TOPOLOGY OPTIMIZATION: FUNDAMENTALS

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LAY-OUT

- Introduction
- Topology problem formulation: Problem statement
- Compliance minimization
- Homogenization method vs SIMP based
- Filtering techniques
- Sensitivity analysis
- Solution of optimization problems using structural approximations and dual maximization
- Vibration Problems
- Applications
INTRODUCTION
What is topology?
TYPES OF VARIABLES
- a/ Sizing
- b/ Shape
- c/ Topology
- (d/ Material)

TYPES OF OPTIMISATION
- structural
- multidisciplinary
  - structural
  - aerodynamics,
  - thermal,
  - manufacturing
One generally distinguishes two approaches of topology optimization:
- Topology optimization of naturally discrete structures (e.g. trusses)
- Topology optimization of continuum structures (eventually after FE discretization)
Why topology optimization?

- CAD approach does not allow topology modifications

A better morphology by topology optimization (Duysinx, 1996)
TOPOLOGY PROBLEM
FORMULATION
Abandon CAD model description based on boundary description

Optimal topology is given by an optimal material distribution problem

Search for the indicator function of the domain occupied by the material

The physical properties write

The problem is intrinsically a binary 0-1 problem \(\Rightarrow\) solution is extremely difficult to solve
MATERIAL DENSITY FUNCTION

- Avoid 0/1 problem and replace by a continuous approximation considering a variable density material running from void (0) to solid (1) $\chi \in [0, 1]$

- Homogenization law of mechanical properties a porous material for any volume fraction (density) of materials

- Mathematical interpolation and regularization
  - SIMP model $E = x^p E^0$
  - RAMP

- Penalization of intermediate densities to end-up with black and white solution

- Efficient solution of optimization problem based on sensitivity analysis and gradient based mathematical programming algorithms

$$E^* = \rho^p E^0$$
Implementation of material density approach is rather easy:

- Fixed mesh
- Design variables are element or nodal densities
- Similar to sizing problem
- SIMP law is easy to code
- Sensitivity of compliance is cheap
- Use efficient gradient based optimization algorithms as MMA
A FIRST EXAMPLE: GENESIS OF A STRUCTURE
TOPOLOGY OPTIMIZATION AS A COMPLIANCE MINIMIZATION PROBLEM
DESIGN PROBLEM FORMULATION

- The fundamental problem of topology optimization deals with the optimal material distribution within a continuum structure subject to a single static loading.

- In addition one can assume that the structure is subject to homogeneous boundary conditions on $\Gamma_u$.

  $$u_i = 0 \quad \text{on} \quad \Gamma_u$$

- The principle of virtual work writes

  $$\mathbf{u} \in V_0 : \quad a_E(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \quad \forall \mathbf{v} \in V_0$$

  $$\delta \mathbf{q}^T \mathbf{Kq} = \delta \mathbf{q}^T \mathbf{g}$$
DESIGN PROBLEM FORMULATION

- A typical topology optimization problem is to find the best subset of the design domain minimizing the volume or alternatively the mass of the structure,

\[ V = \int_{\Omega} \rho(x) \, d\Omega \]

- while achieving a given level of functional (mechanical) performance.

Following Kohn (1988), the problem is well posed from a mathematical point of view if the mechanical behaviour is sufficiently smooth. Typically one can consider:
- Compliance (energy norm)
- A certain norm of the displacement over the domain
- A limitation of the maximum stress
Design Problem Formulation

- Compliance performance: The mechanical work of the external loads
  \[ l(u) = \int_{\Omega} f^T u \, d\Omega + \int_{\Gamma} t^T u \, d\Gamma \]

  - Using finite element formulation
    \[ C(q) = g^T q \]

- Limitation of a given local stress measure \( ||\sigma(x)|| \) over a sub-domain \( \Omega_2 \) excluding some neighborhood of singular points related to some geometrical properties of the domain (reentrant corners) or some applied loads
  \[ \sup_{x \in \Omega_2} ||\sigma(x)|| \]

  - Stress measure \( ||\sigma(x)|| \) → Von Mises, Tresca, Tsai-Hill...
DESIGN PROBLEM FORMULATION

- The average displacement (according to a selected norm) over the domain or a subdomain $\Omega_1$ excluding some irregular points
  \[
  \int_{\Omega_1} ||u|| \, d\Omega
  \]

- If one considers the quadratic norm and if the finite element discretization is used, one reads
  \[
  \int_{\Omega_1} u^T u \, d\Omega = \int_{\Omega_1} q^T N(x)^T N(x) q \, d\Omega = q^T M q
  \]

- Assuming a lumped approximation of the matrix $M$, one can find the simplified equivalent quadratic norm of the displacement vector
  \[
  \int_{\Omega_1} ||u|| \, d\Omega \approx q^T q
  \]
DESIGN PROBLEM FORMULATION

- The choice of the compliance is generally the main choice by designers.
  - At equilibrium, the compliance is also the strain energy of the structure, so that compliance is the energy norm of the displacements giving rise to a smooth displacement field over the optimized structure.

\[
l(u) = a_E(u, u) \quad g^T q = q^T K q
\]

- One can interpret the compliance as the displacement under the loads. For a single local case, it is the displacement under the load.

\[
C(q) = g^T q = f_i u_i
\]
DESIGN PROBLEM FORMULATION

- The choice of the compliance is generally the main choice of designers.
  - The sensitivity of compliance is easy to calculate. Being self adjoined, compliance is self adjoined, and it does not require the solution of any additional load case.
  
