STRESS CONSTRAINED TOPOLOGY OPTIMIZATION

Pierre DUYSINX
LTAS - Automotive Engineering,
University of Liège, Belgium
P.Duysinx@uliege.be
OUTLINE

- INTRODUCTION
- FAILURE IN POROUS MATERIALS
- SINGULARITY OF STRESS CONSTRAINTS
- SENSITIVITY ANALYSIS
- GLOBAL STRESS CONSTRAINTS
- UNEQUAL STRESS LIMITS
- CONCLUSIONS
INTRODUCTION
STRESS CONSTRAINED TOPOLOGY

- Topology optimization based on optimal material distribution is mostly based on compliance design formulation

\[
\begin{align*}
\min_{0 < \rho(x) \leq 1} & \quad \max_k q_k^T f_k \\
\text{s.t.} & \quad V = \int_\Omega \rho(x) dx \leq \bar{V}
\end{align*}
\]

- Where equilibrium writes

\[K q_k = f_k\]
STRESS CONSTRAINED TOPOLOGY

- Compliance design is efficient to predict optimal structural lay-out
  - For single load case
  - Relation between energy minimization and fully stressed design nearly everywhere in the material

- However, theoretical works based on truss lay-out proved that stiffness design and strength designs can be different when
  - Several load cases
  - Several materials
  - Different stress limits in tension and compression
  - Subject to geometrical restrictions...
STRESS CONSTRAINTS IN TOPOLOGY OPTIMIZATION

- Here we focus on the strength problem:

\[
\min_{0 < \rho(x) \leq 1} \int_{\Omega} \rho(x) \, dx \\
\text{s.t.} \quad \langle \| \sigma_{eq}(\rho(x), x) \| \rangle \leq T \quad \text{if} \quad \rho(x) > 0
\]

- Where \( \langle \| \sigma_{eq}(\rho(x), x) \| \rangle \) is an equivalent stress criterion which predicts the failure of the material at point \( x \) while \( T \) is the stress limit.

- The stresses are limited to guarantee:
  - No fail of the material and of the component
  - Life design
STRESS CONSTRAINTS IN TOPOLOGY OPTIMIZATION

\[ \min_{0 < \rho(x) \leq 1} \quad V = \int_{\Omega} \rho(x) \, dx \]

s.t. \[ \langle |\sigma^{eq}(\rho(x), x)| \rangle \leq \sigma_l \quad \text{if} \quad \rho(x) > 0 \]

- Very complex problem:
  - **Definition of a macroscopic (first) failure criterion** for porous materials
  - Extremely **large scale problem**:
    - large number of design variables
    - large number of restrictions: one constraint per finite element
  - ‘**Singularity phenomenon**’ of topology problems with stress constraints
LOCAL STRESS CONSTRAINTS

- Controlling local responses is a very large scale problem:
  - Example control of local von Mises stresses

Bruggi & Duysinx (2012)

Compliance design

Stress design

Duysinx & Bendsoe (1998)
Compliance vs stress design

- L-Shape problem (a) Local stress constraints (b) global stress constraint (c) compliance constraint from Bruggi and Duysinx (2012)
STRESS CONSTRAINED TOPOLOGY

- It is interesting to
  - Investigate topology optimization of continuum structures
  - Illustrate the conjecture that maximum stiffness design may be different from strength design when
    - Several load cases
    - Different stress limits in tension and compression
  - Show how to implement efficiently stress constraints in topology optimization
FAILURE CRITERIA IN POROUS MATERIALS
Layered materials are interesting to investigate the micro-stress state in terms of given macro-stress state and some microstructural parameters (µ, γ) that is also the overall density

\[
E^H = \frac{E}{(1-\mu) + \mu \gamma (1-\nu^2)} \begin{bmatrix}
\gamma & \frac{\mu \gamma \nu}{\mu (1-\mu(1-\gamma))} & 0 \\
\mu \gamma \nu & \mu (1-\mu(1-\gamma)) & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Stress constraints in layered materials

Stresses in a layered material

From Bendsoe (1995), it comes

\[ \sigma_{ij}^+ = \bar{\sigma}_{ij} + c_3 t_i t_j \]
\[ \sigma_{ij}^- = \bar{\sigma}_{ij} - \frac{\delta}{1 - \delta} c_3 t_i t_j \]

\[ c_3 = \frac{1 - \delta}{N(C_{rstut} t_r t_s t_t t_u)} [C_{ijkl}^+ - C_{ijkl}^-] \bar{\sigma}_{kl} \]

\[ N(f) = (1 - \delta) f^+ + \delta f^- \text{ where } f = \begin{cases} f^+ \text{ in material } + \\ f^- \text{ in material } - \end{cases} \]
Stress constraints in rank-2 layered materials

