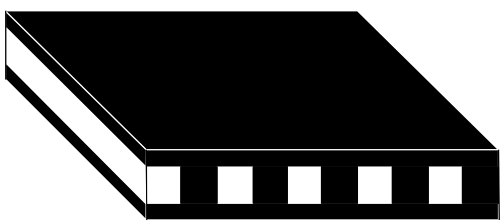
PLATE AND SANDWICH PLATE MODELS



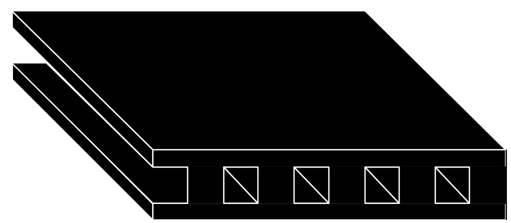
(a)



(b)

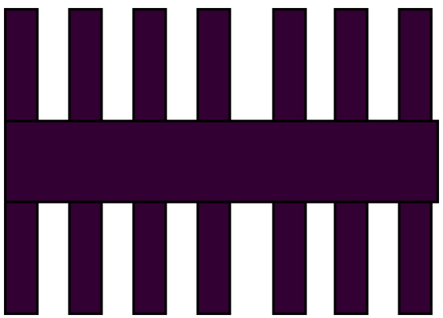


(c)

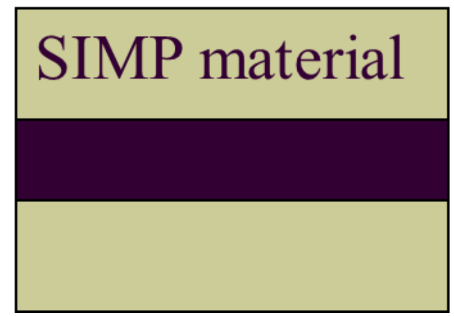


(d)

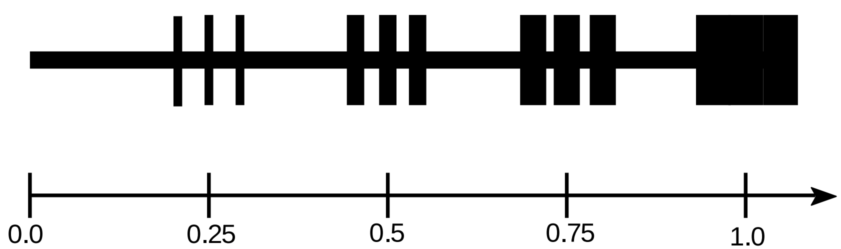
USING SIMP MODEL FOR TOPOLOGY OPTIMISATION OF PLATES AND SHELLS



is replaced by:



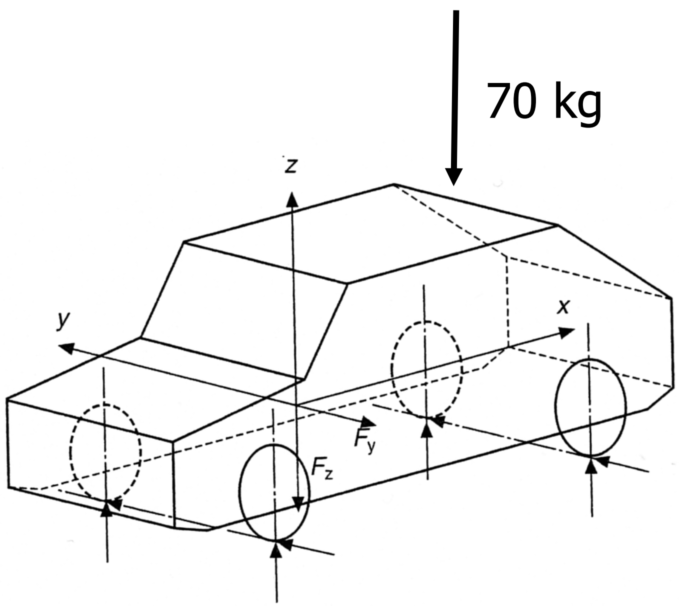
PHYSICAL MEANING OF DENSITY VARIABLE:



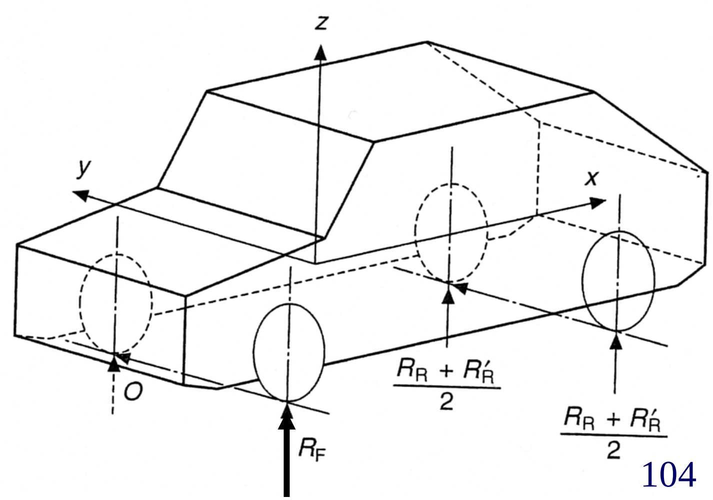
PROTOTYPE CAR BODY OPTIMIZATION

- Load case 1: bending
  - Self weight
  - Components (20 kg)
  - Pilot (50 kg)
  - Roll-over load (70 kg on top of roll cage)

- Load case 2: torsion + bending = curb impact
  - Rear axle clamped
  - Right front wheel free supported
  - Left front wheel withstanding 3 times the weight of the axle



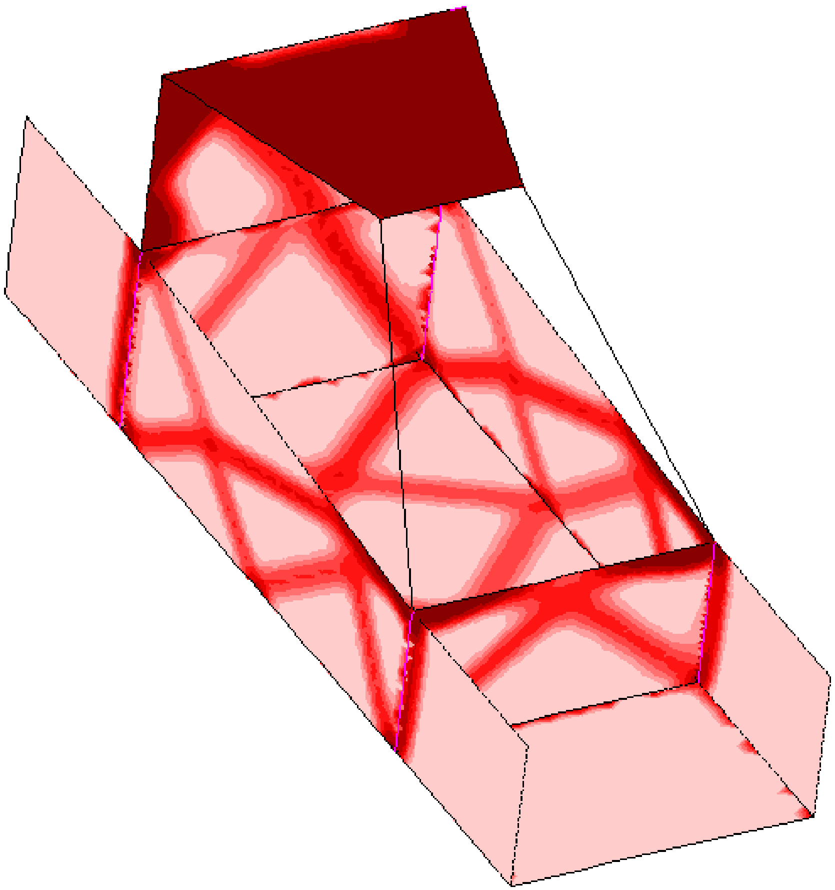
(Figures from Happian-Smith)



