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SEQUENTIAL CONVEX PROGRAMMING FOR STRUCTURAL OPTIMIZATION PROBLEMS

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ABSTRACT. In this Lecture, several recent methods based on convex approximation schemes are discussed, that have demonstrated strong potential for efficient solution of structural optimization problems.

First, the now well established "Approximation Concepts" approach is briefly recalled for sizing as well as shape optimization problems. Next, the "Convex Linearization" method (CONLIN) is described, as well as one of its recent generalization, the "Method of Moving Asymptotes" (MMA). Both CONLIN and MMA can be interpreted as first order convex approximation methods, that attempt to estimate nonlinearity on the basis of semi-empirical rules.

Attention is next directed toward methods that use diagonal second derivatives in order to provide a sound basis for building up high quality explicit approximations of the behaviour constraints. In particular, it is shown how second order information can be effectively used without a prohibitive computational cost.

Various first and second order approaches have been successfully tested on simple problems that can be solved in closed form, on sizing optimization of trusses, and on two-dimensional shape optimal design problems. In most cases convergence is achieved within five to ten structural reanalyses.

1. Introduction

This Lecture describes recent optimization methods particularly well adapted to solve many problems arising in structural design. Mathematically the numerical optimization problem considered herein can be written in the following general form:

$$\text{minimize} \quad c_0(x) \tag{1}$$

$$\text{subject to} \quad c_j(x) \leq 0 \qquad j=1,m \tag{2}$$

Although the objective function (1) is very often a linear function of the design variables x_i (classical weight minimization with sizing variables), herein $c_0(x)$ will be assumed to be possibly a nonlinear function, so that it can represent any structural characteristic to be minimized (e.g. stress concentration), with sizing or shape design variables. The inequalities (2), often called behaviour constraints, impose limitations on structural response quantities such as stresses and displacements under static loading cases. These constraints are usually nonlinear functions, but in some situations, they might also

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include linear functions. Of course the design variables must also be bounded from below and from above, however, these so-called side constraints are not considered explicitly in the sequel, for sake of clarity.

In this Lecture, various methods based on explicit approximation schemes are discussed, that have demonstrated strong potential for efficient solution of design optimization problems. The key idea is to replace the implicit problem (1,2) with a sequence of convex explicit subproblems having a simple algebraic form. After recalling the key role played by "reciprocal variables" in the short history of structural optimization, the "Convex Linearization" method (CONLIN) [1,2] will be described, as well as a recent generalization, the "Method of Moving Asymptotes" (MMA) [3]. In these two methods, each explicit subproblem represents a convex approximation to the primary problem, obtained through first order Taylor series expansion of the objective and constraint functions in terms of intermediate variables (e.g. direct/reciprocal variables). In addition to their theoretical and academic interest, the CONLIN and MMA methods have recently received growing attention from the industrial community. They have been recently implemented in several commercially available finite element systems.

In assessing the quality of an approximation scheme, there are a number of criteria to be considered, such as separability, convexity, conservativeness and accuracy, and a number of objectives to be pursued, namely, overall convergence of the sequence of approximate problems, convergence within a minimum number of stages, minimum effort for generating each approximate subproblem, and efficient solution of the explicit subproblems. Unfortunately, these objectives and approximation criteria often conflict with one another. In order to achieve a better compromise among them one needs more information than just the first order behaviour sensitivity derivatives. For instance, information about the way the design variables change between iterations (oscillation, steady but slow change) has been exploited in [3] as a basis for choosing the asymptotes.

Both CONLIN and MMA can be interpreted as first order convex approximation methods that attempt to estimate the curvature of the problem functions by assuming that these functions have a given separable form. This explicit separable form is governed by semi-empirical rules based on structural considerations (e.g. stress and displacement constraints are almost linear in the reciprocal variables). In other words, the Hessian matrices of these functions are assumed to be diagonal. Based on this observation, second order convex approximation strategies have begun to emerge, where exact curvature information is employed to build up the approximating subproblem.

One possible approach consists in resorting to a sequential quadratic programming strategy, restricted to the diagonal second derivatives of the Lagrangian function. For structural optimization problems involving static constraints, this latter approach can be implemented at the expense of only one additional "virtual load case" in the sensitivity analysis part of the finite element code. An alternative way of using diagonal second derivatives will be briefly described. The idea is to select intermediate linearization variables, so that the diagonal second derivatives of some combined constraint are zero. Finally, still another approach is to use the diagonal second derivatives of each function in order to define the ideal values of the moving asymptotes in the MMA method.

A very attractive feature of all these convex approximation approaches, is that they replace the primary optimization problem with a sequence of separable subproblems that can be solved efficiently by a dual method formulation. In the dual approach, discussed in another Lecture, the constrained primal minimization problem is replaced by

maximizing a quasi-unconstrained dual function depending only on the Lagrangian multipliers associated with the linearized constraints. These multipliers are the dual variables subject to simple non-negativity constraints. The efficiency of the dual formulation is due to the fact that maximization is performed in the dual space, whose dimensionality is relatively low and depends on the number of active constraints at each design iteration.

These various convex approximation techniques have been successfully tested on simple problems that can be solved in closed form, on sizing optimization of trusses, and on two-dimensional shape optimal design problems.

2. First order convex approximation methods

2.1. THE KEY ROLE OF RECIPROCAL VARIABLES

The so-called approximation concepts approach to structural optimization is now widely employed to solve optimal sizing problems [4,5]. Such problems consist in minimizing the weight of thin-walled structures modeled by bar and membrane elements. Because the geometry is fixed, the design variables reduce to the transverse sizes of the structural members (i.e. bar cross-sections and membrane thicknesses). This approach consists basically of the following steps:

- a finite element analysis is performed for the initial trial design;
- from the results of the current structural analysis, an approximate optimization problem is generated (this step implies that sensitivity analysis capabilities be available in the finite element code);
- because the approximate subproblem is fully explicit, convex and separable, it can be efficiently solved by resorting to its dual formulation;
- the solution of the approximate subproblem is adopted as a new starting point in the design space and the optimization process is continued until convergence is achieved.

In the foregoing approximation concepts approach the primary optimization problem is replaced with a sequence of explicit subproblems having a simple algebraic structure. Each subproblem is generated through first order Taylor series expansion of the objective function and constraints in terms of intermediate linearization variables. For example, linearization of the constraints with respect to reciprocal variables is a well recognized technique to solve optimal sizing problems. There is an intuitive explanation for the success of this technique, in that stresses and displacements are exact linear functions of the reciprocal sizing variables in the case of a statically determinate structure. For shape optimal design problems, there is no such physical guideline for the selection of intermediate linearization variables. Nevertheless, this change of variables continues to have a highly beneficial effect on the convergence properties of the shape optimization process [6,7].