  - Conversely local stress constraints call for an important amount of additional CPU to compute the local sensitivities.
  
  - One can find analytical results providing the optimal bounds of composites mixture of materials for a given external strain field. The problem is known as the $G$-closure.
Finally the statement of the basic topology problem writes:

$$\begin{align*}
\min & \quad V(\mathbf{x}) \\
\text{s.t.:} & \quad \underline{x}_i \leq x_i \leq \bar{x}_i \\
& \quad C(\mathbf{q}) = \mathbf{g}^T \mathbf{q} \leq \bar{C} \\
& \quad \mathbf{K} \mathbf{q} = \mathbf{g}
\end{align*}$$

Alternatively it is equivalent for a given bounds on the volume and the compliance to solve the minimum compliance subject to volume constraint

$$\begin{align*}
\min & \quad C(\mathbf{q}) = \mathbf{g}^T \mathbf{q} \\
\text{s.t.:} & \quad \underline{x}_i \leq x_i \leq \bar{x}_i \\
& \quad V(\mathbf{x}) \leq \bar{V} \\
& \quad \mathbf{K} \mathbf{q} = \mathbf{g}
\end{align*}$$
PROBLEM FORMULATION

- For several load cases, average compliance

\[
\begin{align*}
\min \quad & \sum_{k} w_k \ g^{(k)}^T q^{(k)} \\
\text{s.t.:} \quad & x_i \leq x_i \leq \bar{x}_i \\
\end{align*}
\]

- Or better a worst-case approach

\[
\begin{align*}
\min \quad & \max_k \ g^{(k)}^T q^{(k)} \\
\text{s.t.:} \quad & V(x) \leq \bar{V} \\
& K q^{(k)} = g^{(k)} \\
\end{align*}
\]

- Where \( k \) is load case index, \( K \) is the stiffness matrix of FE approximated problem, \( g_k \) and \( q_k \) are the load case and generalized displacement vectors for load case \( k \)
- and \( \rho(x) \) is the local density and \( V \) is the volume
DESIGN PROBLEM FORMULATION

- Minimize compliance
  s.t.
  - Given volume
  - (bounded perimeter)
  - (other constraints)

- Maximize eigenfrequencies
  s.r.
  - Given volume
  - (bounded perimeter)
  - (other constraints)

- Minimize the maximum of the local failure criteria
  s.t.
  - Given volume
  - (bounded perimeter)
  - (other constraints)
TOPOLOGY OPTIMIZATION USING HOMOGENIZATION VS SIMP BASED TOPOLOGY OPTIMIZATION
TOPOLOGY OPTIMIZATION: FORMULATION

- Well-posed ness of problem?
  Discretised problem is ill-posed
  - Mesh-dependent solutions
  - Recreate microstructures
  - Nonexistence and uniqueness of a solution

- Homogenisation Method:
  → Extend the design space to all porous composites of variable density

- Filter method / Perimeter method / Slope constraints:
  → Restrict the design space by eliminating chattering designs from the design space
Select one family of microstructures whose geometry is fully parameterized in terms of a set of design variables [Bendsoe and Kikuchi, 1988]

- G closure: optimal microstructure (full relaxation)
- Suboptimal microstructures (partial relaxation)

Use homogenization theory to compute effective properties: in terms of microstructural geometrical parameters: $E_{ijk} = E_{ijkl}^h(a,b,...)$

Difficult to interpret and fabricate the optimal material distribution as it is

Revival interest with arrival of cellular structures e.g. lattice structures made by additive manufacturing
POWER LAW MODEL (SIMP)

- Simplified model of a microstructured material with a penalisation of intermediate densities [Bendsoe, 1989]
  
- Stiffness properties:
  \[ E = x^p E^0 \]
  \[ \rho = x \rho^0 \]
  \[ 0 \leq x \leq 1 \quad p > 1 \]

- Strength properties:
  \[ \sigma_l = x^p \sigma_l^0 \]

- Modified SIMP should be preferred to avoid singularities
  \[ E(x) = E_{\text{min}} + x^p (E^0 - E_{\text{min}}) \]

- Can be related to actual micro geometries [Bendsoe and Sigmund, 1999]

- 90% of current topology optimization runs
ALTERNATIVE PARAMETRIZATION TO SIMP

- Alternatively **RAMP** parameterization (Stolpe & Svanberg, 2001) enables controlling the slope at zero density:

\[ E(x) = \frac{x}{1 + p(1 - x)} E^0 \]

- Halpin Tsai (1969):

\[ E(x) = \frac{r x}{(1 + r) - x} E^0 \]

- Polynomial penalization (Zhu, 2009):

\[ E(x) = \left( \frac{\alpha - 1}{\alpha} x^p + \frac{1}{\alpha} x \right) E^0 \]
HOMOGENIZATION METHOD

Penalization of intermediate densities

From anisotropic to isotropic materials

- Investigation of the influence of the selected microstructure upon the optimal topology:
  - 1° orthotropic v.s. isotropy
  - 2° penalization of intermediate properties
SIMP : PENALIZATION OF INTERMEDIATE DENSITIES

SIMP with $p=2$

SIMP with $p=3$

SIMP with $p=4$
Prescribing immediately a high penalization may introduce some numerical difficulties:
- Optimization problem becomes difficult to solve because of the sharp variation of material properties close to \( x=1 \)
- Optimization problem includes a lot of local optima and solution procedure may be trapped in one of these.

