TOPOLOGY OPTIMIZATION: FUNDAMENTALS

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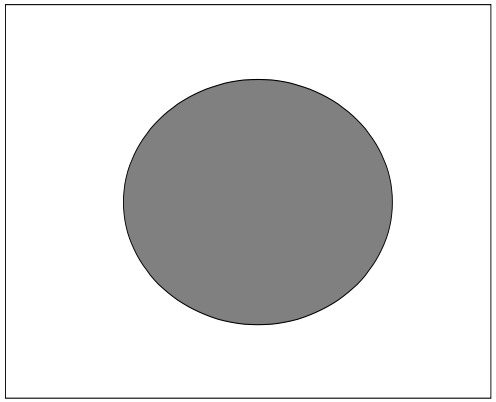
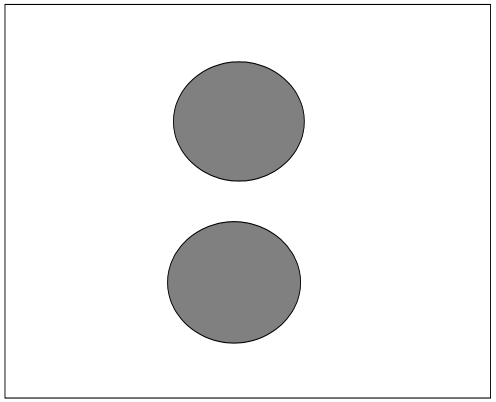
LAY-OUT

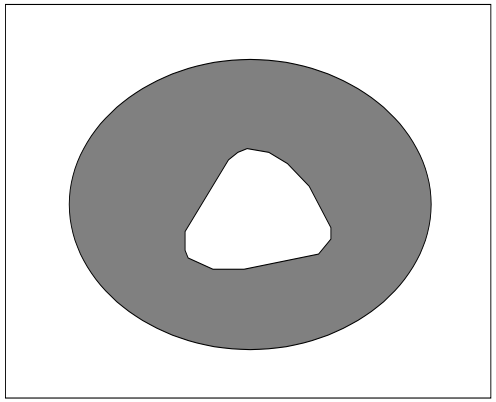
- Introduction
- Topology problem formulation: Problem statement
- Compliance minimization
- Homogenization method vs SIMP based
- Filtering techniques
- Sensitivity analysis
- Solution of optimization problems using structural approximations and dual maximization
- Vibration Problems
- Applications

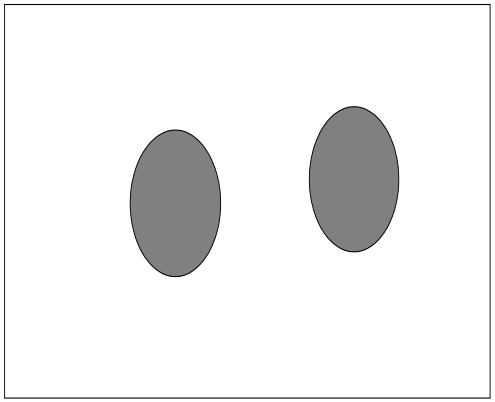
INTRODUCTION

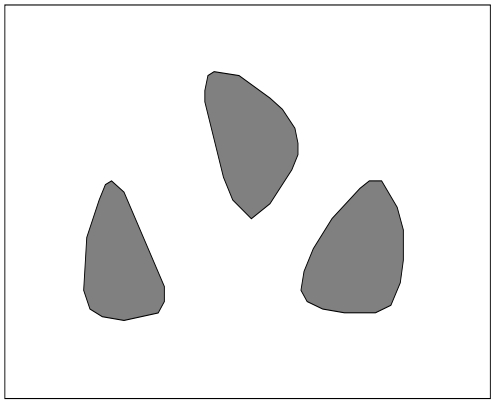
What is topology?