Consequently it can be argued that an efficient strategy consists in linearizing the objective function with respect to the direct variables,

$$\tilde{c}_0(x) = c_0(x^0) + \Sigma\, (\partial c_0/\partial x_i)^0 (x_i - x_i^0) \tag{3}$$

while the constraint functions are linearized with respect to the reciprocal variables:

$$\tilde{c}_j(x) = c_j(x^0) - \Sigma\, (x_i^0)^2 (\partial c_j/\partial x_i)^0\, (1/x_i - 1/x_i^0) \tag{4}$$

From extensive numerical experiments it can be concluded that the approximation concepts approach converges to an optimum design in usually less than ten iterations (i.e. finite element analyses). These remarkable convergence properties are generally attributed to the fact that the behaviour constraints are much less nonlinear in the space of the reciprocal variables.

More general techniques have been developed based on the idea of reciprocal variables. The Convex Linearization method (CONLIN) employs reciprocal variables in the linearization process, but it also uses direct variables, depending upon the sign of the first derivatives of the function being approximated [1]. The initial motivation for CONLIN was to further extend the range of applicability of the approximation concepts and generalized optimality criteria methods, so that these approaches could handle constraints other than upper limits on stresses or displacements, e.g. strictly linear constraints in the sizing variables. Very soon it became apparent that the CONLIN approximation scheme was well adapted to shape optimization problems [6,7].

2.2. THE CONVEX LINEARIZATION METHOD (CONLIN)

The convex linearization method (CONLIN) [1] was initially conceived as an extension to the approximation concepts approach. The key idea in the CONLIN method is to perform the linearization process with respect to mixed variables, either direct or reciprocal, independently for each function involved in the optimization problem. At each successive iteration point, the CONLIN method only requires evaluation of the objective and constraint functions and their first derivatives with respect to the design variables. The optimizer will then select by itself an appropriate approximation scheme on the basis of the signs of the derivatives. This constitutes a major improvement with respect to the regular approximation concept approach, where it is usually assumed that the objective function is linear in the direct variables (e.g., structural weight) and that the constraints can be accurately approximated as linear functions of the reciprocal variables (e.g., stresses and displacements). Furthermore, the CONLIN optimizer has an inherent tendency to generate a sequence of steadily improving feasible designs, in contrast with the previously developed approximation concepts approach using dual methods [5].

The CONLIN method proceeds by linearizing each function defining the optimum design problem with respect to a properly selected mix of direct and reciprocal variables, so that a convex and separable subproblem is generated. The selection of the "intermediate" linearization variables is made on the basis of the signs of the first partial derivatives. It is easily proven that, considering any differentiable function c(x), the following linearization scheme yields a convex approximation (hence the name "convex linearization" initially suggested in Ref. [1]):

$$\tilde{c}(x) = \Sigma_+ (\partial c/\partial x_i)^0 x_i - \Sigma_- (x_i^0)^2 (\partial c/\partial x_i)^0 / x_i + c(x^0) - \Sigma\, |(\partial c/\partial x_i)^0|\, x_i^0 \tag{5}$$

where the symbol Σ_+ (Σ_-) means "summation over positive (negative) terms". Note that the two last terms in this expression represent the contribution of the zeroth order terms in the Taylor series expansion. One of the most interesting feature of the convex linearization scheme is that it also leads to the most conservative approximation amongst all the possible combinations of mixed direct/reciprocal variables. This property was initially demonstrated in [8], where conservative approximation was employed to handle difficult buckling constraints.

Another important property, which is particularly useful in relation to the dual approach is that the explicit approximation (5) is convex. The CONLIN algorithm applies this convex linearization scheme to the objective function and to all the constraint functions. The resulting explicit approximations take on a simpler form if the design variables are normalized so that they become equal to unity at the current point x^0 where the problem is linearized. The following convex subproblem is then generated:

$$\begin{aligned} &\text{minimize} \quad \Sigma_+ c_{i0}\, x_i - \Sigma\, c_{i0}/x_i - d_0 \\ &\text{subject to} \quad \Sigma_- c_{ij}\, x_i - \Sigma\, c_{ij}/x_i \leq d_j \quad j=1, m \end{aligned} \tag{6}$$

where the c_{ij} 's denote the normalized first derivatives of the objective and constraint functions evaluated at the current point x^0. Note that the constants d_j collect the zeroth order contributions in the Taylor series expansion.

In CONLIN, therefore, the linearization process is performed with respect to mixed variables, either direct or reciprocal, independently for each function involved in the optimization problem. Direct variables are used for positive first derivatives, while reciprocal variables are employed for negative first derivatives. A convex, separable subproblem is generated, that can be efficiently solved by using the dual method approach presented in another Lecture.

Because the CONLIN strategy employs conservative approximations, it has an inherent tendency to generate a sequence of steadily improving feasible designs. In addition to this desirable property, the CONLIN optimizer usually produces a nearly optimal design within less than 10 reanalyses. In fact, a rigorous mathematical study has demonstrated that the CONLIN method converges to a local optimum under rather unrestrictive assumptions [9].

2.3. THE METHOD OF MOVING ASYMPTOTES (MMA)

In practice, the CONLIN optimizer exhibits very good convergence properties when dealing with structural optimization problems. It usually converges in less than ten iterations, for sizing problems as well as for shape optimal design problems. However, in some cases, the convex approximation scheme used in CONLIN might not be appropriate, leading to inaccurate approximations, either too conservative (in which case slow convergence occurs), or not sufficiently conservative (in which case oscillations can set in). Some counter-examples illustrating this behaviour can be found in Ref. [3], where a simple and very elegant modification of the convex linearization method is proposed, called the Method of Moving Asymptotes (MMA).

In the MMA method, the intermediate linearization variables are modified such that the degree of convexity, and therefore conservativeness, of the approximation can be adjusted depending upon the problem being solved. Instead of just using direct and reciprocal variables, this method employs the intermediate variables $1/(U_i-x_i)$ and $1/(x_i-L_i)$, respectively, where U_i and L_i are user-selected parameters.

In the first order Taylor series expansion, the selection of the intermediate linearization variable is still based upon the signs of the first derivatives. The MMA approximation has the form:

$$\tilde{c}(x) = \Sigma_+ c_i/(U_i-x_i) - \Sigma_- c_i/(x_i-L_i) - d^0 \qquad (7)$$

where

$$c_i = (U_i-x_i^0)^2(\partial c/\partial x_i)^0 \qquad \text{if } (\partial c/\partial x_i)^0 > 0$$

$$c_i = -(x_i^0-L_i)^2(\partial c/\partial x_i)^0 \qquad \text{if } (\partial c/\partial x_i)^0 < 0$$

The constant d^0 collects the zeroth order terms of the expansion.