- Let’s consider a rank two material made of a solid material $E^+ = E$ and a very soft material (void) $E^- \to 0$

- From Francfort and Murat (1986) and Bendsoe (1995), it comes

\[
\begin{align*}
\sigma_{11}^+ &= \bar{\sigma}_{11} , \\
\sigma_{22}^+ &= \bar{\sigma}_{22} + c_3 , \\
\sigma_{12}^+ &= \bar{\sigma}_{12} \\
\sigma_{11}^- &= \bar{\sigma}_{11} , \\
\sigma_{22}^- &= \bar{\sigma}_{22} - \frac{\mu}{1-\mu} c_3 , \\
\sigma_{12}^- &= \bar{\sigma}_{12}
\end{align*}
\]

\[
c_3 = \frac{1-\mu}{N(C_{2222})} \left[ C_{22kk}^- - C_{22kk}^+ \right] \bar{\sigma}_{kk}
\]

- If $E^- \to 0^+$

\[
\begin{align*}
\frac{1}{N(C_{2222})} [C_{2211}^- - C_{2211}^+] \bar{\sigma}_{11} &\to 0 \\
\frac{1}{N(C_{2222})} [C_{2222}^- - C_{2222}^+] \bar{\sigma}_{22} &\to \frac{\bar{\sigma}_{22}}{\mu}
\end{align*}
\]
Stress constraints in rank-2 layered materials

- If we call $<\sigma_{ij}>$ the given stress state at macroscopic level, the micro stress state of rank-2 layers is given by:

- For the solid layers

  \[
  \begin{align*}
  \sigma_{11}^{L2} &= \sigma_{11}^+ = <\sigma_{11}> \\
  \sigma_{22}^{L2} &= \sigma_{22}^+ = <\sigma_{22}> / \mu \\
  \sigma_{12}^{L2} &= \sigma_{12}^+ = <\sigma_{12}>
  \end{align*}
  \]

- For the rank 1 layers

  \[
  \sigma_{11}^- = <\sigma_{11}> , \quad \sigma_{22}^- = 0 , \quad \sigma_{12}^- = <\sigma_{12}>
  \]
The stress state in rank-1 layers is now determined by using a second time the formula

\[
\sigma_{11}^{+} = \bar{\sigma}_{11} + c_3, \quad \sigma_{22}^{+} = \bar{\sigma}_{22}, \quad \sigma_{12}^{+} = \bar{\sigma}_{12}
\]

\[
\sigma_{11}^{-} = \bar{\sigma}_{11} - \frac{\gamma}{1 - \gamma} c_3, \quad \sigma_{22}^{-} = \bar{\sigma}_{22}, \quad \sigma_{12}^{-} = \bar{\sigma}_{12}
\]

\[
c_3 = (1 - \gamma) \frac{E^+ - E^-}{\gamma E^+ + (1 - \gamma) E^-} \left[-\nu \bar{\sigma}_{11} + \bar{\sigma}_{22}\right]
\]

Passing to the limit if \( E^- \to 0^+ \), one gets the stress state in rank 1 solid layers

\[
\sigma_{11}^{L1} = \sigma_{11}^+ = \langle \sigma_{11} \rangle / \gamma
\]

\[
\sigma_{22}^{L1} = \sigma_{22}^+ = 0
\]

\[
\sigma_{12}^{L1} = \sigma_{12}^+ = \langle \sigma_{12} \rangle
\]
Stress constraints in rank-2 layered materials

The micro stress state in rank-2 materials then reads for a given macro stress field \( <\sigma_{ij}> \)

\[
\begin{align*}
\sigma_{11}^{L2} &= \sigma_{11}^+ = <\sigma_{11}> \\
\sigma_{22}^{L2} &= \sigma_{22}^+ = <\sigma_{22}> / \mu \\
\sigma_{12}^{L2} &= \sigma_{12}^+ = <\sigma_{12}> \\
\sigma_{11}^{L1} &= \sigma_{11}^+ = <\sigma_{11}> / \gamma \\
\sigma_{22}^{L1} &= \sigma_{22}^+ = 0 \\
\sigma_{12}^{L1} &= \sigma_{12}^+ = <\sigma_{12}>
\end{align*}
\]

And von Mises first point failure at microlevel is predicted by

\[
\begin{align*}
|<\sigma_{11}>/\gamma| &\leq \sigma_l \\
\sqrt{<\sigma_{11}>^2 + <\sigma_{22}>^2/\mu^2 - <\sigma_{11}> <\sigma_{22}>/\mu} &\leq \sigma_l
\end{align*}
\]
Asymptotic behavior at zero density:

- The asymptotic behavior of stresses at zero density plays a key role in the singularity phenomenon in truss topology design with stress constraints.