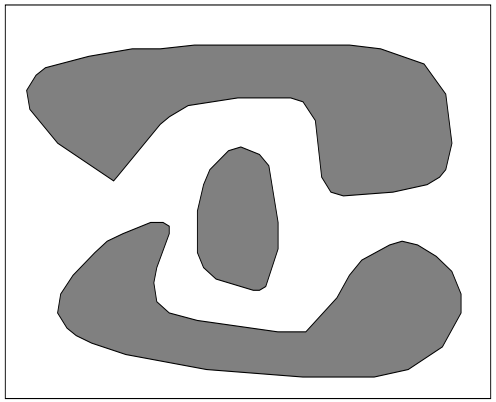






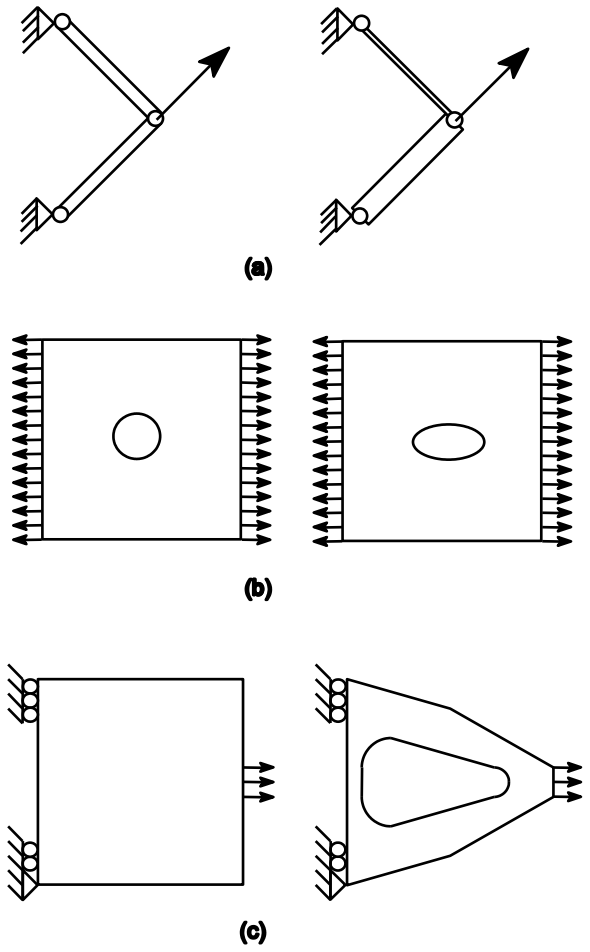






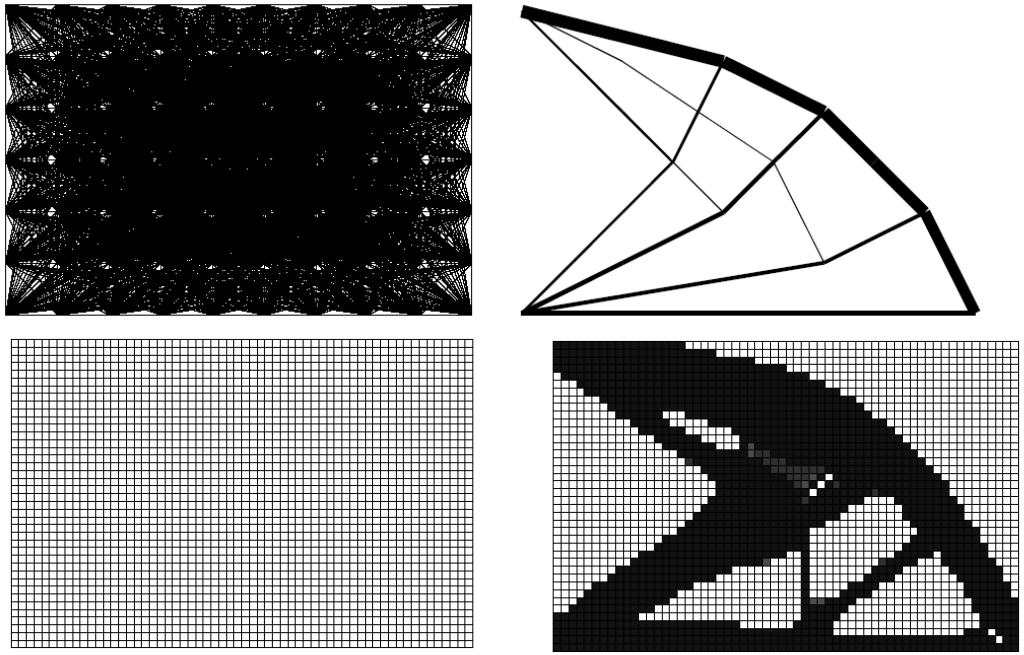
STRUCTURAL & MULTIDISCIPLINARY OPTIMISATION

- TYPES OF VARIABLES
  - a/ Sizing
  - b/ Shape
  - c/ Topology
  - (d/ Material)
- TYPES OF OPTIMISATION
  - structural
  - multidisciplinary
    - structural
    - aerodynamics,
    - thermal,
    - manufacturing



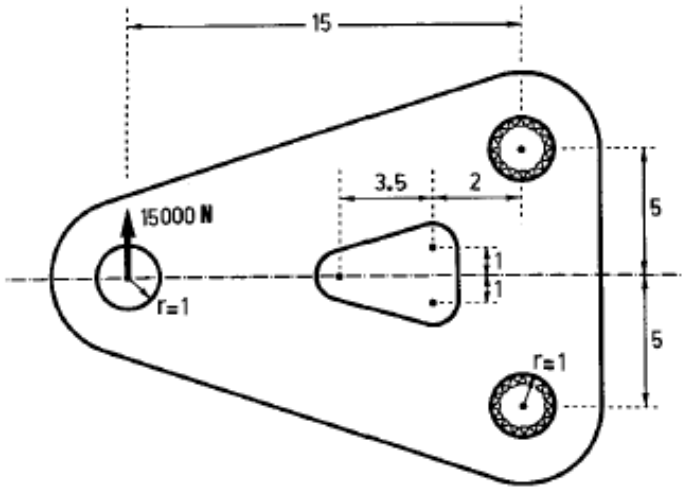
Topology optimization

- One generally distinguishes two approaches of topology optimization:
  - Topology optimization of naturally discrete structures (e.g. trusses)
  - Topology optimization of continuum structures (eventually after FE discretization)

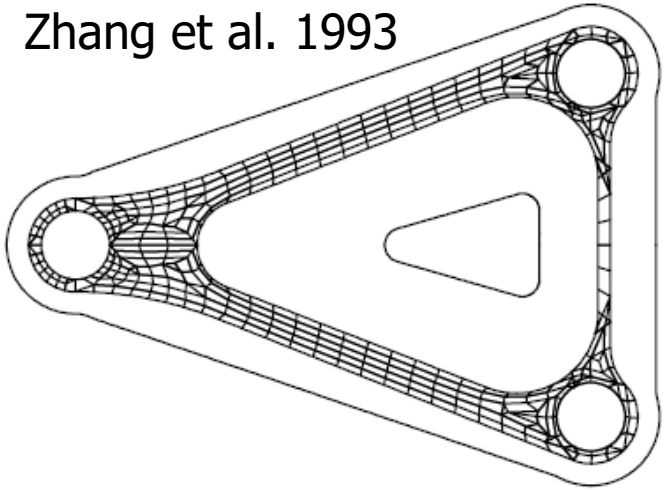


Why topology optimization?

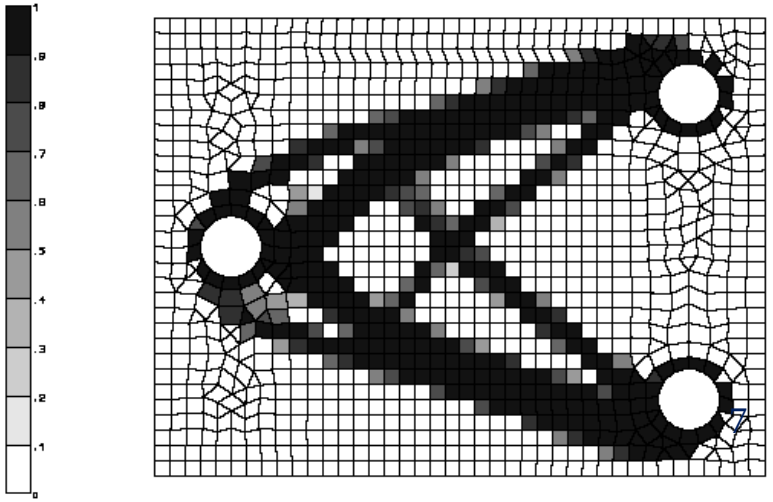
- CAD approach does not allow topology modifications



Zhang et al. 1993

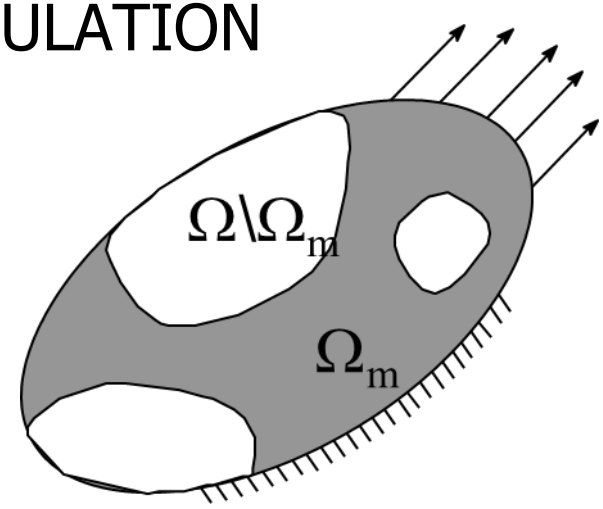


A better morphology by topology optimization (Duysinx, 1996)