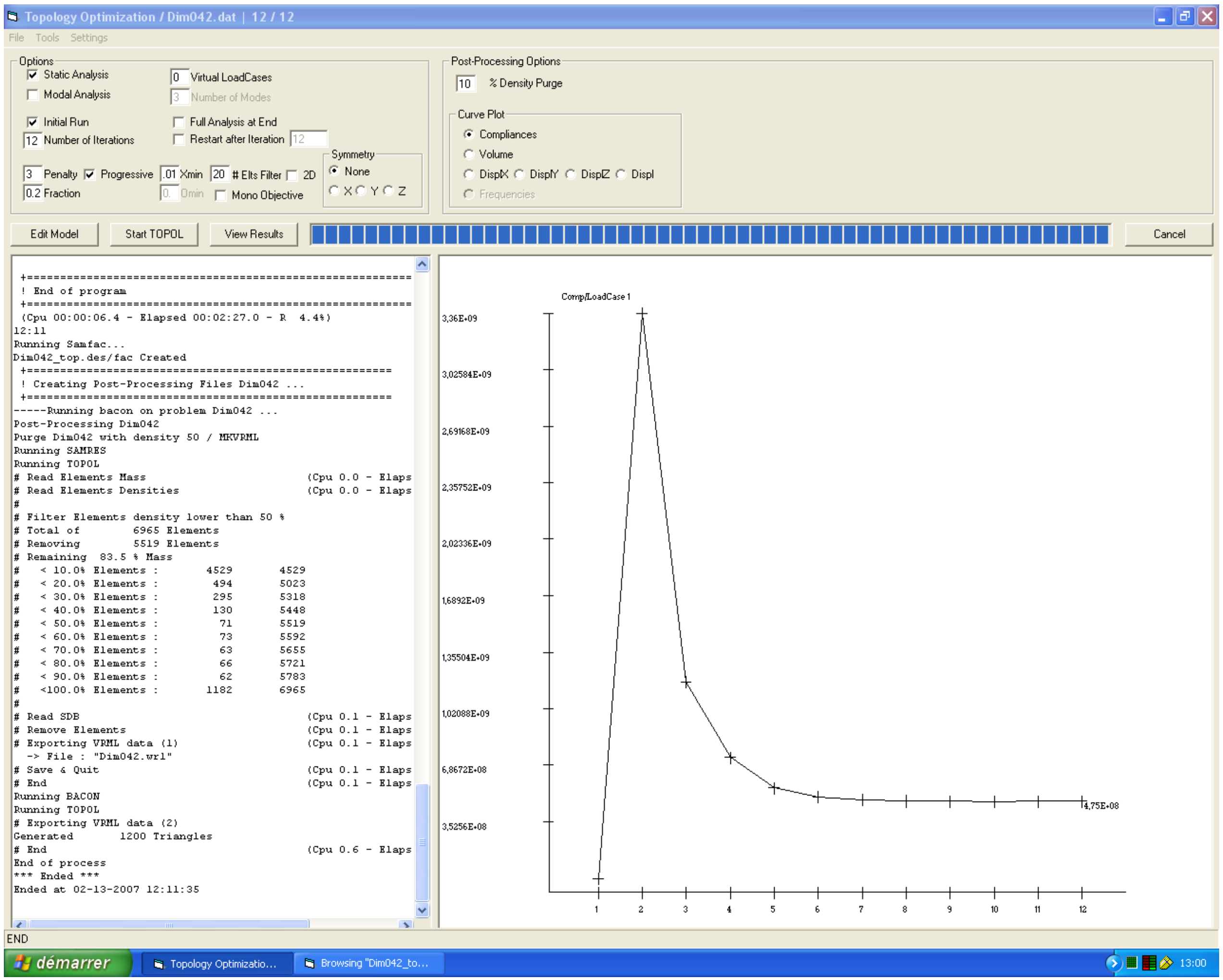
weight x 3= 2700 N

DESIGN OF A URBAN CONCEPT STRUCTURE

- Topology optimization of the truss structure
  - Target mass of 15 kg
  - Minimum compliance
  - Mostly determined by load case 2 (torsion)
  - SIMP material with p=3
  - Left / right symmetry of material distribution
  - Filtering