- As it is shown in Cheng and Jiang (1992) the macroscopic strains of a composite $\langle \varepsilon_{ij} \rangle$ in a point remain continuous and retains a finite value when the density goes to zero:

$$\lim_{\mu, \gamma \to 0^+} \langle \varepsilon_{ij} \rangle = \langle \varepsilon_{ij}^0 \rangle$$

- The macroscopic stresses are also continuous, but they vanish at zero density since the homogenized stiffness tensor tends to zero.

$$\lim_{\mu, \gamma \to 0^+} \langle \sigma_{ij} \rangle = \tilde{E}_{ijkl}^H \langle \varepsilon_{ij} \rangle = 0$$
Asymptotic behavior at zero density:

- However, the behavior of the local stresses is completely different. The local stresses $\sigma_{ij}$ tend to finite (non zero) values at zero density. This is shown with the help of the previous expressions for the local stresses in a rank 2 composite.

- In rank 1 material

$$\lim_{\mu, \gamma \to 0^+} \sigma_{11} = E \langle \varepsilon_{11}^0 \rangle, \quad \lim_{\mu, \gamma \to 0^+} \sigma_{22} = 0, \quad \lim_{\mu, \gamma \to 0^+} \sigma_{12} = 0$$

- In rank 2 material

$$\lim_{\mu, \gamma \to 0^+} \sigma_{11} = 0, \quad \lim_{\mu, \gamma \to 0^+} \sigma_{22} = E \langle \varepsilon_{22}^0 \rangle, \quad \lim_{\mu, \gamma \to 0^+} \sigma_{12} = 0$$
Stress criterion for power law materials

□ In SIMP material the parameterization of the rigidity is given by

\[ \langle E(\rho) \rangle = \rho^p E^0 \]

□ In order to establish a stress criterion for this model at intermediate densities, it is necessary to propose a local stress model, or more exactly, to assume a relationship between mimicked local stresses, the averaged stresses and the density parameter.

□ Assume a simple expression of the micro stresses in terms of the inverse of the density parameter. Finally, the criterion must penalize intermediate densities in order to generate black and white results.

□ One local stress model that satisfies all these requirements is to assume that local stresses are given by:

\[ \sigma_{ij} = \frac{\langle \sigma_{ij} \rangle}{\rho^q} \]
Stress criterion for power law material

- The exponent $q$ is a number (greater than 1) is determined from a requirement that local stresses remain finite and non zero at zero density (as in rank 2 materials):

$$\lim_{\rho \to 0^+} \sigma_{ij} = \frac{\rho^p}{\rho^q} E_{ijkl} <\varepsilon_{kl}^0> \neq 0$$

- This leads to the choice

$$p = q$$
Stress criterion for power law material

- Finally to it is possible to establish that local failure of a porous material of density $\rho$ is controlled by the following macroscopic criteria which bounds the value of the von Mises equivalent stress $<\sigma>_eq$ computed at macroscopic scale:

$$<\sigma>_eq \leq \rho^q \, \sigma_l$$

- From a macroscopic point of view, this criterion predicts that the overall strength of the material is

$$<\sigma>_l = \rho^q \, \sigma_l.$$
STRESS CONSTRAINED TOPOLOGY

- Homogenized failure criteria predicting failure in the microstructure from macroscopic point of view:

\[ \left\| \sigma^{eq}(\rho) \right\| = \frac{\sigma^{eq}}{\rho^p} \leq \sigma_i \]

- With consistency conditions requirements: \( p=q \)

Rank 2 layered material

SIMP (isotropic) material
SINGULARITY OF STRESS CONSTRAINTS
Singularity phenomenon of stress constraints

- **Kirch (1990):** It is impossible to remove or create holes in the material distribution with optimization algorithms.
- **Cheng and Jiang (1992):** The physics of the phenomenon is that low density regions can remain highly strained, and highly stressed even when there is very little density. But when density is zero, stress state is suddenly cancelled, which creates a discontinuity of stress constraints.
- **Rozvany and Birker (1994):** The discontinuities of the stress constraints create non connected parts and zero measure regions in the design space. Optimum configurations are often located in these degenerated parts.
- **Duysinx and Bendsoe (1998):** Optimum, which are often located in degenerated regions, don’t satisfy Slater conditions (they are not a regular points), so algorithms get stocked and can’t reach them.
- **Cheng and Guo (1996):** Perturbation technique (ε-relaxation technique) to alleviate the degeneracy problem in truss topology optimization.
- **Duysinx and Bendsoe, Duysinx and Sigmund (1998):** Application of ε-relaxation technique to topology optimization of continuum structures.
Singularity phenomenon of stress constraints

\[
\begin{align*}
\min_{A_1, A_2 \geq 0} & \quad W_T = \alpha A_1 + A_2 \\
\text{s.t.} & \quad g_1 = A_1 + A_2/3 - 0.5 \leq 0 \\
& \quad g_2 = A_1 + A_2/3 - 0.236 \leq 0 \\
& \quad g_3 = A_1 + A_2/3 - 0.167 \leq 0
\end{align*}
\]
$\varepsilon$-relaxation of stress constraints