TOPOLOGY PROBLEM FORMULATION

TOPOLOGY OPTIMIZATION FORMULATION



- Abandon CAD model description based on boundary description
- Optimal topology is given by an optimal material distribution problem
- Search for the indicator function of the domain occupied by the material

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m \\ 0 & \text{if } \mathbf{x} \in \Omega \setminus \Omega^m \end{cases}$$

- The physical properties write

$$E_{ijkl}(\mathbf{x}) = \chi(\mathbf{x})\, E^0_{ijkl}$$

$$\rho(\mathbf{x}) = \chi(\mathbf{x})\, \rho^0$$

- The problem is intrinsically a binary 0-1 problem ➔ solution is extremely difficult to solve

$$\chi \in \{0, 1\}$$

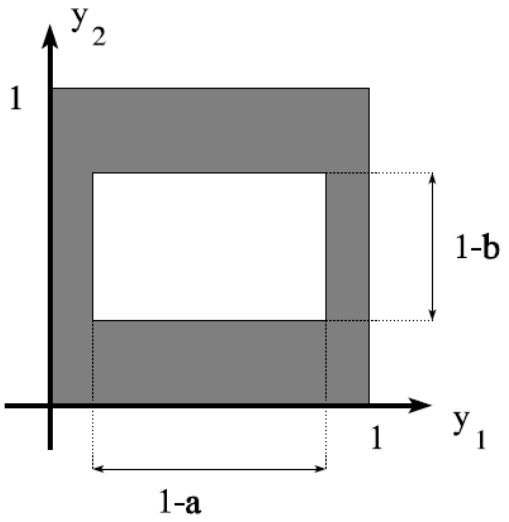
MATERIAL DENSITY FUNCTION

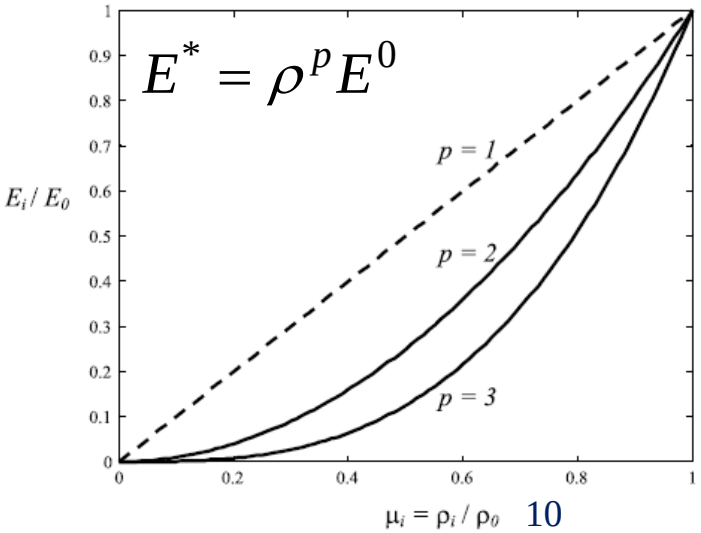
- Avoid 0/1 problem and replace by a continuous approximation considering a variable density material running from void (0) to solid (1)

$$\chi \in [0, 1]$$

  - Homogenization law of mechanical properties a porous material for any volume fraction (density) of materials
  - Mathematical interpolation and regularization
    - SIMP model $E = x^p \, E^0$
    - RAMP

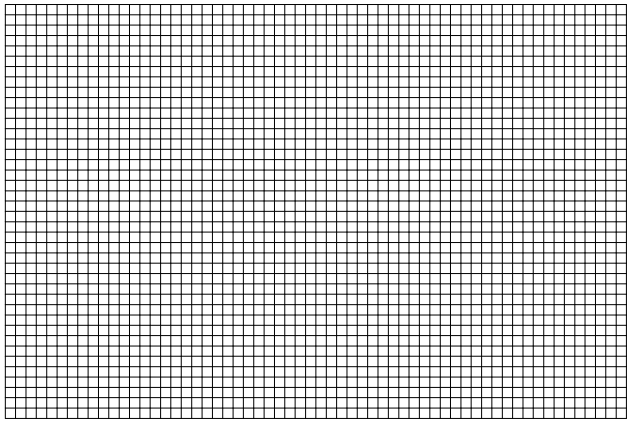
- Penalization of intermediate densities to end-up with black and white solution
- Efficient solution of optimization problem based on sensitivity analysis and gradient based mathematical programming algorithms