As in CONLIN, the MMA method replaces the primary problem (1,2) with a sequence of convex subproblems:

$$\begin{aligned} &\text{minimize} \quad \Sigma_+ c_{i0}/(U_i-x_i) - \Sigma_- c_{i0}/(x_i-L_i) \\ &\text{subject to} \quad \Sigma_+ c_{ij}/(U_i-x_i) - \Sigma_- c_{ij}/(x_i-L_i) \leq d_j \end{aligned} \qquad (8)$$

In this technique the form of each approximated function is driven by the selected values for the constants L_i and U_i, which act as "asymptotes". Taking $L_i = 0$ and $U_i = +\infty$ gives back the initial CONLIN approach; $L_i = -\infty$ and $U_i = +\infty$ leads to a sequential linear programming technique. Other reasonable values of L_i and U_i are acceptable, and these values may even be modified from one iteration to the next. That is why this method has been called the "Method of Moving Asymptotes".

2.4. OTHER CONVEX APPROXIMATIONS (POWER EXPANSIONS)

The approximation scheme used in CONLIN is powerful for conventional sizing problems with static response constraints. For a wider class of problems, for example those involving bending elements or shape design variables, as well as for problems with dynamic constraints, more elaborate approximation schemes are necessary in order to taylor the explicit approximate function to the type of nonlinearity in each problem.

In Ref. [10] the sizing of frames subject to frequency constraints is investigated using approximations of the form:

$$\tilde{c}(x) = c(x^0) + \Sigma_+ (\partial c/\partial x_i)^0(x_i-x_i^0)(x_i/x_i^0)^{p+} + \Sigma_- (\partial c/\partial x_i)^0(x_i-x_i^0)(x_i/x_i^0)^{p-} \qquad (9)$$

where $p^+ \geq 0$ and $p^- < 0$ are adjustable parameters. This approximation is sometimes referred to as "hybrid power expansion". It should be pointed out that it coincides with the CONLIN approximation (5) for the values $p^+ = 0$ and $p^- = -1$, while for $p^+ = p^- = 0$, it reduces to a conventional linear Taylor series expansion in the direct variables [see Eq. (3)]. It is shown in [10] that this approximation can be made convex over a fairly large domain around the base point x^0.

On the other hand, this idea of reciprocal variables has been extended to structural synthesis problems involving bending elements, like plates and shells [11], frames [12], etc... For example it is clear that the best linearization variable for a plate element in pure bending is the reciprocal of the cube of the thickness. For more general situations, where bending and extension effects are of the same order of magnitude, the selection of the intermediate variable is not that obvious [13]. Much research has been devoted to this topic. In particular Ref. [14] presents a systematic investigation of which intermediate variables should be used for an extensive class of structural models and behaviour constraints. A rather general change of variables is introduced, and the approximation consists of a first order Taylor series expansion in terms of the new variables. These intermediate variables are chosen in the form:

$$z_i = 1/(p-1)\, I_i^{1-p} \qquad \text{if } p \neq 1$$

$$z_i = -\ln(I_i) \qquad \text{if } p = 1$$

where $I_i = I_i(x_i)$ is a structural parameter, usually a cross-sectional property such as the moment of inertia or the area for a beam member. The exponent p is a parameter which provides a means to control the degree of curvature of the approximation. Note that the Taylor series approximations expressed in Eq. 3 (in direct variables) and in Eq. 4 (in reciprocal variables) correspond respectively to $p = 0$ and $p = 2$.

3. Second order convex approximation methods

It is important to realize that both CONLIN, MMA, and the other approximation schemes discussed in the previous Section are based on first order approximations that attempt to simulate the curvature of the objective function and constraint functions. These approximations are convex. They are also conservative, in that they tend to overestimate the true value of the approximated function.

In CONLIN the degree of conservativeness is fixed, while MMA offers much more flexibility through the moving asymptotes L and U. In other words CONLIN will provide some curvature to the approximated function, but the artificial curvature introduced by the reciprocal variables will generally be different from the true curvature.

On the other hand MMA is capable of matching the explicit curvature to the exact curvature, provided that the asymptotes are appropriately selected. Obviously, when only first derivatives are available, it is not at all straightforward to find suitable values for the asymptotes, so as to accurately represent the curvature of each function. Empirical techniques can be used to gradually update the values of L and U, depending upon the results generated at each iteration in the optimization process. For example, in Ref. [3], where the MMA method was first proposed as a generalization to the CONLIN method, it is suggested that the asymptotes be moved away from the current point x^0 if the optimization process is converging slowly. On the other hand, if the process tends to oscillate, then the asymptotes are moved closer to x^0.

The schemes that have been reported up to now to adjust the curvatures tend to be heuristic and problem dependent. In this Section, more general and rational schemes are described, based on second order sensitivity information. The methods discussed below can indeed be viewed as further generalizations of the convex approximation strategies, such as CONLIN and MMA. Because these new methods use the diagonal second derivatives of the Lagrangian function, they inherently build up the required information on the problem curvature. An important advantage of the proposed approach resides in its simplicity of use and implementation. There is no need to set up a sophisticated linearization scheme using well selected intermediate variables, nor to update control parameters in order to simulate the curvature of the constraint and objective functions. Numerical experiments on well known test problems indicate that these methods perform equally well whatever design variables are selected as base variables (direct, reciprocal, mixed, etc...).

3.1. PURE SEQUENTIAL QUADRATIC PROGRAMMING (SQP)

The approach to structural optimization which is presented in the sequel is based on a well known method in mathematical programming, first proposed and referred to by Wilson [15] as the SOLVER method, and also known as the Lagrange-Newton method [16]. Indeed, this method can be interpreted as applying Newton's method to find the stationary point of the Lagrangian function

$$L(x,\lambda) = \sum_{j=0}^{m} \lambda_j c_j(x) \qquad (10)$$

where λ_0 is fixed to unity and $\{\lambda_j, j=1,m\}$ denote the Lagrangian multipliers associated with the constraints $c_j(x)$. This approach can also appropriately be called a Sequential Quadratic Programming (SQP) method, because it replaces the primary optimization problem (1,2) with a sequence of quadratic approximations of the form:

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2} (x - x^0)^T A (x - x^0) + b^T (x - x^0) \\ &\text{subject to} \quad c(x^0) + C^T (x - x^0) \geq 0 \end{aligned} \qquad (11)$$

where the vector b and the matrix C respectively contain the first derivatives of the objective function and those of the constraint functions, evaluated at x^0:

$$b_i = \partial c_0/\partial x_i \mid x^0$$

$$c_{ij} = \partial c_j/\partial x_i \mid x^0$$

The symmetric matrix A represents the Hessian of the Lagrangian function, and consequently, it contains information on the functions curvature:

$$A_{ik} \equiv \partial^2 L/\partial x_i \partial x_k = \sum_{j=0}^{m} \lambda_j^0 \, \partial^2 c_j/\partial x_i \partial x_k \qquad (12)$$

where the λ_j^0's denote the current values of the Lagrangian multipliers associated with the constraints (λ_j^0 is zero if the constraint c_j is inactive).