To mitigate these problems, one resorts to the so-called **continuation procedure** in which \( p \) is gradually increased from a small initial value till the desired high penalization.

Typically:
- \( p(0) = 1.6 \)
- \( p(k+1) := p(k) + \Delta p \) after a given number of iterations or when a convergence criteria is OK
FILTERING TECHNIQUES AND MESH INDEPENDENCY STRATEGIES
Two numerical difficulties

- **Checkerboard patterns**: numerical instabilities related to the inconsistency between the displacement and density fields.
  - Appearance of alternate black-white patterns
  - Checkerboard patterns replaces intermediate densities

- **Mesh dependency**: the solution depends on the computing mesh.
  - New members appears when refining the mesh
  - Number of holes and structural features is modified when changing the mesh.
  - Stability (and meaning) of solutions?
Checkerboard patterns

- Babuska Brezzi conditions of discretization schemes
- Checkerboard free numerical schemes
  - High order FE elements
  - Filtering density field solutions ➔ lower order density fields
  - Perimeter constraint
Checkerboard patterns

Solution with checkerboards
SIMP with $p=2$
FE $u$: degree 1 / Density: constant

FE $u$: degree 2 / Density: constant

With perimeter constraint
Checkerboard patterns

SIMP with p=3. F u: degree 1

Perimeter < 60

FE u: degree 2
Checkerboard patterns

Perimeter < 60

SIMP with $p=3$. FE $u$: degree 1

FE $u$: degree 2
Mesh dependency

- Mesh independent solution: insure mesh independent filtering of lower size details
  - Low pass filter [Sigmund (1998)]

\[
\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X) v_i x_i}{\sum_i w_i(X_i) v_i}
\]

- Perimeter constraint [Ambrosio & Butazzo (1993)]

\[
\min_{0 \leq \rho(x) \leq 1} g^T q \\
\text{s.t. } P(\rho(x)) \leq \bar{P} \\
V = \int_\Omega \rho(x) d\Omega \leq \bar{V} \\
g_j(x) \leq \bar{g}_j \quad j = 1 \ldots m
\]
Mesh independency

Perimeter < 60
FE u: degree 1
PERIMETER METHOD

- Continuous version of perimeter measure

\[ P(\rho) = \int_{\Omega} ||\nabla \rho|| d\Omega + \int_{\Gamma_j} ||[\rho]||_j \ d\Gamma \]

- With the gradient of the density field and the jump \([\rho]_j\) of the density across discontinuity surfaces \(j\)

- The continuous approximation of the modulus of the gradient

\[ P(\rho) \approx \int_{\Omega} \sqrt{\nabla \rho^T \nabla \rho + \frac{\varepsilon^2}{h^2} - \frac{\varepsilon}{h}} \ d\Omega \]

\[ + \int_{\Gamma_j} \sqrt{[\rho]^2_j + \varepsilon^2 - \varepsilon} \ d\Gamma \]
PERIMETER METHOD

- The structural complexity of the structure is controlled with a bound over the perimeter.
- An efficient numerical strategy has been elaborated to cater with the difficult perimeter constraint *(Zhang & Duysinx, 1998)*.
FILTERING MATERIAL DENSITIES

To avoid mesh dependency and numerical instabilities like checkerboards patterns, one approach consists in restricting the design space of solutions by forbidding high frequency variations of the density field.

Basic filtering by Bruns and Tortorelli (2001), proven by Bourdin (2001)

\[
\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X) v_i x_i}{\sum_i w_i(X_i) v_i}
\]

\[N_e = \{ i \mid ||X_i - X_e|| \leq R \}\]

\[w_i(X_i) = \begin{cases} 
  R - ||X_i - X_e|| & i \in N_e \\
  0 & i \notin N_e 
\end{cases}\]
FILTERING MATERIAL DENSITIES

- Other weighting functions
  - Gaussian
    \[ w(X_i) = \exp \left[ -\frac{1}{2} \left( \frac{||X_i - X_e||}{\sigma_d} \right)^2 \right] \]
  - Constant
    \[ w(X_i) = 1 \]

- Density filter is equivalent to solving a Helmotz equation
  Lazarov et al.

\[ -R_{min} \nabla^2 \bar{\delta} + \bar{\delta} = \delta \]

- With the following Neuman boundary conditions

\[ \frac{\partial \bar{\delta}}{\partial n} = 0 \]
FILTERING MATERIAL DENSITIES

Historically Ole Sigmund (1994, 1997) introduced a filter of the sensitivities

\[
\frac{\partial \tilde{f}}{\partial x_e} = \frac{\sum_{i \in N_e} w(X_i) x_i \frac{\partial f}{\partial x_i}}{x_e \sum_{i \in N_e} w(X_i)}
\]

with

\[w(X_i) = R - ||X_i - X_e||\]

For non uniform meshes, Sigmund proposed to use

\[
\frac{\partial \tilde{f}}{\partial x_e} = \frac{\sum_{i \in N_e} w(X_i) \frac{x_i}{v_i} \frac{\partial f}{\partial x_i}}{\frac{x_e}{v_e} \sum_{i \in N_e} w(X_i)}
\]

The smoothed sensitivities correspond to the sensitivities of a smoothed version of the objective function (as well as the constraints)
FILTERING MATERIAL DENSITIES