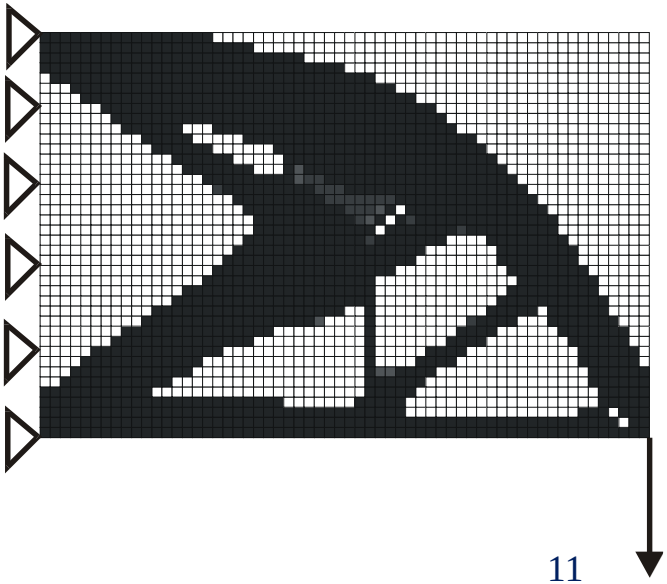




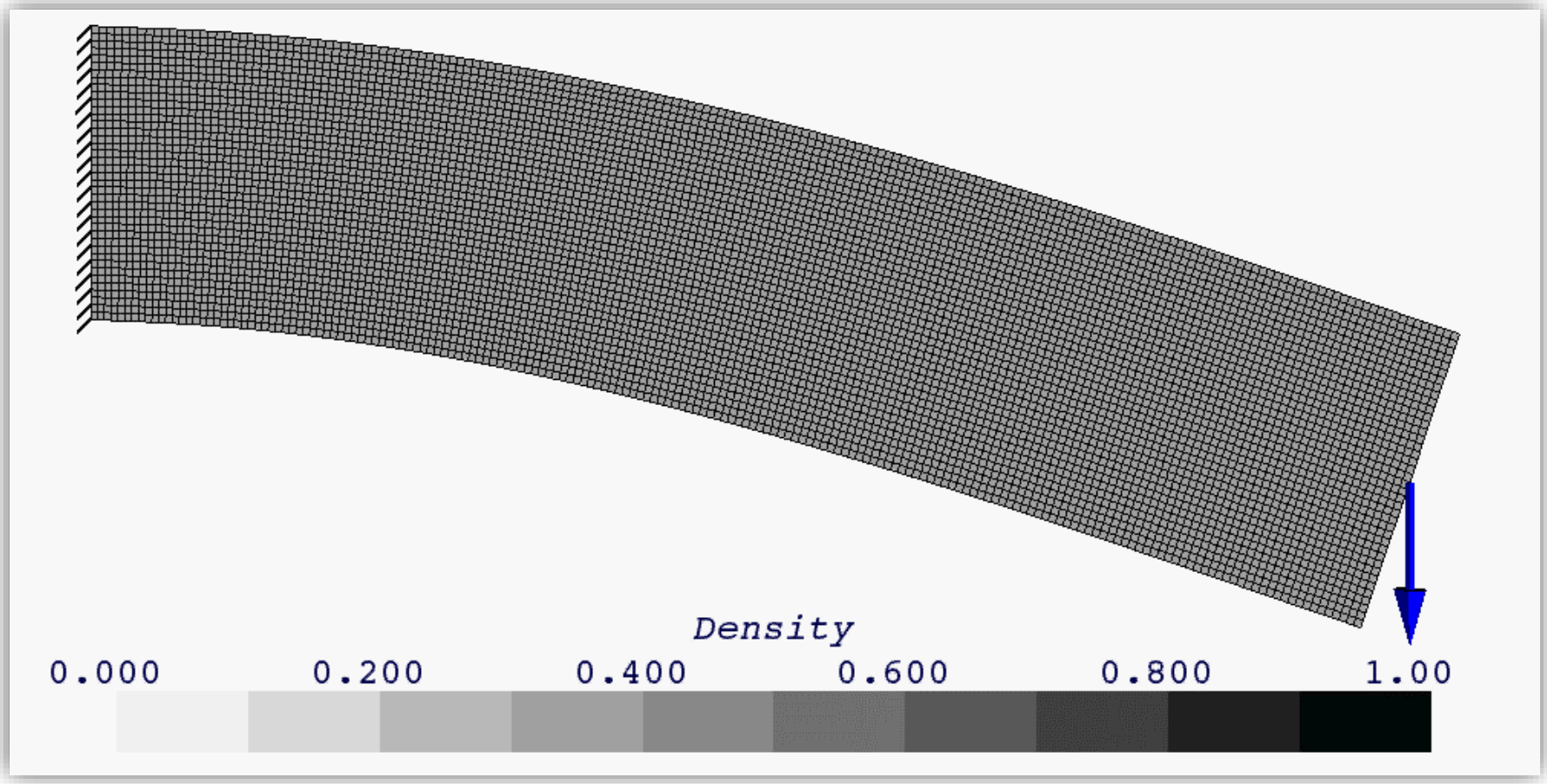
IMPLEMENTION OF MATERIAL DENSITY FUNCTION

- Implementation of material density approach is rather easy:
  - Fixed mesh
  - Design variables are element or nodal densities
  - Similar to sizing problem
  - SIMP law is easy to code
  - Sensitivity of compliance is cheap
  - Use efficient gradient based optimization algorithms as MMA





A FIRST EXAMPLE: GENESIS OF A STRUCTURE



E. Lemaire, PhD Thesis, Uliege, 2013

TOPOLOGY OPTIMIZATION AS A COMPLIANCE MINIMIZATION PROBLEM

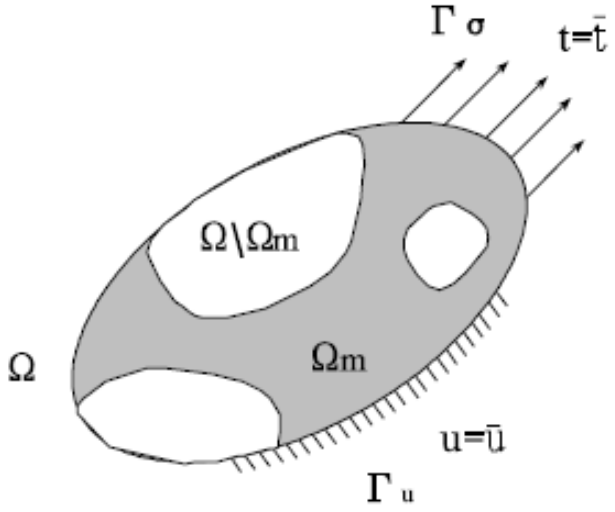
DESIGN PROBLEM FORMULATION

- The fundamental problem of topology optimization deals with the optimal material distribution within a continuum structure subject to a single static loading.

- In addition one can assume that

the structure is subject to homogeneous boundary conditions on Γ_u.

$$u_i = 0 \quad \text{on} \quad \Gamma_u$$



- The principle of virtual work writes

$$\mathbf{u} \in V_0 \; : \quad a_E(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \quad \forall\, \mathbf{v} \in V_0$$

$$\delta \mathbf{q}^T \mathbf{K} \mathbf{q} = \delta \mathbf{q}^T \mathbf{g}$$

DESIGN PROBLEM FORMULATION

- A typical topology optimization problem is to find the best subset of the design domain minimizing the volume or alternatively the mass of the structure,

$$V = \int_{\Omega} \rho(\mathbf{x})\, d\Omega$$

- while achieving a given level of functional (mechanical) performance.
- Following Kohn (1988), the problem is well posed from a mathematical point of view if the mechanical behaviour is sufficiently smooth. Typically one can consider :
  - Compliance (energy norm)
  - A certain norm of the displacement over the domain
  - A limitation of the maximum stress

DESIGN PROBLEM FORMULATION

- Compliance performance: The mechanical work of the external loads

$$l(\boldsymbol{u}) = \int_{\Omega} \boldsymbol{f}^T \boldsymbol{u}\, d\Omega + \int_{\Gamma} \boldsymbol{t}^T \boldsymbol{u}\, d\Gamma$$

  - Using finite element formulation

$$C(\boldsymbol{q}) = \boldsymbol{g}^T \boldsymbol{q}$$

- Limitation of a given local stress measure $||\sigma(x)||$ over a sub-domain Ω_2 excluding some neighborhood of singular points related to some geometrical properties of the domain (reentrant corners) or some applied loads

$$\sup_{\boldsymbol{x} \in \Omega_2} ||\sigma(\boldsymbol{x})||$$

  - Stress measure $||\sigma(x)||$ → Von Mises, Tresca, Tsai-Hill…

DESIGN PROBLEM FORMULATION

- The average displacement (according to a selected norm) over the domain or a subdomain Ω_1 excluding some irregular points

$$\int_{\Omega_1} ||\boldsymbol{u}||\, d\Omega$$

- If one considers the quadratic norm and if the finite element discretization is used, one reads

$$\int_{\Omega_1} \boldsymbol{u}^T\, \boldsymbol{u}\, d\Omega \;=\; \int_{\Omega_1} \boldsymbol{q}^T \boldsymbol{N}(\boldsymbol{x})^T\, \boldsymbol{N}(\boldsymbol{x})\, \boldsymbol{q}\, d\Omega \;=\; \boldsymbol{q}^T \boldsymbol{M}\, \boldsymbol{q}$$