The application of the method needs, as usual, a sensitivity analysis providing the first derivatives of the problem functions. In addition the A matrix needs evaluation of the second derivatives for the objective function and all the currently active constraints (i.e the constraints associated with non-zero Lagrange multipliers). In many cases the required second order sensitivity analysis is not available, or it is computationally too expensive. For that reason, the most common SQP implementations resort to a quasi-Newton approach, in which the A matrix is only an approximation to the Hessian of the Lagrangian function, gradually built up at each iteration from the first order derivatives (DFP or BFGS update formulas).

Obviously the SQP approach based upon quasi-Newton approximation will not, in general, converge as fast as a pure Newton approach using true second derivatives. The motivation of the study presented in [17], and briefly discussed below, was therefore to keep the initial formulation of the Lagrange-Newton method, which requires second order sensitivity analysis. The key idea is to take advantage of the special mathematical structure exhibited by structural optimization problems in order to generate the required second order sensitivity information with the smallest possible computational time.

The evaluation of sensitivity derivatives for static displacement and stress constraints is now well established and documented (see e.g. [18]). Differentiating the equilibrium equations of the finite element model,

$$K\, q = g \qquad (13)$$

where K is the stiffness matrix, g a load vector, and q the nodal displacements, we obtain:

$$K\, \partial q/\partial x_i = \partial g/\partial x_i - (\partial K/\partial x_i)\, q \qquad (14)$$

Differentiating again:

$$K\, \partial^2 q/\partial x_i \partial x_k = \partial^2 g/\partial x_i \partial x_k - (\partial^2 K/\partial x_i \partial x_k) q - (\partial K/\partial x_i)\, \partial q/\partial x_k - (\partial K/\partial x_k)\, \partial q/\partial x_i \qquad (15)$$

The "reduced" second order sensitivity analysis presented in [17] consists of using the first order sensitivity equation in its direct form to get the displacement first derivatives, from which the constraint gradients can be readily evaluated. On the other hand the second order sensitivity equation is employed in its adjoint form, with the addition of only one virtual load case. It can indeed be observed that the constraint curvature term appearing in the expression (12) of the A matrix, does not require evaluation of the second derivatives for each individual constraint. Only the Lagrangian function (10) need to be considered. Therefore it is sufficient to build one virtual load case corresponding to a linear combination of the constraints with coefficients λ_j (non zero Lagrangian multipliers).

To help fix ideas, let us consider the familiar 10-bar truss example, used in this Lecture to illustrate various SCP methods. The problem consists of minimizing the weight of the truss with limits on the stresses and vertical displacements under the loading shown in Fig. 1. In this particular case the problem involves 10 design variables, so that the sensitivity analysis needs consideration of 10 pseudo-load cases, plus one virtual load case. Fig. 1 summarizes very clearly the substantial benefits that can be gained from using the foregoing approach: for only a small increase in computer time (one more right hand side in the equilibrium equations), the total number of iterations required for convergence of the optimization process is cut in half.

Table 1 provides a more detailed iteration history for this problem. The "feasible weight" is obtained by multiplying the actual weight generated by the SQP algorithm, by the ratio of the most critical constraint value to its upper bound. This "scaling factor" is also tabulated. From these results it can be clearly observed that the ultimate rate of convergence is of order two: the value of the most critical constraint approaches its upper bound in a quadratic rate in the last three iterations. It is important to mention that, in a practical application, the optimization process could already be terminated after four or five iterations. However it was interesting to continue up to convergence within more than six digits accuracy, in order to demonstrate numerically the quadratic rate of convergence of the Newton-Lagrange method.

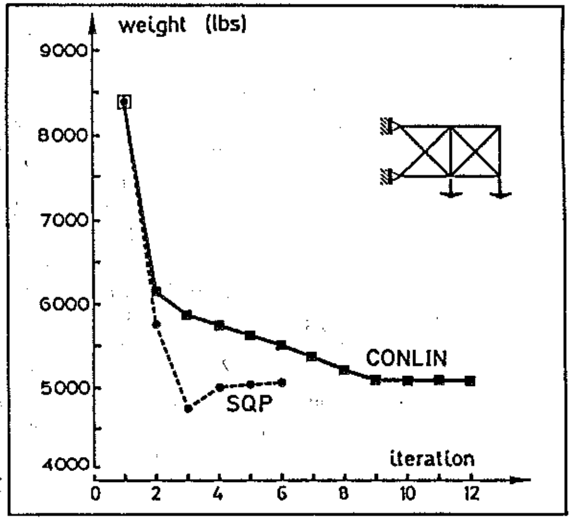

Figure 1. 10-Bar Truss Iteration History

Table 1. 10-bar Truss - Iteration History with Full Hessian Matrix

Iteration	Scaling Factor	Feasible Weight
1	0.98489	8266.15
2	1.06246	6109.40
3	1.08141	5132.36
4	1.01424	5076.59
5	1.00470	5061.76
6	1.00004	5060.87
7	1.00000003	5060.85

3.2. THE DIAGONAL SQP METHOD

In the 10-bar truss example the method described above is quite efficient. However, for practical sizing problems, where the number of design variables might be large, computing and storing the nxn A matrix could become burdensome. On the other hand, for shape optimal design applications, the stiffness matrix can no longer be expressed as a simple explicit form of the design variables (e.g. positions of Bezier control nodes), and its second derivatives can be difficult to evaluate. Hence the idea to neglect the coupling between design variables, and to restrict the A matrix to its diagonal terms.

The quadratic subproblem now considered has the following separable form:

$$\text{minimize} \quad \tfrac{1}{2} \Sigma\, a_i(x_i - x_i^0)^2 + \Sigma\, b_i(x_i - x_i^0)$$
$$\text{subject to} \quad \Sigma\, c_{ij}\, x_i \geq d_j \qquad (16)$$

where

$$a_i = A_{ii} + \delta_i \qquad (17)$$

and A_{ii} are the diagonal elements of the Hessian matrix of the Lagrangian function (see Eq. 12). The c_{ij} and b_i have the same meaning as in problem (11), i.e. they represent the first derivatives of the constraints and of the objective function, respectively. Note that the zeroth order contributions in the linear Taylor series approximations of the constraints have been collected in the form of lower bounds d_j.