$\rho$ \hspace{1cm} $\tilde{\rho}(r_{fil} = 8)$ \hspace{1cm} $\tilde{\rho}(r_{fil} = 15)$
FILTERING MATERIAL DENSITIES

\[ \rho \quad \tilde{\rho}(r_{fil} = 8) \quad \tilde{\rho}(r_{fil} = 15) \]

\[ E_i = \rho_i^p E_0 \]

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FILTERING MATERIAL DENSITIES

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\[ E_i = \rho_i^p E_0 \]

- Mesh dependent
- Checkerboard
- Non-Discrete Solut.

\[ E_i = \tilde{\rho}_i^p E_0 \]

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HEAVISIDE FILTER

- To obtain 0/1 solutions, Guest et al. (2014) modifies the density filter with a Heaviside function such that if $x_e > 0$, the Heaviside gives a physical value of the density equal to ‘1’ and if $x_e = 0$, the Heaviside gives a density ‘0’

\[
\hat{\rho}_i = \begin{cases} 
1 & \text{if } \tilde{\rho}_i > 0 \\
0 & \text{otherwise}
\end{cases}
\]
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- Heaviside smooth approximation

$$\hat{\rho}_i = \begin{cases} 1 & \text{if } \tilde{\rho}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\rho}_i = 1 - e^{-\beta \tilde{\rho}_i} + \tilde{\rho}_i e^{-\beta}$$

- For $\beta \to 0$, the filter gives the original filter
- For $\beta \to \infty$, the function reproduces the max operator, that is the density becomes 1 if there is any element in the neighborhood that is non zero.
HEAVISIDE FILTER

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\[ \hat{\rho}_i = \begin{cases} 
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\end{cases} \]

\[ \hat{\rho}_i = 1 - e^{-\beta \tilde{\rho}_i} + \tilde{\rho}_i e^{-\beta} \]

- Mesh dependent
- Checkerboard
- Non-Discrete Solution
- Need of continuation / Large number of iterations (>100)
Heaviside function can be extended (Wang, Lazarov, Sigmund, 2011) to control minimum and maximum length scale.

\[
\hat{x}_e = \frac{\tanh(\beta \eta) + \tanh(\beta (\bar{x}_e - \eta))}{\tanh(\beta \eta) + \tanh(\beta (1.0 - \eta))}
\]
HEAVISIDE FILTER

- Heaviside function enables a control of manufacturing tolerant designs ➔ robust design

\[ \hat{x}_e = \frac{\tanh(\beta \eta) + \tanh(\beta (\hat{x}_e - \eta))}{\tanh(\beta \eta) + \tanh(\beta (1.0 - \eta))} \]
Combining density filtering and Heaviside filter give rise to the so called **three field topology optimization scheme** proposed by Wang et al. (2011), one uses a design field, a filtered field and a physical field whose relations are defined though the following filter and thresholding processes

- **Filtering**

\[
\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X) v_i x_i}{\sum_i w_i(X_i) v_i}
\]

- **Heaviside**

\[
\hat{x}_e = \frac{\tanh(\beta \eta) + \tanh(\beta (\tilde{x}_e - \eta))}{\tanh(\beta \eta) + \tanh(\beta (1.0 - \eta))}
\]
SENSITIVITY ANALYSIS
SENSITIVITY ANALYSIS

- Study of the derivatives of the structure under linear static analysis when discretized by finite elements.
- The study is carried out for one load case, but it can be easily extended to multiple load cases.

- Equilibrium equation of the discretized structure:

\[ K \mathbf{q} = \mathbf{g} \]

- \( \mathbf{q} \) generalized displacement of the structure
- \( K \) stiffness matrix of the structure discretized into F.E.
- \( \mathbf{g} \) generalized load vector consistent with the F.E. discretization
SENSITIVITY ANALYSIS

- Let \( \mathbf{x} \) be the vector of design variables in number \( n \).

- The differentiation of the equilibrium equation yields the sensitivity of the generalized displacements:

\[
\frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q} + \mathbf{K} \frac{\partial \mathbf{q}}{\partial x_i} = \frac{\partial \mathbf{g}}{\partial x_i} \quad \Rightarrow \quad \mathbf{K} \frac{\partial \mathbf{q}}{\partial x_i} = \left\{ \frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q} \right\}
\]

- The right-hand side term is called pseudo load vector

\[
\tilde{g}_i = \frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q}
\]

- Physical interpretation of the pseudo load (Irons): load that is necessary to re-establish the equilibrium when perturbing the design.
SENSITIVITY ANALYSIS

- A central issue is the calculation of the derivatives of the stiffness matrix and of the load vector.

- In some cases the structure of the stiffness matrix makes it easy to have the sensitivity of the matrix with respect to the design variable.