- **Original Constraint**
  $\|\sigma^{eq}\| \leq \sigma_l$ if $\rho > 0$

- **Rewritten**
  $\rho \left( \frac{\|\sigma^{eq}\|}{\sigma_l} - 1 \right) \leq 0$

- **Relaxed Constraint**
  (Cheng and Guo, 1997)
  $\rho \left( \frac{\|\sigma^{eq}\|}{\sigma_l} - 1 \right) \leq \varepsilon$
  $\frac{\varepsilon^2}{\rho} \leq \rho$

- **Normalized form**
  (Duysinx and Sigmund, 1998)
  $\frac{\|\sigma^{eq}\|}{\sigma_l} - \frac{c}{\rho} + \varepsilon \leq 1$
  $\frac{\varepsilon^2}{\rho} \leq \rho$

- **Interpretation**
  $\|\sigma^{eq}\| \leq \sigma_l \left( 1 - \varepsilon + \frac{\varepsilon}{\rho} \right)$
\[ \varepsilon \text{-relaxation of stress constraints} \]

- Relaxation for truss topology (Cheng and Guo, 1997)

\[ \rho \left( \frac{\| \sigma^{eq} \|}{\sigma_l - 1} \right) \leq \varepsilon \]
\[ \varepsilon^2 \leq \rho \]

- Relaxation for continuum topology (Duysinx and Sigmund, 1998)

\[ \frac{\| \sigma^{eq} \|}{\sigma_l} - \frac{\varepsilon}{\rho} + \varepsilon \leq 1 \]
\[ \varepsilon^2 \leq \rho \]

- Interpretation of relaxation

\[ \| \sigma^{eq} \| \leq \sigma_l \left( 1 - \varepsilon + \varepsilon / \rho \right) \]
ε-relaxation of stress constraints

- Original Problem

\[
\begin{align*}
\min_{0 \leq \rho(x) \leq 1} & \quad \int_{\Omega} \rho(x) \, dx \\
\text{s.t.:} & \quad \sigma_{eq}^c / \rho^p \leq \sigma_l \quad \text{if} \quad \rho(x) > 0
\end{align*}
\]

- ε-relaxed problem (Cheng and Guo, 1997)

\[
\begin{align*}
\min_{\epsilon^2 \leq \rho(x) \leq 1} & \quad \int_{\Omega} \rho(x) \, dx \\
\text{s.t.:} & \quad \sigma_{eq}^c / \rho^p \leq \sigma_l \left(1 - \epsilon + \epsilon / \rho\right)
\end{align*}
\]

- For a sequence of decreasing perturbations ε (going to zero), optimization problems and their optimal solutions converge continuously to the original problem and its singular solution.
\(\varepsilon\)-relaxation: illustration

- Simple two-bar truss example
  - One load parallel to support line: intuitive solution
  - \(E=1\text{N/m}^2\), \(\nu=0.3\), \(t=1\text{m}\)
  - \(P=1\text{ N}\)
\( \varepsilon \)-relaxation: illustration

- Material relaxation: \( p=3 \) and \( q=2 \)

\[ \text{p=3, q=2 and } \varepsilon \text{-relaxation} \]

- Material distribution with \( p=q=3 \) and \( \varepsilon \)-relaxation

\[ \text{p=q=3 and } \varepsilon \text{-relaxation} \]
$\varepsilon$-relaxation: illustration

- Topology designs with decreasing $\rho_{\text{min}} = \varepsilon^2$ parameters: $10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$

![Illustration](image_url)
\( \epsilon \)-relaxation: illustration

- Relaxed stress constraint for a minimum density of 10-1

\[
\|\sigma^*_e\| = \frac{\sqrt{q_e^T M_e^0 q_e}}{\sigma_l} + \epsilon - \frac{\epsilon}{\rho_e}
\]

- Overall on Mises stress criterion for a minimum density of 10-1

\[
\sigma_{VM}^*/\rho^n = \sqrt{q^T M^0 q}
\]
Algorithm for $\varepsilon$ parameter reduction

- Procedure: solving a sequence of optimization problems relative to sequence of decreasing values of $\varepsilon$ parameters

- Managing $\varepsilon$ parameter:
  - Perturbation parameter $\varepsilon = $ additional parameter
  - Continuation approach similar to what is done for parameters in barrier and penalty functions (sequential unconstrained minimization)