- Assuming a lumped approximation of the matrix **M,** one can find the simplified equivalent quadratic norm of the displacement vector

$$\int_{\Omega_1} ||\boldsymbol{u}||\, d\Omega \;\approx\; \boldsymbol{q}^T \boldsymbol{q}$$

DESIGN PROBLEM FORMULATION

- The choice of the compliance is generally the main choice by designers.
  - At equilibrium, the compliance is also the strain energy of the structure, so that compliance is the energy norm of the displacements giving rise to a smooth displacement field over the optimized structure.

$$l(\boldsymbol{u}) = a_E(\boldsymbol{u}, \boldsymbol{u}) \qquad \boldsymbol{g}^T \boldsymbol{q} = \boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q}$$

  - One can interpret the compliance as the displacement under the loads. For a single local case, it is the displacement under the load.

$$C(\boldsymbol{q}) = \boldsymbol{g}^T \boldsymbol{q} = f_i \, u_i$$

DESIGN PROBLEM FORMULATION

- The choice of the compliance is generally the main choice of designers.
  - The sensitivity of compliance is easy to calculate. Being self adjoined, compliance is self adjoined, and it does not require the solution of any additional load case.
  - Conversely local stress constraints call for an important amount of additional CPU to compute the local sensitivities.
  - One can find analytical results providing the optimal bounds of composites mixture of materials for a given external strain field. The problem is known as the *G-closure.*

DESIGN PROBLEM FORMULATION

- Finally the statement of the basic topology problem writes:

$$\begin{aligned} &\min && V(\mathbf{x}) \\ &\underline{x}_i \le x_i \le \overline{x}_i \\ &\text{s.t.:} && C(\mathbf{q}) = \mathbf{g}^T\mathbf{q} \le \bar{C} \\ & && \mathbf{K}\,\mathbf{q} = \mathbf{g} \end{aligned}$$

- Alternatively it is equivalent for a given bounds on the volume and the compliance to solve the minimum compliance subject to volume constraint

$$\begin{aligned} &\min && C(\mathbf{q}) = \mathbf{g}^T\mathbf{q} \\ &\underline{x}_i \le x_i \le \overline{x}_i \\ &\text{s.t.:} && V(\mathbf{x}) \le \bar{V} \\ & && \mathbf{K}\,\mathbf{q} = \mathbf{g} \end{aligned}$$

PROBLEM FORMULATION

- For several load cases, average compliance

$$\begin{array}{cc} \min & \sum_k w_k \, \mathbf{g}^{(k)\,T} \mathbf{q}^{(k)} \\ \underline{x}_i \le x_i \le \overline{x}_i & \\ \text{s.t.:} & V(\mathbf{x}) \le \bar{V} \\ & \mathbf{K}\,\mathbf{q}^{(k)} = \mathbf{g}^{(k)} \end{array}$$

- Or better a worst-case approach

$$\begin{array}{cc} \min & \max_k \mathbf{g}^{(k)\,T} \mathbf{q}^{(k)} \\ \underline{x}_i \le x_i \le \overline{x}_i & \\ \text{s.t.:} & V(\mathbf{x}) \le \bar{V} \\ & \mathbf{K}\,\mathbf{q}^{(k)} = \mathbf{g}^{(k)} \end{array}$$

  - Where k is load case index, K is the stiffness matrix of FE approximated problem, g_k, and q_k are the load case and generalized displacement vectors for load case k
  - and $\rho(x)$ is the local density and V is the volume

DESIGN PROBLEM FORMULATION

- Minimize compliance

s.t.

  - Given volume
  - (bounded perimeter)
  - (other constraints)

- Maximize eigenfrequencies

s.r.

  - Given volume
  - (bounded perimeter)
  - (other constraints)

- Minimize the maximum of the local failure criteria

s.t.

  - Given volume
  - (bounded perimeter)
  - (other constraints)

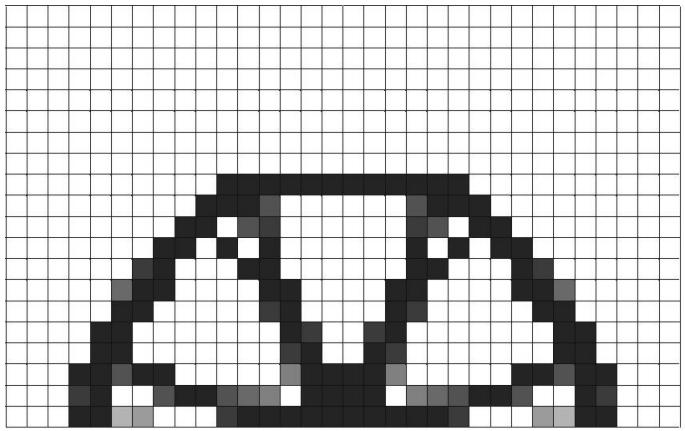
TOPOLOGY OPTIMIZATION USING HOMOGENIZATION VS SIMP BASED TOPOLOGY OPTIMIZATION

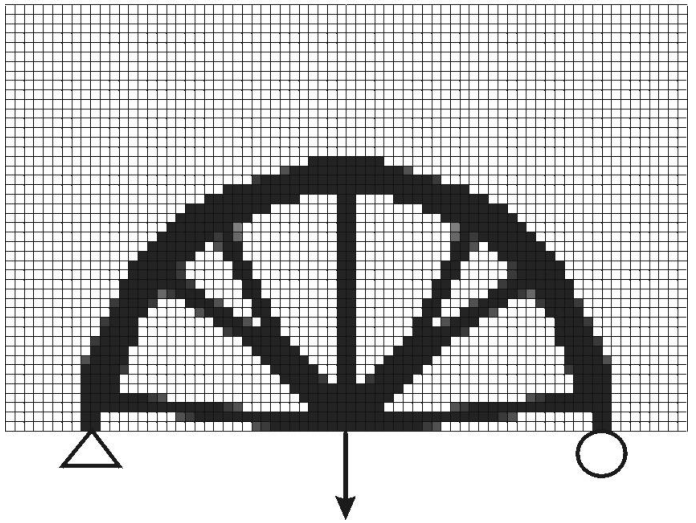
TOPOLOGY OPTIMIZATION: FORMULATION

- Well-posed ness of problem?
  Discretised problem is ill-posed
  - Mesh-dependent solutions
  - Recreate microstructures
  - Nonexistence and uniqueness of a solution
- Homogenisation Method:

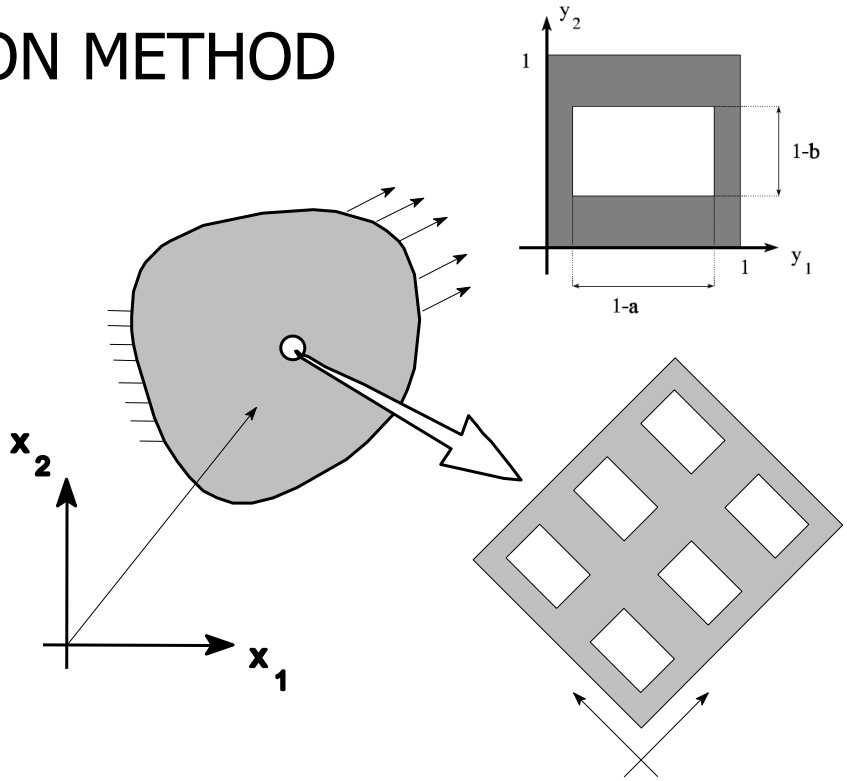
  → Extend the design space to all porous composites of variable density
- Filter method / Perimeter method / Slope constraints:

  → Restrict the design space by eliminating chattering designs from the design space





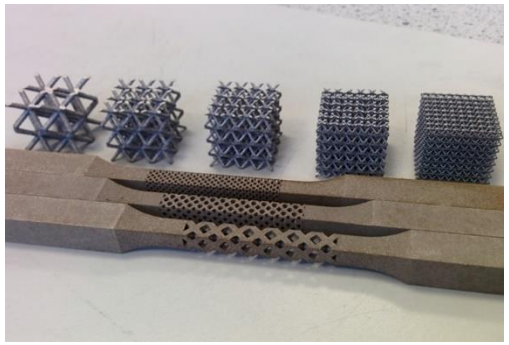
HOMOGENIZATION METHOD



- Select one family of microstructures whose geometry is fully parameterized in terms of a set of design variables [Bendsoe and Kikuchi, 1988]
  - G closure : optimal microstructure (full relaxation)
  - Suboptimal microstructures (partial relaxation)

- Use homogenization theory to compute effectives properties: in terms of microstructural geometrical parameters: $E_{ijk} = E^{h}_{ijkl}(a,b,\ldots)$
- Difficult to interpret and fabricate the optimal material distribution as it is

- Revival interest with arrival of cellular structures e.g. lattice structures made by additive manufacturing



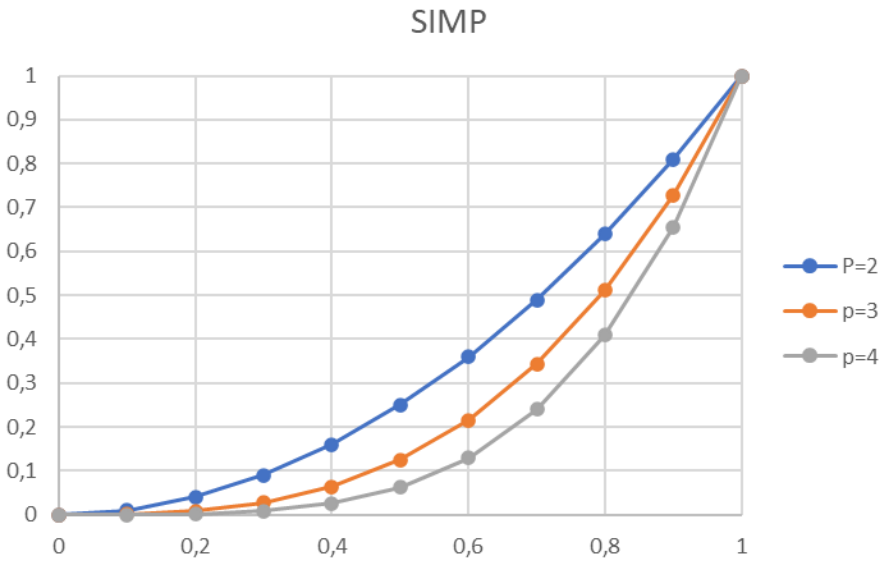
POWER LAW MODEL (SIMP)

- Simplified model of a microstructured material with a penalisation of intermediate densities [Bendsoe, 1989]
- Stiffness properties:

$$E = x^p \, E^0$$

$$\rho = x \, \rho^0$$

$$0 \leq x \leq 1 \quad p > 1$$



- Strength properties:

$$\sigma_l = x^p \, \sigma_l^0$$

- Modified SIMP should be preferred to avoid singularities

$$E(x) = E_{min} + x^p(E^0 - E_{min})$$

- Can be related to actual micro geometries [Bendsoe and Sigmund, 1999]
- 90% of current topology optimization runs

ALTERNATIVE PARAMETRIZATION TO SIMP

- Alternatively RAMP parameterization (Stolpe & Svanberg, 2001) enables controlling the slope at zero density

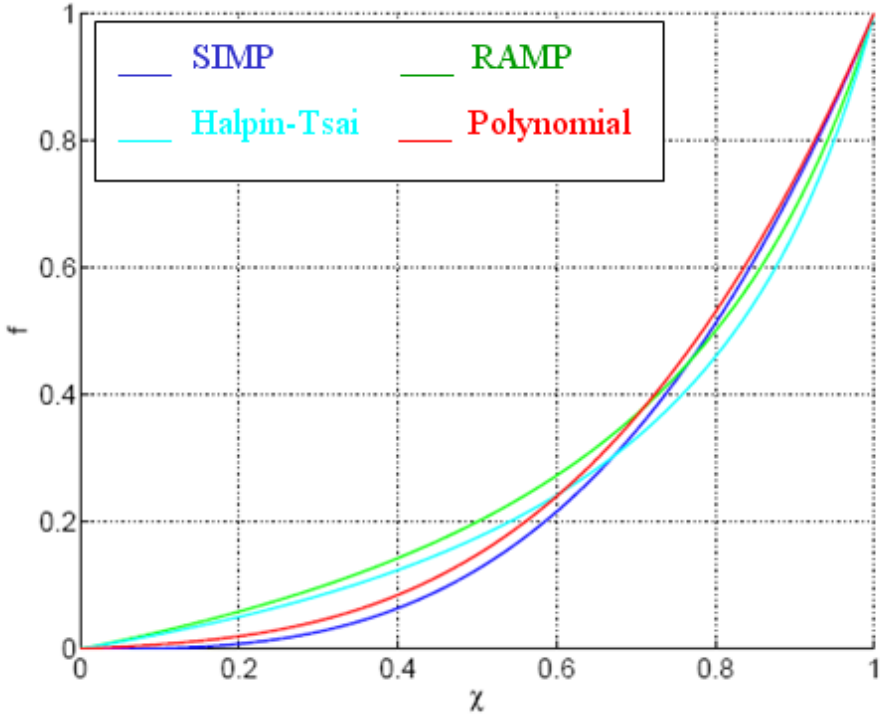
$$E(x) = \frac{x}{1 + p(1 - x)} E^0$$

- Halpin Tsai (1969)