- In topology optimization using SIMP model:
  \[ E = x^p E^0 \]

- The stiffness matrix
  \[ K_e = x_e^p \bar{K}_e \]

- And its derivatives
  \[ \frac{\partial K_e}{\partial x_e} = p x_e^{p-1} \bar{K}_e \]
SENSITIVITY OF COMPLIANCE

- The compliance is defined as the work of the applied load.

\[ C = g^T q \]

- It is equal to the twice the deformation energy

\[ C = g^T q = q^T K q \]

- The derivative of the compliance constraint gives:

\[ \frac{dC}{dx_i} = \frac{\partial g^T}{\partial x_i} q + g^T \frac{\partial q}{\partial x_i} \]

- Introducing the value of the derivatives of the generalized displacements:

\[ \frac{\partial q}{\partial x_i} = K^{-1} \left\{ \frac{\partial g}{\partial x_i} - \frac{\partial K}{\partial x_i} q \right\} \]

\[ \frac{dC}{dx_i} = \frac{\partial g^T}{\partial x_i} q + g^T K^{-1} \left\{ \frac{\partial g}{\partial x_i} - \frac{\partial K}{\partial x_i} q \right\} \]
SENSITIVITY OF COMPLIANCE

- The expression of the sensitivity of the compliance writes

\[
\frac{dC}{dx_i} = \frac{\partial g^T}{\partial x_i} q + g^T K^{-1} \left\{ \frac{\partial g}{\partial x_i} - \frac{\partial K}{\partial x_i} q \right\}
\]

\[
= \frac{\partial g^T}{\partial x_i} q + q^T \left\{ \frac{\partial g}{\partial x_i} - \frac{\partial K}{\partial x_i} q \right\}
\]

\[
\frac{dC}{dx_i} = -q^T \frac{\partial K}{\partial x_i} q + 2 \frac{\partial g^T}{\partial x_i} q
\]

- Generally the load vector derivative is zero (case of no body load), it comes:

\[
\frac{dC}{dx_i} = -q^T \frac{\partial K}{\partial x_i} q
\]
NUMERICAL SOLUTION OF TOPOLOGY PROBLEMS USING GRADIENT BASED MATH PROGRAMMING
Optimal material distribution = very large scale problem
- Large number of design variables: 1 000 → 100 000
- Number of restrictions:
  - 1 → 10 (for stiffness problems)
  - 1 000 → 10 000 (for strength problem with local constraints)

Solution approach based on the sequential programming approach and mathematical programming
- Sequence of convex separable problems based on structural approximations
- Efficient solution of sub problems based on dual maximization

Major reduction of solution time of optimization problem
Generalization of problems that can be solved
SEQUENTIAL CONVEX PROGRAMMING APPROACH

Direct solution of the original optimisation problem which is generally non-linear, implicit in the design variables

$$\min_{\mathbf{x}} \quad g_0(\mathbf{x})$$

$$\text{s.t.} \quad g_j(\mathbf{x}) \leq \bar{g}_j \quad j = 1 \ldots m$$

is replaced by a sequence of optimisation sub-problems

$$\min_{\mathbf{x}} \quad \tilde{g}_0(\mathbf{x})$$

$$\text{s.t.} \quad \tilde{g}_j(\mathbf{x}) \leq \bar{g}_j \quad j = 1 \ldots m$$

by using approximations of the responses and using powerful mathematical programming algorithms
SEQUENTIAL CONVEX PROGRAMMING APPROACH

- Two basic concepts:
  - **Structural approximations** replace the implicit problem by an explicit optimisation sub-problem using convex, separable, conservative approximations; e.g. CONLIN, MMA
  - **Solution of the convex sub-problems**: efficient solution using dual methods algorithms or SQP method.

- Advantages of SCP:
  - Optimised design reached in a reduced number of iterations: typically **100 F.E. analyses** in topology optimization
  - Efficiency, robustness, generality, and flexibility, small computation time
  - Large scale problems in terms of number of design constraints and variables
Linear approximation and Sequential Linear Programming

- Linear approximation = first order Taylor expansion around $x^0$:

$$
\tilde{g}_j(x) = g_j(x^0) + \sum_{i=1}^{n} \frac{\partial g(x^0)}{\partial x_i} (x_i - x_i^0)
$$

- When linear approximation is applied to each function of the problem, one transforms the problem into a sequence of linear programming problems (SLP):

$$
\min_{x_i} \quad g_0(x^0) + \nabla g_0(x^0)^T (x - x^0)
$$

s.t. \hspace{1cm} g_j(x^0) + \nabla g_j(x^0)^T (x - x^0) \leq 0

\underline{x_i} \leq x_i \leq \bar{x}_i$$
SEQUENTIAL LINEAR PROGRAMMING METHOD

- The current design point is $x^{(k)}$. Using the first order Taylor expansion of $f(x)$, $h_j(x)$, we can get a linear approximation of the NL problem in $x^{(k)}$:

$$
\begin{align*}
\min_x & \quad f(x^{(k)}) + (x - x^{(k)})^T \nabla f(x^{(k)}) \\
\text{s.t.} & \quad h_j(x^{(k)}) + (x - x^{(k)})^T \nabla h_j(x^{(k)}) \leq 0 \quad j = 1 \ldots q \\
& \quad x_i \leq x_i \leq \bar{x}_i \quad i = 1, \ldots, n
\end{align*}
$$

- Solving this LP problem, we get a new point in $x^{(k+1)}$ and start again.

\[\text{DOESN'T WORK!}\]
SEQUENTIAL LINEAR PROGRAMMING METHOD

MOVE LIMIT STRATEGY

- Introduce a box constraint around the current design point to limit the variation domain of the design variables

\[
\hat{x}_i - \alpha_i \leq x_i \leq \hat{x}_i + \beta_i
\]

- Of course take the most restrictive constraints with the

\[
\max\{x_i, \hat{x}_i - \alpha_i\} \leq x_i \leq \min\{\bar{x}_i, \hat{x}_i + \beta_i\}
\]
STRUCTURAL APPROXIMATIONS