The parameters δ_i in (17) are important. They convexify the problem in the case where some of the diagonal second derivatives happen to be negative, or zero, which frequently occur in the early stages of the optimization process. In addition the control parameters δ_i act as "move limits". What has been done was simply to add to the objective function a term that represents a weighed distance with respect to x^0, i.e., the design point where the problem has been approximated:

$$\tfrac{1}{2} \Sigma\, \delta_i\, (x_i - x_i^0)^2 \qquad (18)$$

One simple way to select the parameters δ_i is to ignore the linear constraints. We are left with an unconstrained problem whose solution is:

$$x_i = x_i^0 - b_i/(A_{ii} + \delta_i)$$

It is then straightforward to find δ_i such that, for example, each design variable x_i will remain within some percentage of its initial value x_i^0. In particular, in weight minimization problems (e.g. 10-bar truss), the objective function is linear, and it does not contribute to the Hessian matrix A. Therefore, at the first iteration, where the Lagrangian multipliers are usually set to zero, the A_{ii} are zero and the move limits δ_i must be given. From numerical experiments it appears that 100 % move limits are appropriate values for the first iteration. For the subsequent iterations, some of the Lagrangian multipliers become positive, and the constraint curvature starts to build up in the diagonal Hessian matrix, so that the move limits δ_i become less critical.

With this minor modification, the a_i's are positive and the quadratic subproblem (16) is strictly convex. This problem involves only linear constraints, and many algorithms are available to solve it. However, because the problem is also convex and separable, a dual approach constitutes probably the best choice.

3.3. ZERO CURVATURE APPROXIMATION

As previously mentioned reciprocal variables have played a significant role in the history of structural optimization. It is now well recognized that, for sizing problems involving bar and membrane elements, high quality approximation of the behaviour constraints can be achieved through linearization in terms of the reciprocal of the design variables (cross-sectional areas of bars; thicknesses of membrane elements). This idea has been extended to structural synthesis problems involving bending elements, like plates and shells [11], frames [12], as well as more general situations, where bending and extension effects are of the same order of magnitude [13,14].

Table 2. 10-bar Truss - Iteration History for Various Linearization Variables (1st entry: Weight; 2nd entry: Scaling Factor)

Iter.	p = 1		p = ½		p = -1		p = -½		p(i)	
1	8393	0.985	8393	0.985	8393	0.985	8393	0.985	8393	0.985
2	6113	1.093	4658	1.354	6946	1.000	6608	1.030	5732	1.040
3	5544	1.181	5143	1.105	6457	0.998	6250	1.004	5805	0.992
4	5145	1.014	5458	1.009	6229	0.994	6040	0.995	5688	0.988
5	4951	1.126	5361	0.992	6044	0.995	5859	0.995	5548	0.987
6	5020	1.014	5153	1.003	5898	0.995	5713	0.995	5388	0.985
7	5033	1.007	5055	1.003	5768	0.994	5575	0.994	5193	0.998
8	5047	1.003	5059	1.002	5645	0.995	5441	0.994	5030	1.007
9	5058	1.001	5059	1.000	5526	0.995	5305	0.994	5062	1.000
10	5060	1.000	5061	1.000	5412	0.995	5187	1.000	5061	1.000
11	5061	1.000			5300	0.996	5126	1.000		
12					5202	1.000	5100	1.000		
13					5146	1.001	5084	1.000		
14					5119	1.000	5077	1.000		
15					5101	1.000				
16					5089	1.000				
17					5081	1.000				
18					5077	1.000				

Most first order methods commonly used in structural optimization are very sensitive to the choice of the intermediate linearization variables. However, because it uses limited second order information, the method described below does not suffer from this drawback. In the sequel our discussion will be focused on sizing problems, where the goal is to minimize the structural weight. The following change of variables covers most situations:

$$z_i = x_i^p$$

For example, p = +1 does not change anything and gives back the direct variables; the choice p = -1 leads to reciprocal variables; pure plate bending elements should use p = -3. Square root variables (p = +½) can also give interesting results, because they transform the linear weight objective function into a quadratic function.

The first and second derivatives of any function $c[x(z)]$ can be obtained with respect to the new variables z_i by using the following transformations (omitting the subscript i):

$$\partial c/\partial z = 1/p\, x^{1-p}\, \partial c/\partial x$$

$$\partial^2 c/\partial z^2 = (1-p)/p^2\, x^{1-2p}\, \partial c/\partial x + 1/p^2\, x^{2(1-p)}\, \partial^2 c/\partial x^2 \quad (19)$$

Various values of the exponent p have been tried out for the classical 10-bar truss example shown in Fig. 1. The results given in Table 2 demonstrate that the method converges rapidly for any value of p. Surprisingly, direct variables (p=1), and even square root variables (p=½), lead to faster convergence than reciprocal variables (p=-1). This is probably because the objective function is represented exactly in those two cases: direct variables preserve the linear form of the weight objective function, while square root variables transform it into a quadratic separable function. However, as it might be expected, reciprocal variables yield a sequence of almost feasible designs: at each iteration, the scaling factor needed to bring back the design point on the boundary of the feasible domain, is very close to one. This behaviour is also true in the case p=-½. On the other hand, some of the designs in the p=+1 and p=+½ sequences, seriously violate the behaviour constraints.

The last column in Table 2 provides interesting results obtained with a variation of the method. The idea is to select the exponent p independently for each design variable (i), so that the second diagonal derivatives of the Lagrangian function are zero (see Eqs. 10 and 12). In other words the constraints are linearized in some space of minimum global curvature (zero curvature in the case of truly separable constraints). Because in the present case the objective function is known explicitly, the initial diagonal SQP method can be modified to further speed up convergence. Instead of solving the quadratic subproblem (16), it is indeed much better to keep the explicit nonlinear form of the weight objective function, and to solve the following linearly constrained subproblem:

$$\text{minimize} \quad f(z) = \Sigma\, b_i\, z_i^{1/p(i)}$$
$$\text{subject to} \quad \Sigma\, c_{ij}\, z_i \geq \underline{c}_j \quad (20)$$

In this particular truss example, the coefficients b_i are related to the lengths of the bars. The exponents p(i) are selected so that the global constraint curvature is zero. Applying (19) to the Lagrangian function (10), and setting the resulting second derivative to zero, we obtain:

$$0 = (1-p)/p^2\, x^{1-2p}\, \partial L/\partial x + 1/p^2\, x^{2(1-p)}\, \partial^2 L/\partial x^2$$

Solving this equation for the exponent p yields, for each design variable (i):

$$p(i) = 1 + x_i^0\, (\partial^2 L/\partial x_i^2)/(\partial L/\partial x_i)$$

where the first and second derivatives are evaluated at the linearization point x^0.