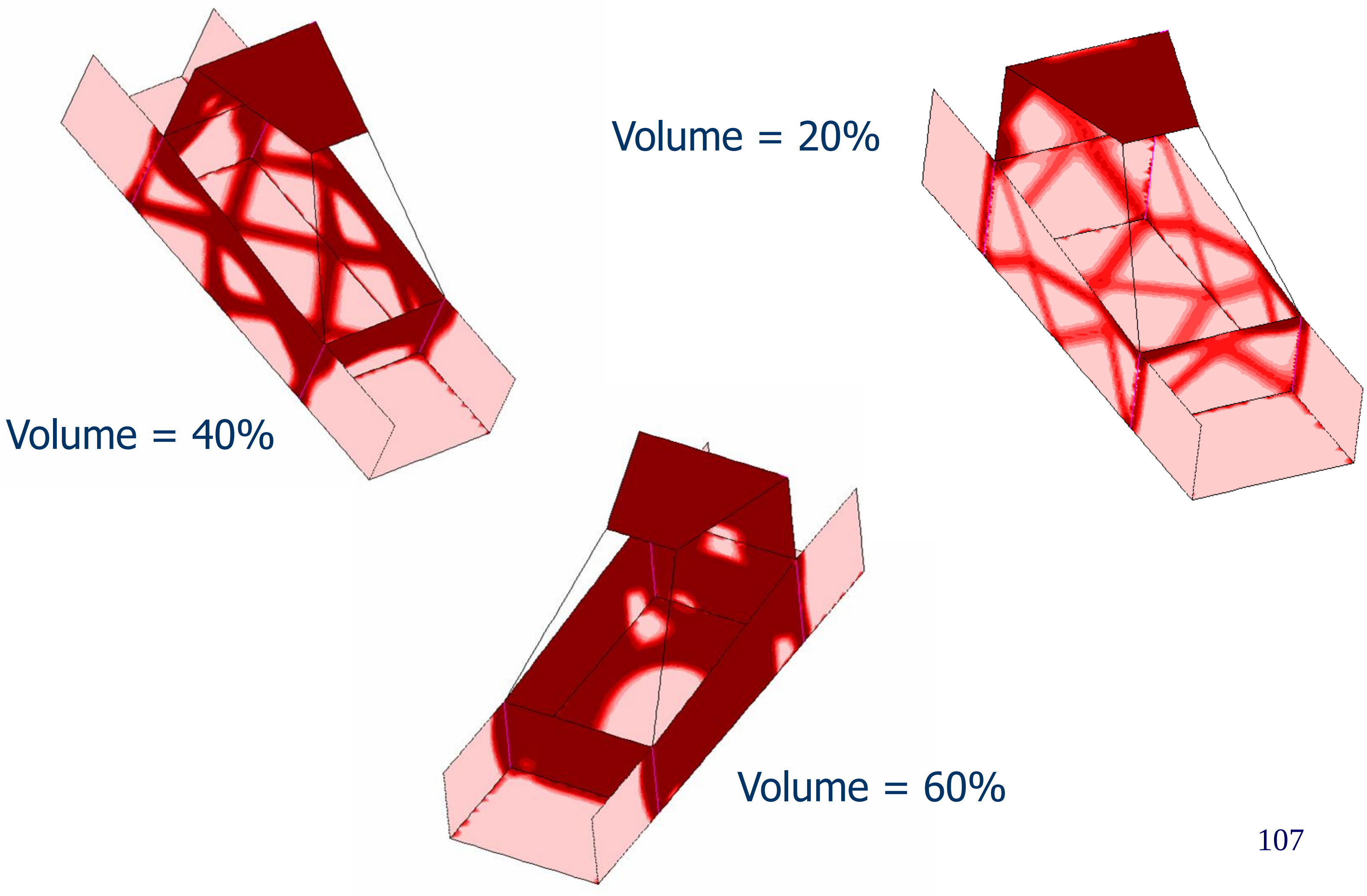


DESIGN OF AN URBAN CONCEPT STRUCTURE



Convergence history

DESIGN OF AN URBAN CONCEPT STRUCTURE



DESIGN OF AN URBAN CONCEPT STRUCTURE