- Process driven by minimum density $\rho_{min} = \varepsilon^2$
  Minimum density is reduced from 0.1 to 0.001 or 0.0001
  Choosing a quite large initial perturbation is necessary to capture singular optima from most initial starting points
Algorithm for $\varepsilon$ parameter reduction

- Automatic procedure to reduce the parameter $\varepsilon$:
  - Based on numerical experience, highly perturbated problems need not to be solved with a high precision
  - Reduction of parameter $\varepsilon$ as soon as a “loose” convergence criterion is satisfied
  - Reduction criterion based on Euclidian norm of gradient vector of Lagrangian function for free design variables

$$
\| \nabla L \|_2 = \| \nabla g_0 / g_0 - \sum_j \lambda_j \nabla g_j / g_j \|_2.
$$

- Reduction algorithm:

  If $\| \nabla L \|_2 \leq \alpha$  Then  $\varepsilon := \varepsilon / \beta.$

with typically $\alpha = 0.005$ and $\beta = 1.05$
Solving optimization problems with a large number of stress constraints

- Use a normalized version of the stress constraints
  \[ \frac{\|\sigma\|}{\sigma_i} - \frac{\varepsilon}{\rho} + \varepsilon \leq 1 \]

- CONLIN approximation of constraints is generally quite conservative because the perturbation term \(-\varepsilon/\rho\) brings a concave contribution

- CONLIN solvers is robust even for a large number of constraints

- Solution time of optimization problem is of the same order of magnitude as the Finite Element analysis

- An active constraint deletion strategy is highly recommended
Solving optimization problems with a large number of stress constraints
3-bar truss problem

- Famous benchmark problem with 3 independent load cases
  - $F_1 = 40$ N
  - $F_2 = 30$ N
  - $F_3 = 20$ N

- Material and geometrical data
  - $L = 1$ m
  - $W = 2.5$ m
  - $E = 100$ N/m²
  - $\nu = 0.3$
  - $\sigma_l = 150$ N/m²
  - $V_{\text{max}} = 25\%$

- Finite Element mesh
  - 50 x 20 finite elements
Model of Material Properties

POWER LAW MODEL or SIMP MODEL
(Bendsoe 1989, Zhou and Rozvany, 1992)

- **Stiffness properties**
  \[ E^* = \rho^p E^0 \quad 0 \leq \rho \leq 1 \quad p > 1 \]

- **Stresses**
  \[ \sigma_{ij} = \sigma_{ij}^* / \rho^p \]

- **Failure criterion**
  \[ ||\sigma^{eq}(\rho)|| = *\sigma^{eq} / \rho^p \leq \sigma_l \]
  \[ *\sigma^{eq} \leq \rho^p \sigma_l \]
3-bar truss: comparison of compliance design and stress constrained design

- **Minimum compliance design**
  
  Compliance $(1,2,3) = 73.3 \text{ Nm}$

  Max von Mises:
  1) $229 \text{ N/m}^2$
  2) $571 \text{ N/m}^2$
  3) $555 \text{ N/m}^2$

  Volume = 25%

- **Stress constrained design**

  Compliance
  1) $91.2 \text{ Nm}$
  2) $45.6 \text{ Nm}$
  3) $45.0 \text{ Nm}$

  Max Von Mises $(1,2,3) = 150 \text{ N/m}^2$

  Volume = 26.4 %
3-bar truss: comparison of compliance design and stress constrained design

**Local stresses: bound = 150. N/m²**

Max str. crit. = 150. N/m²

Max str. crit. = 150. N/m²

Max str. crit. = 150. N/m²

Min max compliance

Max str. crit. = 228. N/m²

Max str. crit. = 571 N/m²

Max str. crit. = 555. N/m²
SENSITIVITY ANALYSIS
SENSITIVITY ANALYSIS

- Discretized equilibrium

\[ K q = f \]

- The stress vector can be cast under the matrix form using the stress matrix \( T \) of the element.

\[ \sigma = T q \]

- The sensitivity of a component \( \sigma_k \) of the stress with respect to a design variable \( z \) is given by:

\[
\frac{\partial \sigma_k}{\partial z} = \frac{\partial t_k}{\partial z} q + t_k^T K^{-1} \left( \frac{\partial f}{\partial z} - \frac{\partial K_k}{\partial z} q \right)
\]

It is clear that in order to evaluate the sensitivity expression, one has to compute the direct load cases (one per design variable) or the adjoin load cases (one per constraint)

\[ K^{-1} \left( \frac{\partial f}{\partial z} - \frac{\partial K}{\partial z} q \right) + K^{-1} t_k \]
SENSITIVITY ANALYSIS

- For an equivalent stress criterion, an economical way of computing is based on the expression

\[
\sigma_{VM}^2 = q^T T^T V T q = q^T M q
\]