The foregoing optimization strategy is interesting from a practical point of view, because it produces a sequence of nearly feasible designs while keeping the fast convergence of the diagonal SQP approach. An additional benefit gained from this modified method lies in its ability to dynamically select the "best" intermediate linearization variables for any structural optimization problem. For a statically determinate truss, the method will inherently determine that the exponent p must be equal to -1 for each design variable (i.e. automatic selection of reciprocal variables). For a statically indeterminate truss, the method will obtain different values for each p_i, most likely (but not necessarily) negative and close to -1. In the case of structural models involving in-plane as well as bending behaviour, the method would probably select values for p(i) between -1 and -3.

3.4. SECOND ORDER MMA METHOD

As shown in the previous Sections, a natural way of using second order information consists of constructing Taylor series expansions. However, from the results reported in Ref. [10], it can be argued that convex approximation schemes such as those used in the CONLIN and MMA methods, can reduce the number of explicit subproblems when compared to second order Taylor series. This suggests that convex approximation strategies, although they are based on first order information only, might have the ability to represent the exact functions even better than higher order standard methods. The drawback of generalized convex linearization methods like MMA is that they necessitate the setting of parameters with a high degree of accuracy. Hence the idea has emerged, to combine the advantages of pure SQP methods and the potential of convex approximations for high quality. This can be achieved by using second order information in order to select proper values for the tuning parameters (e.g. moving asymptotes in MMA).

The flexibility of the generalized convex approximations such as MMA is associated with adjustable parameters which control the curvature of the approximation. However the basis for selecting their appropriate values remains empirical. Indeed, the suggested approximations are all based on first order sensitivity, which does not provide any information about the curvatures. This information is either built up during the optimization process by means of empirical rules [3], or gathered from parametric studies on specific classes of problems [14].

A more rational and systematic approach based on second order information is proposed in [19]. In particular the Method of Moving Asymptotes (MMA) was implemented and successfully tested with an automated strategy for selecting the best possible values for the moving asymptotes. It is shown how restricted second order information can be used as a basis for a general and systematic rule for selecting the asymptotes of the MMA approximations. Since the approximations dealt with are separable, only diagonal terms of the second derivative matrices are needed in the process.

Distinct asymptotes for each constraint and each design variable are selected so that positive diagonal second derivatives of the constraint function and its approximate representation are identical. The explicit approximation of a function c(x) at the current design point x^0 has now the form:

$$\tilde{c}(x) = \Sigma\, a_i/(x_i-b_i) - c \qquad (21)$$

where, in order to match the first and second derivatives of the exact and approximate functions:

$$a_i = -(x_i^0-b_i)^2\, (\partial c/\partial x_i)^0$$

and

$$b_i = x_i^0 + 2\, (\partial c/\partial x_i)^0/(\partial^2 c/\partial x_i^2)^0 \qquad (22)$$

The zero order term c is given by

$$c = \Sigma\, a_i/(x_i^0-b_i) - c(x^0)$$

To help fix ideas, the reader is invited to construct the following explicit approximations (at $x^0 = 1$) for two simple functions:

$$c(x) = x^3 \quad ===> \quad \tilde{c}(x) = 3/(2-x) - 2$$

$$c(x) = 1/x^3 \quad ===> \quad \tilde{c}(x) = 3/(4x-2) - 1/2$$

It can be seen that for the increasing cubic function, an upper asymptote is selected, while for the inverse cubic function, which is decreasing, a lower asymptote is used.

In order to preserve convexity, in case of zero or negative curvature, $(\partial^2 c/\partial x_i^2)^0$ is replaced in Eq. (22) with a small positive number. It is important to notice that this procedure permits obtaining a convex explicit approximation for a linear function. For example, at $x^0 = 0$, the increasing function c(x) = x is approximated by:

$$\tilde{c}(x) = U[(U/(U-x) - 1] \; (= x/[1-x/U])$$

where U denotes a big number (upper asymptote). On the other hand, the decreasing function c(x) = -x is approximated by:

$$\tilde{c}(x) = L[(L/(x-L) + 1] \; (= -x/[1-x/L])$$

where L denotes a negative number with big absolute value (lower asymptote). These two simple examples illustrate well how the MMA approximation is capable of representing linear functions. Coming back to the original MMA method (Section 2.3) it is seen that the limiting case where the asymptotes are $L_i = -\infty$ and $U_i = +\infty$ for all the approximated functions leads to a sequential linear programming technique.

Applying this second order convex approximation scheme to all the functions defining the optimization problem leads to the following explicit subproblem:

$$\begin{aligned} &\text{minimize} \quad \Sigma\, a_{i0}/(x_i-b_{i0}) - c_0 \\ &\text{subject to} \quad \Sigma\, a_{ij}/(x_i-b_{ij}) \leq c_j \qquad j=1, m \end{aligned} \qquad (23)$$

It should be observed that the subscript j has been added to the asymptotes, now denoted b_{ij}. This is different from the original MMA method, where a unique pair (U_i and L_i) is associated to each primal variable, but remains the same for each approximated function. As explained in another Lecture, the explicit subproblem (23) can be efficiently solved by dual methods. However, because each asymptote depends now on both the primal variable (index i) and the constraint (index j), the Lagrangian problem becomes implicit and requires a numerical one-dimensional solution. The effort demanded by the numerical solution of the implicit Lagrangian problem, as well as the significant cost of evaluating second derivatives, represent the price one has to pay for the high quality achieved for the approximations. Nevertheless, numerical results for several test problems indicate that the approximation based on second order information requires fewer approximate problems than the original method of moving asymptotes.

4. The conlin optimizer

Based on the concepts outlined in this Lecture, as well as on the dual method approach presented in another Lecture, an advanced optimizer was developed, that has been adopted in several FEM based optimization systems. This optimizer is called CONLIN, because it was initially restricted to the pure "CONvex LINearization" method presented in Section 2.2. The CONLIN optimizer solves problems of the form:

Minimize an objective function $c_0(x) + p\,\delta^2$

subject to constraints:

$$c_j(x) \le c_j^{max} + (\delta-1)\, c_j^{add} \qquad j=1,\ldots,m$$

$$x_i^{min} \le x_i \le x_i^{max} \qquad i=1,\ldots,n$$

$$1 \le \delta \le 2$$

where $x = (x_1,\ldots,x_n)^T$ is the vector of design variables x_i, δ represents an additional scalar variable, and p is a user-controllable weighing factor.