$$E(x) = \frac{r\, x}{(1 + r) - x} E^0$$

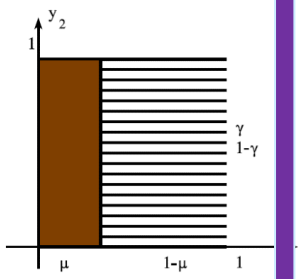
- Polynomial penalization (Zhu, 2009):

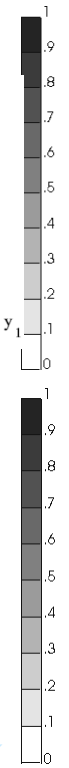
$$E(x) = \left(\frac{\alpha - 1}{\alpha} x^p + \frac{1}{\alpha} x \right) E^0$$

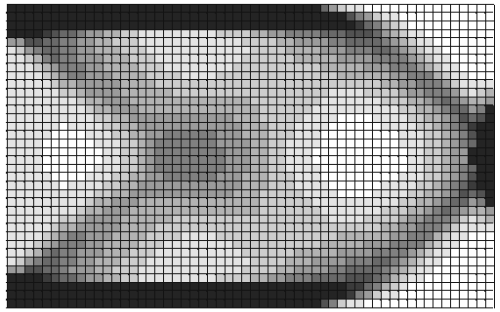


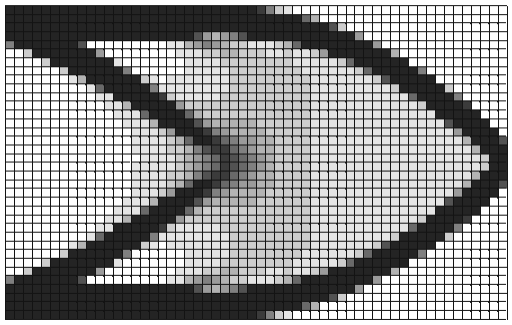
HOMOGENIZATION METHOD

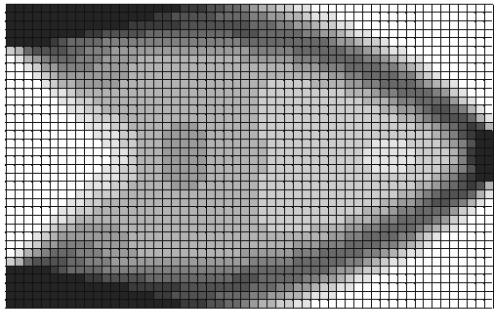
Penalization of intermediate densities

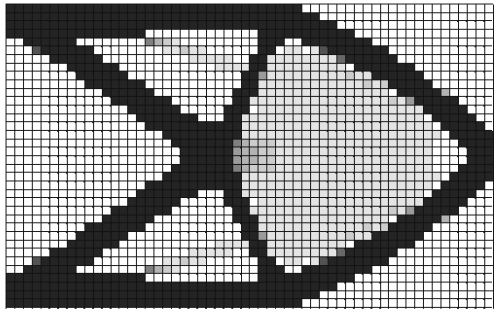


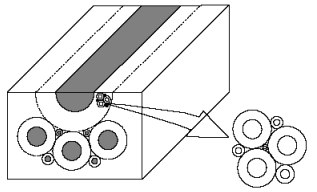


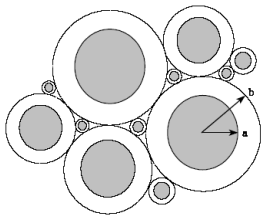


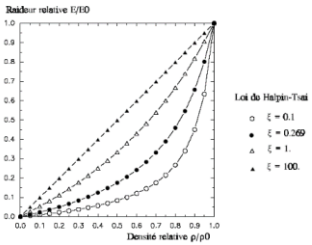








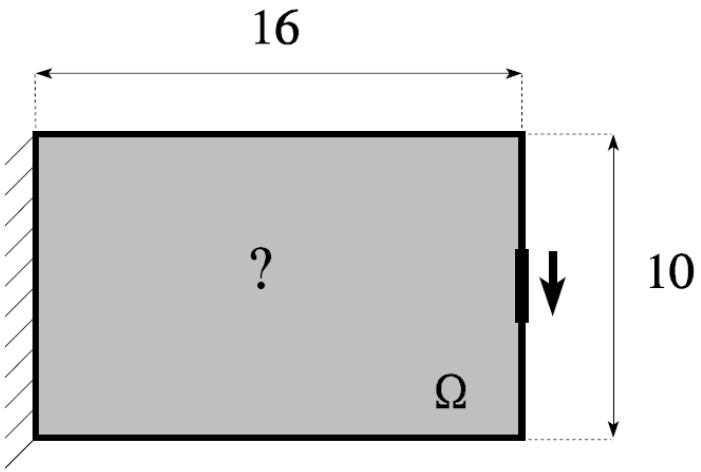


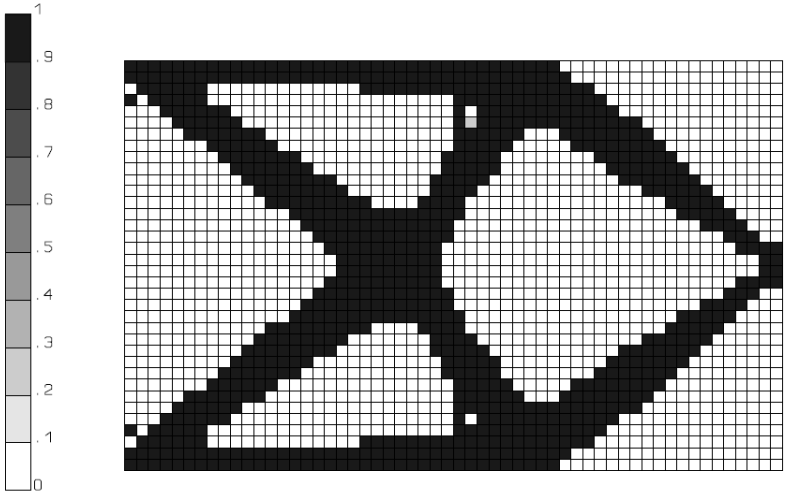


From anisotropic to isotropic materials

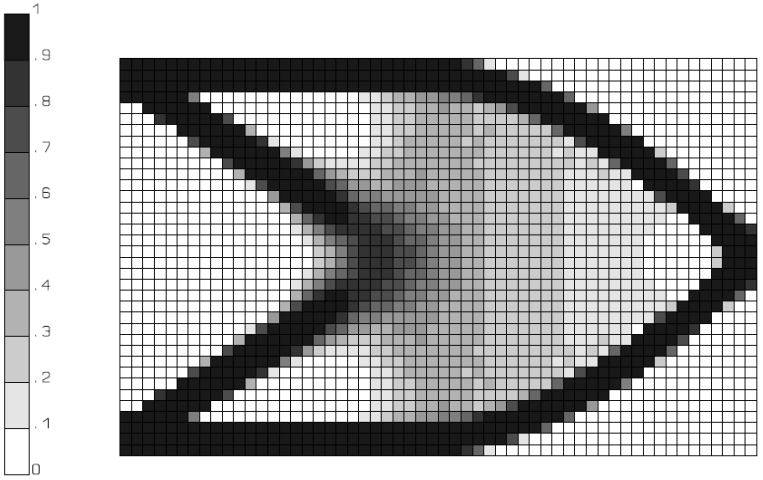
- Investigation of the influence of the selected microstructure upon the optimal topology :
  - 1° orthotropic v.s. isotropy
  - 2° penalization of intermediate properties

SIMP : PENALIZATION OF INTERMEDIATE DENSITIES

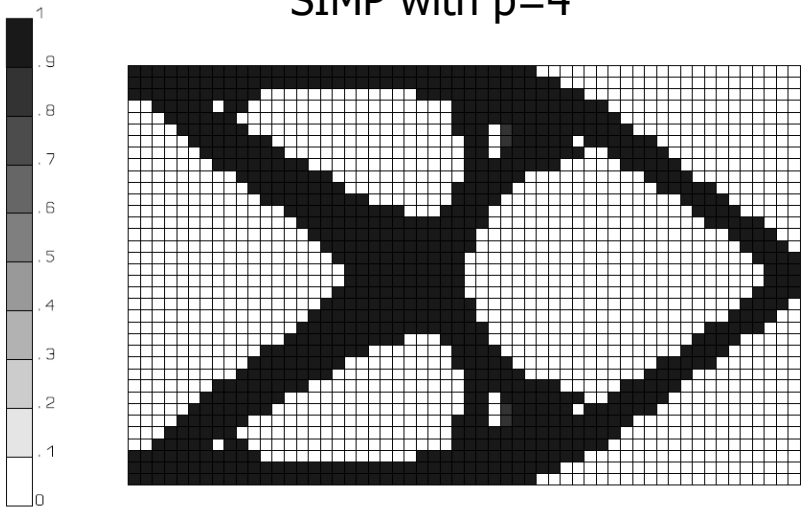




SIMP with p=4



SIMP with p=2



SIMP with p=3

POWER LAW MODEL (SIMP)