- It can then be shown by simple algebra that the sensitivity of the von Mises stress can be written as:

\[
\frac{\partial \sigma_{VM}}{\partial z} = \frac{1}{\sigma_{VM}} \left( q^T \frac{\partial M}{\partial z} q + 2 q^T M K^{-1} \left( \frac{\partial f}{\partial z} - \frac{\partial K}{\partial z} q \right) \right)
\]

- That can solve the adjoin load case

\[
\tilde{g} = M q
\]
GLOBAL STRESS CONSTRAINTS
Integrated (aggregated) stress constraint

- Local stress constraints
  - Introduce a huge number of restrictions
  - Computing effort for optimization problem solving growths as the cube of number of constraints (Fleury, 2007)

- Use aggregate restriction of relaxed stress constraints
  - $q$-norm
    $$\left[ \sum_{e=1}^{N} \left( \max \left\{ 0, \frac{1}{T} \left( \frac{\sigma_{eq}^p}{\rho_e} - \frac{\varepsilon}{\rho_e} + \varepsilon \right) \right\} \right)^q \right]^{1/q} \leq 1$$
  - $q$-mean
    $$\left[ \frac{1}{N} \sum_{e=1}^{N} \left( \max \left\{ 0, \frac{1}{T} \left( \frac{\sigma_{eq}^p}{\rho_e} - \frac{\varepsilon}{\rho_e} + \varepsilon \right) \right\} \right)^q \right]^{1/q} \leq 1$$
Integrated (aggregated) stress constraint

- Asymptotic behavior

\[ \|\sigma^*_e\| = \max \left\{ 0, \frac{\sigma^*_{e,eq}}{\rho_e \sigma_1} + \varepsilon - \frac{\varepsilon}{\rho_e} \right\} \]

\[ \lim_{q \to \infty} \left[ \sum_{e=1}^{N} \|\sigma^*_e\|^q \right]^{1/q} = \max_{e=1..N} \|\sigma^*_e\| \]

\[ \lim_{q \to \infty} \left[ \frac{1}{N} \sum_{e=1}^{N} \|\sigma^*_e\|^q \right]^{1/q} = \max_{e=1..N} \|\sigma^*_e\| \]

- Ordering relationships

\[ q_1 \leq q_2 \leq \infty \]

\[ \left[ \sum_{e=1}^{N} \|\sigma^*_e\|^q \right]^{1/q_1} \geq \left[ \sum_{e=1}^{N} \|\sigma^*_e\|^q \right]^{1/q_2} \geq \max_{e=1..N} \|\sigma^*_e\| \]

\[ \left[ \frac{1}{N} \sum_{e=1}^{N} \|\sigma^*_e\|^q \right]^{1/q_1} \leq \left[ \frac{1}{N} \sum_{e=1}^{N} \|\sigma^*_e\|^q \right]^{1/q_2} \leq \max_{e=1..N} \|\sigma^*_e\| \]

- Bounding maximum stress level

\[ \left[ \frac{1}{N} \sum_{e=1}^{N} \|\sigma^*_e\|^q \right]^{1/q} \leq \max_{e=1..N} \|\sigma^*_e\|^q \leq \left[ \sum_{e=1}^{N} \|\sigma^*_e\|^q \right]^{1/q} \]
SENSITIVITY ANALYSIS

- For an equivalent stress criterion

\[
\sigma_{VM}^2 = q^T \mathbf{T}^T \mathbf{V} \mathbf{T} q = q^T \mathbf{M} q
\]

- Dependency of the stress matrix with density

\[
\mathbf{T} = \rho^n \mathbf{T}_0
\]

- So we find the local stress criterion

\[
\sigma_{VM}^* / \rho^n = \sqrt{q^T \mathbf{M}^* q}
\]

\[
\mathbf{M}^* = \mathbf{T}^o \mathbf{V} \mathbf{T}^o
\]

- The relaxed stress criterion

\[
\|\sigma_e^*\| = \frac{\sqrt{q_e^T \mathbf{M}^o q_e}}{\sigma_l} + \epsilon - \frac{\epsilon}{\rho_e}
\]
SENSITIVITY ANALYSIS

- The p-norm stress criterion

\[
g = \left[ \sum_{e \in E^+} \left( \frac{\sqrt{q_e^T M_e^o q_e}}{\sigma_l} + \epsilon - \frac{\epsilon}{\rho_e} \right)^p \right]^{1/p}
\]

where the set \(E^+\) is the set of all elements in which the relaxed stress criterion \(\|\sigma^*_e\|\) is positive.