- Convex Linearisation (CONLIN)

\[
\tilde{g}_j(x) = g_j(x^0) + \sum_{+} \frac{\partial g_j}{\partial x_i} (x_i - x_i^0) - \sum_{-} (x_i^0)^2 \frac{\partial g_j}{\partial x_i} \left( \frac{1}{x_i} - \frac{1}{x_i^0} \right)
\]

- Method of Moving Asymptotes (MMA)

\[
\tilde{g}_j(x) = r_j^0 + \sum_{i=1}^{n} \frac{p_{ij}}{U_i - x_i} + \sum_{i=1}^{n} \frac{q_{ij}}{x_i - L_i}
\]

\[
p_{ij} = \max\{0, (U_i - x_i^0)^2 \frac{\partial g_j}{\partial x_i}\}
\]

\[
q_{ij} = \max\{0, -(x_i^0 - L_i)^2 \frac{\partial g_j}{\partial x_i}\}
\]
CONLIN approximation

Approximation of the strain energy in a two plies symmetric laminate subject to shear load and torsion (Bruyneel and Fleury, 2000)
MMA approximation

Approximation of the strain energy in a two plies symmetric laminate subject to shear load and torsion (Bruyneel and Fleury, 2000)
DUAL METHODS

- **Primal problem**

  $$\min_{x} \quad f(x)$$
  
  s.t. 
  $$g_j(x) \leq 0 \quad j = 1, \ldots, m$$

- **Lagrange function:**

  $$L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x)$$

- **If the problem is convex...**

  $$\min_{x} \max_{\lambda \geq 0} L(x, \lambda) \iff \max_{\lambda \geq 0} \min_{x} L(x, \lambda)$$
DUAL METHODS

- Dual problem
  \[
  \max_{\lambda_j} \ell(\lambda)
  \]
  \[
  \text{s.t.} \quad \lambda_j \geq 0 \quad j = 1, \ldots, m
  \]
  - with
  \[
  \ell(\lambda) = \min_{x \in X} L(x, \lambda)
  \]

- Solve Lagrangian problem
  \[
  x = x(\lambda) = \arg \min_x L(x, \lambda)
  \]

- Lagrangian problem
  \[
  \ell(\lambda) = L(x(\lambda), \lambda)
  \]
  \[
  = f(x(\lambda)) + \sum_{j=1}^{m} \lambda_j g_j(x(\lambda))
  \]
NUMERICAL APPLICATIONS OF COMPLIANCE MINIMIZATION BASED TOPOLOGY OPTIMIZATION
Optimization of a maximum stiffness bicycle frame

Load cases

Optimum topology

Conventional design

Nonconventional design
Design of a Crash Barrier Pillar (SOLLAC)
Design of a Crash Barrier Pillar (SOLLAC)
Topology Optimization of a Parasismic Building (DOMECO)
3D cantilever beam problem

\[ E = 100 \text{ N/m}^2, \quad \nu = 0.3 \]

\[ 20 \times 32 \times 4 = 2560 \text{ F.E.s} \]
3D cantilever beam problem

Perimeter = 1400

Perimeter = 1000
An industrial application: Airbus engine pylon

- **Application**
  - carried out by **SAMTECH** and ordered by **AIRBUS**

- **Engine pylon**
  - structure fixing engines to the wing

- **Initial Model**
  - CATIA V5 import → Samcef Model
  - BC’s: through shell and beam FE
  - 10 load cases:
    - GUSTS
    - FBO (Fan blade out)
    - WUL (Without undercarriage landing)

Over 250,000 tetraedral FE
An industrial application: Airbus engine pylon

- Target mass: 10%

- Additional constraints:
  - Engine CoG position

- Optimization parameters
  - Sensitivity filtering: (Sigmund’s filter)
  - Symmetry (left right) condition
  - Penalty factor

- CONLIN optimizer: special version for topology optimization

Sensitivities filtering
Penalty factor from 2 to 4
Airbus engine pylon

Topology Optimization - ...
IAPA2 : Loads GUST PEO WUL UNIY UNIZ SYMMETRIC

With courtesy by Samtech and Airbus Industries
Airbus engine pylon

With courtesy by Samtech and Airbus Industries
Sandwich panel optimization

- Geometry of the sandwich panel reinforcement problem

![Diagram of sandwich panel with forces and dimensions]

- Optimal topology

![Optimal topology images]
Sandwich panel optimization

- Geometry of the sandwich panel reinforcement problem

- Optimal topology
PHYSICAL MEANING OF DENSITY VARIABLE:
PROTOTYPE CAR BODY OPTIMIZATION

- **Load case 1: bending**
  - Self weight
  - Components (20 kg)
  - Pilot (50 kg)
  - Roll over load (70 kg on top of roll cage)

- **Load case 2: torsion + bending**
  - Curb impact
    - Rear axle clamped
    - Right front wheel free supported
    - Left front wheel withstand 3 times the weight of the axle

(Figures from Happian-Smith)

70 kg

70 kg on top of roll cage

weight x 3 = 2700 N
Topeology optimization of the truss structure

- Target mass of 15 kg
- Minimum compliance
- Mostly determined by load case 2 (torsion)
- SIMP material with p=3
- Left / right symmetry of material distribution
- Filtering
DESIGN OF AN URBAN CONCEPT STRUCTURE
DESIGN OF AN URBAN CONCEPT STRUCTURE