The additional variable δ is optional; it becomes useful only when the optimization problem does not have any solution. In that case the feasible domain, initially empty for $\delta = 1$, is opened up for $\delta > 1$, by "relaxing" the upper bounds c_j^{max}. For this reason δ is called the relaxation variable. Note that the maximum possible relaxation, corresponding to $\delta = 2$, can be controlled by the user via the weighing factor p in the objective function, as well as the increments d_j in the constraints.

At each stage of the iterative optimization process, the user must evaluate and provide the following quantities, which are employed by CONLIN to build first order explicit approximations:

- the values of the functions defining the design optimization problem (i.e. the objective function $c_0(x)$ and the constraint functions $c_j(x)$, evaluated at the current design point);
- the first derivatives of these functions:

$$c_{ij} = \partial c_j / \partial x_i \qquad i=1,2,\ldots,n \quad j=0,1,\ldots,m$$

In addition the user must decide which type of approximation will be used for each function: convex, reciprocal, or direct (linear) first Taylor series expansion.

CONLIN also contains a provision for some of the more sophisticated explicit approximations previously reviewed in this Lecture: shifted convex (MMA), diagonal quadratic or power Taylor series expansions. For these higher order separable approximations, it is necessary to input additional information related to the diagonal second derivatives of the functions (true values or some estimation).

Of course the user must also input the current values of the design variables x_i, as well as the other parameters defining the optimization problem: x_i^{min}, x_i^{max}, c_j^{max} and c_j^{add}.

4.1. TYPE OF EXPLICIT APPROXIMATIONS

The CONLIN optimizer can currently handle six types of separable explicit approximations: types 0 through 2 are first order Taylor series expansions in mixed, reciprocal or direct variables, while types 3 through 5 correspond to higher order approximations. The dual optimizers implemented in CONLIN are general, and they could easily accommodate other types of separable approximations.

Type 0: convex expansion

First order Taylor series expansion in mixed variables x_i and $1/x_i$, according to the sign of the first derivatives:

$$c(x) = \Sigma_+\, c_i x_i - \Sigma_-\, c_i/x_i$$

Type 1: reciprocal expansion

First order Taylor series expansion in the reciprocal variables $1/x_i$:

$$c(x) = \Sigma\, c_i/x_i$$

Type 2: direct (linear) expansion

First order Taylor series expansion in the direct variables x_i:

$$c(x) = \Sigma\, c_i x_i$$

Type 3: diagonal quadratic expansion

Diagonal second order Taylor series expansion in the direct variables x_i (in addition to the first derivatives, the user must supply the diagonal second derivatives):

$$c(x) = \tfrac{1}{2}\Sigma\, a_i x_i^2 + \Sigma\, b_i x_i$$

Type 4: shifted convex expansion (MMA)

First order Taylor series expansion in the intermediate variables $1/(x_i-m_i)$ or $1/(m_i-x_i)$ according to the sign of the first derivatives, where the m_i's are user supplied "moving asymptotes" for each design variable i. The explicit approximation has the form:

$$c(x) = \Sigma_+\, c_i/(m_i-x_i) - \Sigma_-\, c_i/(x_i-m_i)$$

Type 5: power expansion

First order Taylor series expansion in the intermediate variables $x_i^{p_i}$ where p_i is a user supplied exponent for each design variable i. The explicit approximation has the form:

$$c(x) = \Sigma\, c_i x_i^{p_i}$$

Note:

- for types 0 through 2, the user must only input the first derivatives of the function $c(x)$;
- for types 3, 4 and 5, the user must provide an additional vector of n values: a_i, m_i, and p_i, respectively.

4.2. OPTIMIZATION STRATEGIES AND ALGORITHMS

Two different optimization strategies are implemented into the CONLIN program. The first strategy uses a pure convex linearization approach combined with a dual method formulation. The second strategy, based on a primal-dual formulation, is much more general, but it is less efficient for large scale problems and possibly less reliable.

Pure dual method approach

This approach is similar to the one used in previous versions of CONLIN, however it employs more efficient maximization algorithms. The explicit dual problem is replaced with a sequence of quadratic subproblems. Each subproblem is itself partially solved by either a first order or a second order maximization algorithm in the dual space. For very large scale problems, or when running on machines with small memory availability, a special out-of-core first order algorithm is also provided.

This strategy is efficient and highly reliable, but it is rather restrictive: the objective function and the constraints are ultimately approximated by convex linearization (Type 0). Therefore no equality constraints are permitted. Also only a limited number of approximation types can be accommodated. The objective function must be of type 0, while the constraints can be of types 0, 1, or 2. Higher order approximations (types 3 through 5) are absolutely not accepted.

Constraints of type 1 (reciprocal expansion) and type 2 (linear expansion) are internally replaced with a convex expansion of type 0, and a restart capability is implemented to iteratively solve the primary problem if it involves linear and/or reciprocal constraints. From these considerations, it appears that this first strategy is mostly recommended when all the problem functions are of type 0 (convex approximation).

It is important to notice that, when the user selects this strategy, he can influence the optimization results only externally, via the well known "move limit" technique. Indeed no convexification (or trust region) capabilities are provided.

Primal-dual method approach

In this second strategy, the primal explicit problem is solved by resorting to a sequential quadratic programming approach. Because all the approximating schemes used in CONLIN are separable, each quadratic subproblem will be separable too. When considering the dual of the quadratic subproblem, it appears to have the same form as in the first strategy. Therefore, similar first and second order dual algorithms can be employed to solve the sequence of subproblems. This strategy is very general and it can handle all the approximation types discussed above, as well as equality constraints.

In this primal-dual strategy, a strictly convex quadratic term, similar to (18), is added to the objective function. This term represents the distance to the current design point x0:

$$\delta\ \Sigma(x_i-x_i^0)^2$$

A large value of the constant δ leads to a small move in the design space, while a small value affects little the optimal solution to the current approximate subproblem. The constant δ can therefore be interpreted as a control parameter that acts as a "move limit" imposed on the explicit subproblem.

In summary, the derivation of a wide variety of convex approximating functions, coupled with the development of a highly efficient dual solver, can lead to a reliable general purpose optimization method.

5. Numerical examples

In addition to the 10-bar truss example that has been used as a support throughout the text, the methods using second derivatives proposed in this paper have been applied to the three problems used in [3] to assess the validity of the MMA method. Although these problems look simple and can be solved in closed form, they exhibit the typical characteristics of difficult structural optimization problems. These examples demonstrate the limitations of the methods using reciprocal variables, including CONLIN [1]. In fact, as stated in Ref. [3], when applied to these problems, the "traditional" method (i.e. of the CONLIN type) does not converge at all. The MMA method is capable of solving each problem efficiently, however the moving asymptotes L and U have to be adjusted very carefully at each iteration of the optimization process. Determining the right values for L and U might become cumbersome for larger problems. On the other hand the diagonal SQP method presented in this paper converges as fast as MMA for all the examples, without requiring any control parameters to be gradually adjusted.