- The derivative of the relaxed effective von Mises criterion \(\|\sigma^*_e\|\) in element ‘e’:

\[
\frac{\partial \|\sigma^*_e\|}{\partial \rho_i} = \frac{q^T M_e^o}{\sigma_l \sqrt{q^T M_e^o q}} K^{-1} \left( \frac{\partial f}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} q \right) + \frac{\epsilon}{\rho_i^2} \delta_{ie}
\]
SENSITIVITY ANALYSIS

- The derivative of the p-norm stress criterion

\[
\frac{\partial g}{\partial \rho_i} = \left[ \sum_{e \in E^+} (\|\sigma_e^*\|)^p \right]^{(1/p)-1} \left[ \sum_{e \in E^+} (\|\sigma_e^*\|)^{p-1} \frac{\partial \|\sigma_e^*\|}{\partial \rho_i} \right]
\]

- By combining the two last results

\[
\frac{\partial g}{\partial \rho_i} = \left[ \sum_{e \in E^+} (\|\sigma_e^*\|)^p \right]^{(1/p)-1} \left[ \tilde{q}^T \left( \frac{\partial f}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} q \right) + (\|\sigma_e^*\|)^{p-1} \frac{\varepsilon}{\rho_e^2} \delta_{ie} \right]
\]

- With the solution of the adjoint system:

\[
K \tilde{q} = \left[ \sum_{e \in E^+} \frac{(\|\sigma_e^*\|)^{p-1}}{\sigma_l \sqrt{q^T M_e^o q}} M_e^o q \right]
\]
UNEQUAL STRESS LIMITS
Unequal stress limits in tension and compression

- Extending Von Mises criterion to other failure criteria to cope with unequal stress limits behaviors ($T \neq C$, $s=C/T$)

- **Raghava criterion** (parabolic criterion from Tsai-Wu criterion family)

  $$\sigma^*_{RAG} = \frac{J_1(s - 1) + \sqrt{J_1^2(s - 1)^2 + 12sJ_{2D}}}{2s} \leq T$$

- **Ishai criterion** (hyperbolic criterion from Prager-Drucker family)

  $$\sigma^*_{ISH} = \frac{(s + 1)\sqrt{3J_{2D}} + (s - 1)J_1}{2s} \leq T$$

  - with

    $$J_1 = \sigma_I + \sigma_{II} + \sigma_{III}$$

    $$J_{2D} = \frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{6}$$
Unequal stress limits in tension and compression

Raghava criterion

Ishai criterion
SENSITIVITY ANALYSIS

- Discretized equilibrium

\[ K q = f \]

- Sensitivity of displacement vector

\[ \frac{\partial q}{\partial \rho_i} = K^{-1} \left( \frac{\partial g}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} q \right) \]

- Direct approach: solve for every design variable

- Stress constraint

\[ \sigma = Tq \]

\[ J_1^* = 3\sigma_*^h = w^T \sigma = Wq \]

\[ J_{2p} = \frac{1}{3} (\sigma_{VM}^*)^2 = \frac{1}{3} q^T V q \]

\[ T = \rho^p T^0 \quad V = \rho^{2p} V^0 \quad W = \rho^p W^0 \]
SENSITIVITY ANALYSIS

- Sensitivity of unequal stress constraints: Ishai

\[
||\sigma_{ISH}^{eq}|| = \frac{\sigma_{ISH}^{*eq}}{\rho^p} = \frac{s - 1}{2s} W^0 q + \frac{s + 1}{2s} \sqrt{q^T V^0 q}
\]

- Derivative of criteria

\[
\frac{\partial ||\sigma_{ISH}^{eq}||}{\partial \rho_i} = \left\{ \frac{s - 1}{2s} W^0 + \frac{s + 1}{2s} \frac{1}{\sqrt{q^T V^0 q}} V^0 q \right\}^T \frac{\partial q}{\partial \rho_i}
\]

- Adjoin approach (for every constraint)

\[
\lambda = K^{-1} \left\{ \frac{s - 1}{2s} W^0 + \frac{s + 1}{2s} \frac{1}{\sqrt{q^T V^0 q}} V^0 q \right\}
\]

\[
\frac{\partial ||\sigma_{ISH}^{eq}||}{\partial \rho_i} = \lambda^T \left( \frac{\partial g}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} q \right)
\]


SENSITIVITY ANALYSIS

- Direct approach: solve \( n \) (#dv) load cases

\[
\frac{\partial q}{\partial \rho_i} = K^{-1} \left( \frac{\partial g}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} q \right)
\]

- Adjoin method: solve \( m \) (#constraint) load cases

\[
\lambda = K^{-1} \left\{ \frac{s-1}{2s} W^0 + \frac{s+1}{2s} \frac{1}{\sqrt{q^T V^0 q}} V^0 q \right\}
\]

  - For one load case: \( m = \#FE \sim n \)
  - For several load cases: \( m = \#FE \times \#load \ cases \geq n \)
Classical benchmark problem: Support a single shearing load from support

Analytical solution for compliance and equal stress limits: two-bar truss with 45° angles

Analytical solution for unequal stress limits $T=3C$ (Rozvany, 1996) is a two-bar truss with 30° (60°) angles

Numerical solution matches perfectly theoretical result!