Volume = 40%

Volume = 20%

Volume = 60%
DESIGN OF AN URBAN CONCEPT STRUCTURE
VIBRATION PROBLEMS
NATURAL VIBRATION PROBLEMS

- Finite element discretization of the system

\[ u(X) = N(X) \mathbf{q} \]
\[ \varepsilon(X) = \partial N(X) \mathbf{q} = B \mathbf{q} \]
\[ \sigma = H \varepsilon = HB \mathbf{q} = T \mathbf{q} \]

- Kinetic energy

\[ K = \frac{1}{2} \int_{\Omega} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} d\Omega = \frac{1}{2} \mathbf{q}^T M \mathbf{q} \]
\[ M = \int_{\Omega} N^T N \rho \ d\Omega \]

- Strain energy

\[ U = \frac{1}{2} \int_{\Omega} \sigma^T \varepsilon \ d\Omega = \frac{1}{2} \mathbf{q}^T K \mathbf{q} \]
\[ K = \int_{\Omega} B^T H B \ d\Omega \]
NATURAL VIBRATION PROBLEMS

- Hamilton principle

\[ \delta \int_{t_1}^{t_2} (K - U) \, dt = 0 \]

- Dynamic equation of the system

\[ M\ddot{q} + Kq = g(t) \]

- Free vibrations: assume periodic solutions

\[ q(t) = q \, e^{i\omega t} \]

- Nontrivial solutions are solutions of the eigenvalue problem

\[ -\omega^2 M \, q + Kq = 0 \]
NATURAL VIBRATION PROBLEMS

- Eigenvalue problem

\[ (K - \omega^2 M)q = 0 \]

- Rayleigh ratio

\[ \omega_{min}^2 = \min_q \frac{q^T K q}{q^T M q} \]
**DESIGN PROBLEM FORMULATION**

- **Fundamental topology optimization problem of vibrating structures**
  \[
  \min_x \omega_{\text{min}}^2
  \]
  \[
  \text{s.t.} \quad V = \int_{\Omega} \rho \, d\Omega \leq \bar{V} \]
  \[
  (K - \omega_{\text{min}}^2 M)q = 0
  \]

- To avoid mode crossing, it is better to select several eigenvalues and to maximize the minimum of the first NF frequencies
  \[
  \max_x \min_{k=1\ldots NF} \omega_k^2
  \]
  \[
  \text{s.t.} \quad V = \int_{\Omega} \rho \, d\Omega \leq \bar{V} \]
  \[
  (K - \omega_k^2 M)q_k = 0
  \]
DESIGN PROBLEM FORMULATION

- However, the fact that both mass and stiffness depends on the density design variables, the trivial solution 0=0 is feasible.
- Therefore, best eigenfrequency design problem is formulated as a reinforcement problem, i.e. there exist some non design mass or stiffness

\[
\begin{aligned}
\min_{x} \quad & \omega_{\text{min}}^2 \\
\text{s.t.} \quad & V = \int_{\Omega} \rho \, d\Omega \leq \bar{V} \\
& ([K + K_0] - \omega_{\text{min}}^2[M + M_0])q = 0 \\
\omega_{\text{min}}^2 &= \min_{q} \frac{q^T[K + K_0]q}{q^T[M + M_0]q}
\end{aligned}
\]
REINFORCEMENT DESIGN PROBLEM

- Topology optimization of sheet of steel:
  - Basic sheet: \( t=1\text{mm} \)
  - Reinforcement sheet \( t=1\text{mm} \)

\[ \begin{align*}
E &= 210 \times 10^9 \\
\nu &= 0.3 \\
\rho &= 7800 \text{ Kg/m}^3
\end{align*} \]
REINFORCEMENT DESIGN PROBLEM

- Topology optimization of sheet of steel:
  - Basic sheet: t=1mm
  - Reinforcement sheet t=1mm

<table>
<thead>
<tr>
<th>Mode propre</th>
<th>Fréquences propres avant optimisation (Hz)</th>
<th>Fréquences propres après optimisation (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1394,22</td>
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<tr>
<td>5</td>
<td>860,40</td>
<td>1583,39</td>
</tr>
</tbody>
</table>
SENSITIVITY OF EIGENVALUE PROBLEMS

- **Eigenvalue problem**
  - $\mathbf{K}$ stiffness matrix, $\mathbf{M}$ mass matrix
  - $\mathbf{q}$ the eigenmode vector
  - And $\omega$ the eigenfrequency

$$\begin{align*}
  (\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{q} &= 0 \\
  \mathbf{q}^T \mathbf{W} \mathbf{q} &= 1
\end{align*}$$

- The magnitude of the modes is arbitrary, so they are normalized according to a given matrix $\mathbf{W}$ (generally the mass matrix $\mathbf{M}$)

- At first let’s consider the simplified approach: we assume that all eigenvalues are distinct and ordered from the smallest to the largest:

$$0 < \omega^{(1)} < \omega^{(2)} < \ldots < \omega^{(N)}$$

$$0, \mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \ldots \mathbf{q}^{(N)}$$
SENSITIVITY OF EIGENVALUE PROBLEMS

Let’s differentiate the eigenvalue equation

\[
\left( \mathbf{K} - \omega_{(k)}^2 \mathbf{M} \right) \frac{\partial \mathbf{q}^{(k)}}{\partial x_i} = \frac{\partial \omega_{(k)}^2}{\partial x_i} \mathbf{M} \mathbf{q}^{(k)} - \left( \frac{\partial \mathbf{K}}{\partial x_i} - \omega_{(k)}^2 \frac{\partial \mathbf{M}}{\partial x_i} \right) \mathbf{q}^{(k)}
\]