5.1. CANTILEVER BEAM

The weight of a cantilever beam is to be minimized while assigning an upper limit on the tip displacement due to a given concentrated load. The beam is built up of five elements. Each beam element has a hollow square cross-section with constant thickness. The design variable associated to each element is the height (and width) of the square cross-section. This beam is simple from a structural analysis point of view, and therefore the optimization problem can be stated in closed form:

$$\text{minimize} \quad 0.0624\,(x_1 + x_2 + x_3 + x_4 + x_5)$$

$$\text{subject to} \quad 61/x_1^3 + 37/x_2^3 + 19/x_3^3 + 7/x_4^3 + 1/x_5^3 \leq 1$$

The only side constraints are that the design variables xi must remain non negative. The solution to this problem can easily be found analytically [3].

Table 3. Iteration History for Cantilever Beam
(1st entry: Weight; 2nd entry: Infeasibility)

Iter.	CONLIN	MMA (best)	Diagonal SQP (Direct)	Diagonal SQP (Reciprocal)
1	1.56 1.00	1.56 1.00	1.56 1.00	1.56 1.00
2	1.27 1.40	1.31 1.10	1.22 1.43	1.35 1.14
3	1.25 1.43	1.34 1.01	1.27 1.18	1.33 1.05
4	1.26 1.43	1.34 1.00	1.33 1.02	1.34 1.01
5	1.25 1.44		1.34 1.00	1.34 1.00
...	... ...			
13	1.26 1.42			

The diagonal SQP method was first used in the space of the direct variables, and then in the space of the reciprocal variables. The iteration history produced in both cases is given in Table 3. The initial Lagrangian multiplier is set to zero, and 100 % move limits are employed at the first iteration; otherwise the ai's would be zero in the quadratic function (18). The diagonal SQP method performs very well both in the direct space and in the reciprocal space. In the two cases, convergence is achieved in five iterations, however the constraint violation at intermediate design points is much smaller when working in the reciprocal space. It should be noted that, for this particular example, the space of zero constraint curvature would of course be the best choice. The method proposed in the preceding Section will automatically select the exponent p(i) = -3 for each design variable (i), and one single iteration is then sufficient for generating the optimum design.

For comparison the best and worst results obtained in [3] with the MMA method are also given in Table 3. The worst results correspond to using the CONLIN approach (i.e. L=0 and U=+∞ in MMA). The poor performance of CONLIN is due to the fact that the reciprocal variables do not introduce enough curvature in the approximate constraint. As a result the optimization process oscillates indefinitely between two highly infeasible designs. This behaviour can be stabilized using MMA, provided that very tight asymptotes are selected.

It should be noted that, for this particular example, the space of zero constraint curvature would of course be the best choice. The method proposed in the preceding Section will automatically select the exponent p(i) = -3 for each design variable (i), and one single iteration is then sufficient for generating the optimum design.

5.2. Two-Bar Truss

The 2-bar truss problem now considered is interesting, because it involves one sizing variable (cross-section of the bars) and one shape variable (width of the truss). The design optimization goal is to minimize the weight of the truss with upper limits on the stresses under one single load case. Again the optimization problem can be stated in closed form (see [3] for more details):

minimize $x_1 \sqrt{1 + x_2^2}$

subject to $0.124 \sqrt{1 + x_2^2}\,(8/x_1 + 1/x_1x_2) \le 1$

$0.124 \sqrt{1 + x_2^2}\,(8/x_1 - 1/x_1x_2) \le 1$

$0.2 \le x_1 \le 4.0 \qquad 0.1 \le x_2 \le 1.6$

Table 4 summarizes the iteration histories produced by the diagonal SQP method (direct and reciprocal spaces), MMA (best results presented in [3]), and CONLIN. The diagonal SQP approach performs equally well in the direct and reciprocal spaces (slightly better than MMA). Again CONLIN does not converge because it uses approximations that are not sufficiently conservative.

Table 4. Iteration History for 2-bar Truss Example
(1st entry: Weight; 2nd entry: Infeasibility)

Iter.	CONLIN	MMA (best)	Diagonal SQP (Direct)	Diagonal SQP (Reciprocal)
1	1.68 0.92	1.68 0.92	1.68 0.92	1.68 0.92
2	1.43 1.11	1.43 1.10	1.42 1.11	1.43 1.11
3	1.49 1.04	1.37 1.13	1.48 1.03	1.49 1.02
4	1.43 1.11	1.44 1.10	1.50 1.01	1.50 1.01
5	1.49 1.04	1.47 1.03	1.51 1.00	1.51 1.00
6	1.43 1.11	1.51 1.00		

5.3. Eight-Bar Truss

The eight-bar truss problem exhibits several interesting features. As shown in Table 5 the diagonal SQP approach converges rather slowly, much faster than CONLIN, but not as fast as the best results obtained by MMA [3]. In this problem there is much coupling between design variables, so that the separability assumption does not apply well. Therefore the full SQP approach leads to very good results.

In this specific example, it is possible to greatly improve the behaviour of the diagonal SQP method. Instead of positive move limits, slowing down convergence, let us introduce a negative δi in (20), in order to speed up convergence. It should be observed that the limiting case where δi = - Aii corresponds to a sequential linear programming (SLP) approach, that is known to perform well if the solution is at a vertex in the design space. This is the case for the 8-bar truss problem. The best results generated by MMA confirm this idea: the moving asymptotes are taken larger and larger, so that in the limit, MMA behaves as a SLP approach. However, in general, it is not recommended to attempt accelerating the convergence speed by introducing negative control parameters δi. This procedure can indeed produce oscillations, or even divergence.

Table 5. Iteration History for 8-bar Trus Example

Iter	Full SQP	Diag. SQP ($\delta = 0$)	Diag. SQP ($\delta < 0$)	MMA (best)	CONLIN
1	13.05	13.05	13.05	13.05	13.05
2	11.44	11.73	11.73	12.10	11.68
3	11.25	11.62	11.62	11.67	11.66
4	11.23	11.62	11.45	11.65	11.64
5	——	11.57	11.23	11.61	11.62
6		11.52	——	11.52	11.59
7		11.47		11.42	11.57
8		11.42		11.28	11.55
9		11.37		11.23	11.53
10		11.33		——	11.52
..		...			...
15		11.23			11.43
16		——			11.42
..					...
40					11.23

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