- $E=100$ GPa, $\nu=0.3$
- $T=12$ MPa, $C=4$ MPa
ITERATIVE SOLUTION PROCEDURE

- Initial $\varepsilon$-relaxation parameter ($\varepsilon=0.1$)
- Density distribution
- F.E. analysis
- Select active constraint set
  - Sensitivity analysis
  - Estimation of second order diagonal derivatives if necessary
- Solve optimization sub-problem
  - CONLIN (Fleury, 1985) or MMA (Svanberg, 1987) or GCMMA approximation (Bruyneel et al., 2002)
  - Dual solver (Lagrangian maximization)
- Update density distribution
- Convergence test (KKT): update $\varepsilon$: $\varepsilon:=\varepsilon/2$.

\[ \| \sigma^{eq}(\rho) \| \leq 0.85 \sigma_i \left( 1 - \varepsilon + \frac{\varepsilon}{\rho} \right) \]
NUMERICAL APPLICATIONS: 2-BAR TRUSS

Density maps

Stress criterion maps
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- Famous benchmark problem with 3 independent load cases
  - $F_1 = 40 \text{ N}$
  - $F_2 = 30 \text{ N}$
  - $F_3 = 20 \text{ N}$

- Material and geometrical data
  - $L = 1 \text{ m}$
  - $W = 2.5 \text{ m}$
  - $E = 100 \text{ N/m}^2$
  - $\nu = 0.3$
  - $\sigma_l = 150 \text{ N/m}^2$
  - $V_{\text{max}} = 25\%$

- Finite Element mesh
  - 50 x 20 finite elements

- Design variables: 1000
- Load cases: 3
- Stress constraints: 3000
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- Minimum compliance design
  - Compliance \((1,2,3) = 73.3 \text{ Nm}\)
  - Max von Mises:
    1) 229 N/m\(^2\)
    2) 571 N/m\(^2\)
    3) 555 N/m\(^2\)
  - Volume = 25%

- Stress constrained design
  - Compliance
    1) 91.2 Nm
    2) 45.6 Nm
    3) 45.0 Nm
  - Max Von Mises \((1,2,3) = 150 \text{ N/m}^2\)
  - Volume = 26.4 %
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- q-norm of stresses (q=4):
  
  Bound: 500 N/m²
  
  Compliance: 87.3, 59.3, 67.9 Nm
  
  Max von Mises (local) for load case 1, 2, 3:
  
  230, 235, 231 N/m²
  
  Volume = 24.8%

- q-mean of stresses (q=4):
  
  Bound: 92 N/m²
  
  Compliance: 90.6, 50.3, 53.8 Nm
  
  Max von Mises (local) for load case 1, 2, 3:
  
  237, 215, 207 N/m²
  
  Volume = 22.4%
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- High compressive strength (s=C/T=3):
  \( C=450 \text{ N/m}^2, T=150 \text{ N/m}^2 \)

  Volume = 25.6 %

  Compliance (1,2,3): 92.8, 47.3, 46.0 N*m

- High tensile strength (s=C/T=1/3):
  \( C=150 \text{ N/m}^2, T=450 \text{ N/m}^2 \)

  Volume = 12.4 %
NUMERICAL APPLICATIONS: 4-BAR TRUSS

Von Mises
T=C=6 N/m²

Ishai
T=6 & C=24 N/m²

Ishai
T=24 & C=6 N/m²

E=100 N/m², \( \nu=0.3 \), F =1 N, L =1 m

From Swan and Kosaka (1997)
NUMERICAL APPLICATIONS: 4-BAR TRUSS

Similar results for Raghava and Ishai
CONCLUSIONS
CONCLUSIONS

- We demonstrated the specific character of optimal layout based on stress constraints

- Difficulties of stress constraints
  - Local constraints → huge size optimization problem characterized by large number of active constraints
  
  - Stress constraints are subject to so called singularity phenomena
    - Epsilon relaxation (Chang and Guo)
    - QP relaxation (Bruggi)
    - Fish relaxation (Achtziger)

  - Properties of first failure criteria in porous / composite solids is not completed known.
PERSPECTIVES

- Investigation of the fundamental aspects of the stress constraints problem (not only SIMP approach)

- Large scale optimization problems call for new algorithms (not dual maximization)

- Integrated stress constraint is an interesting way to reduce the computational effort but it is not a breakthrough
  - Active constraint selection strategy
  - Clustering / domain decomposition techniques are interesting perspectives (Tortorelli et al., Wang et al)
  - Revisiting other techniques eg OC methods

- Other local constraints: local buckling