Differentiating the normalization equation gives

\[
\mathbf{q}^{(k) T} \mathbf{W} \frac{\partial \mathbf{q}^{(k)}}{\partial x_i} = -\frac{1}{2} \mathbf{q}^{(k) T} \frac{\partial \mathbf{W}}{\partial x_i} \mathbf{q}^{(k)}
\]

To obtain the derivatives of the eigenvalue \( \lambda^{(k)} \), one has to premultiply the first equation by the eigenmode \( \mathbf{q}^{(k)} \)

\[
\mathbf{q}^{(k) T} \left[ \left( \mathbf{K} - \omega_{(k)}^2 \mathbf{M} \right) \frac{\partial \mathbf{q}^{(k)}}{\partial x_i} - \frac{\partial \omega_{(k)}^2}{\partial x_i} \mathbf{M} \mathbf{q}^{(k)} + \left( \frac{\partial \mathbf{K}}{\partial x_i} - \omega_{(k)}^2 \frac{\partial \mathbf{M}}{\partial x_i} \right) \mathbf{q}^{(k)} \right] = 0
\]
SENSITIVITY OF EIGENVALUE PROBLEMS

- Since \( q^{(k)} \) is an eigenmode
  \[
  q^{(k)T} \left( K - \omega_{(k)}^2 M \right) = 0
  \]

- And one gets
  \[
  - \left( q^{(k)T} M q^{(k)} \right) \frac{\partial \omega_{(k)}^2}{\partial x_i} + q^{(k)T} \left( \frac{\partial K}{\partial x_i} - \omega_{(k)}^2 \frac{\partial M}{\partial x_i} \right) q^{(k)} = 0
  \]

- With the scaling factor
  \[
  m^{(k)} = q^{(k)T} M q^{(k)}
  \]

- We finally obtain the final expression of the sensitivity of the eigen values:
  \[
  \frac{\partial \omega_{(k)}^2}{\partial x_i} = \frac{1}{m^{(k)}} \left( q^{(k)T} \frac{\partial K}{\partial x_i} q^{(k)} - \omega_{(k)}^2 q^{(k)T} \frac{\partial M}{\partial x_i} q^{(k)} \right)
  \]
DIFFICULTIES IN VIBRATION PROBLEMS

- Topology optimization of vibrating structures presents a major difficulty:
  - Appearance of dummy eigenmodes, i.e. local modes with zero frequency

- Require special strategy
  - Modification of material interpolation \(\Rightarrow\) modified SIMP or RAMP
  - Filtering the dummy modes
DIFFICULTIES IN VIBRATION PROBLEMS

- Illustration using a numerical example

- $T=1\times10^{-3}$ m
- Interpolation law: SIMP $p=3$
- Max volume = 80%
- Design domain = Support only

Material: Steel $E=210$ GPa, $n=0.3$, $\rho=7800$ kg/m$^3$
DIFFICULTIES IN VIBRATION PROBLEMS

- Illustration using a numerical example

- Material: Steel $E=210$ Gpa, $\nu=0.3$, $\rho=7800$ kg/m$^3$
- $T=1e^{-3}$ m
- Interpolation law: SIMP $p=3$
- Max volume = 80%
- Design domain = Support only
DIFFICULTIES IN VIBRATION PROBLEMS

- Convergence curve after 10 iterations
DIFFICULTIES IN VIBRATION PROBLEMS

- Convergence broken after 4 iterations when SIMP p=4

\[ f_1 = 0.9802 \text{ Hz} \]
\[ f_2 = 1.2247 \text{ Hz} \]
\[ f_3 = 1.2618 \]
\[ f_4 = 1.2984 \]
DIFFICULTIES IN VIBRATION PROBLEMS

- Low frequency modes are present in low density regions

\[
\omega^2_{\text{min}} = \min_q \frac{q^T K q}{q^T M_0 q} = \frac{x^p E_0}{x \rho_0} \frac{q^T K_0 q}{q^T M_0 q}
\]

\[
= x^{p-1} \frac{E_0}{\rho_0} \frac{q^T K_0 q}{q^T M_0 q}
\]

\[
\omega^2_{\text{min}} = x^{p-1} \frac{E_0}{\rho_0} \frac{q^T K_0 q}{q^T M_0 q} \to 0 \quad \text{if} \quad x \to 0
\]

- Modify SIMP to give a lower bound in low density to the ratio \( E/\rho \)

\[
E = \min(E_{\text{min}}, x^p E_0)
\]

- Modified SIMP

\[
E = E_{\text{min}} + x^p (E_0 - E_{\text{min}})
\]
DIFFICULTIES IN VIBRATION PROBLEMS

- Ignore / filter eigenmodes which are not significant
  - Selection criteria: generalized mass $\Rightarrow$ local character of the mode

$$\mu^{(k)} = q^{(k)T}Mq^{(k)}$$

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fréquence propre (Hz)</th>
<th>Energie cinétique $\mu$ (m² kg)</th>
<th>Rapport à l’énergie cinétique maximale</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1,907E-1</td>
<td>1,3810E-3</td>
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<tr>
<td>3</td>
<td>1,2618</td>
<td>1,4265E+2</td>
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<td>4</td>
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<tr>
<td>5</td>
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<td>1,6051E-1</td>
<td>1,0946E-3</td>
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