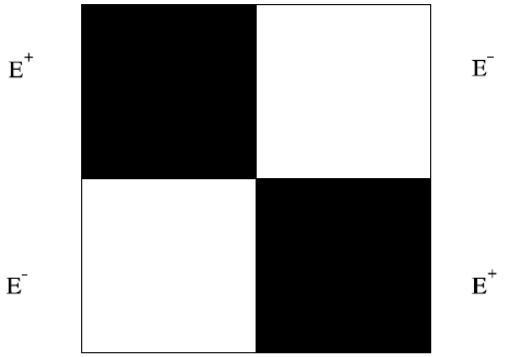
- Prescribing immediately a high penalization may introduce some numerical difficulties:
  - Optimization problem becomes difficult to solve because of the sharp variation of material properties close to x=1
  - Optimization problem includes a lot of local optima and solution procedure may be trapped in one of these.

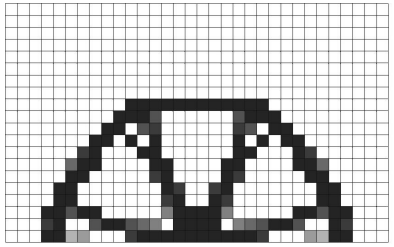
- To mitigate these problems, one resorts to the so-called continuation procedure in which p is gradually increased from a small initial value till the desired high penalization.
- Typically:
  - p(0) = 1.6
  - p(k+1) := p(k) + Δp after a given number of iterations or when a convergence criteria is OK

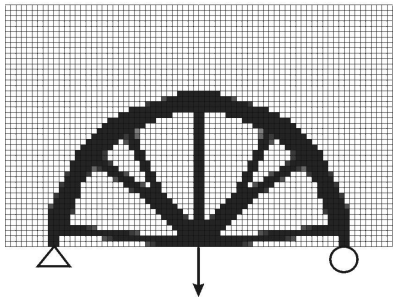
FILTERING TECHNIQUES AND MESH INDEPENDENCY STRATEGIES

Two numerical difficulties

- Checkerboard patterns: numerical instabilities related to the inconsistency between the displacement and density fields.
  - Appearance of alternate black-white patterns
  - Checkerboard patterns replaces intermediate densities



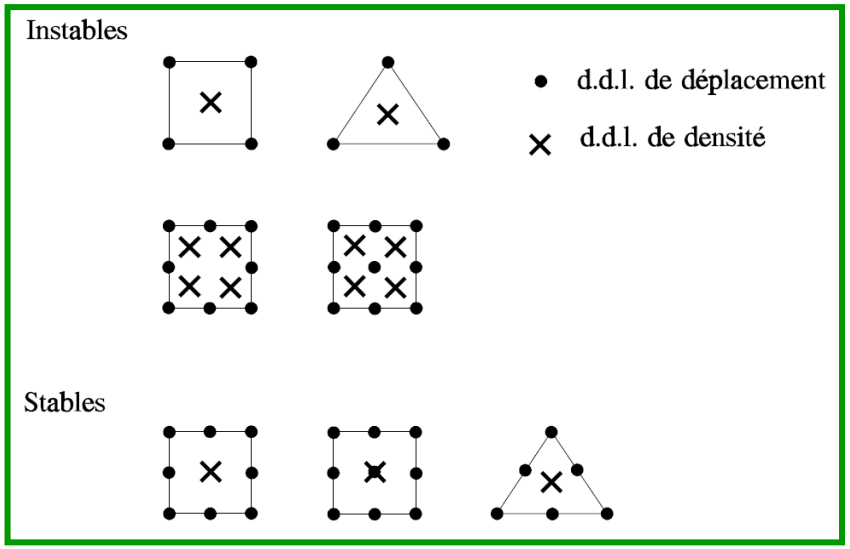


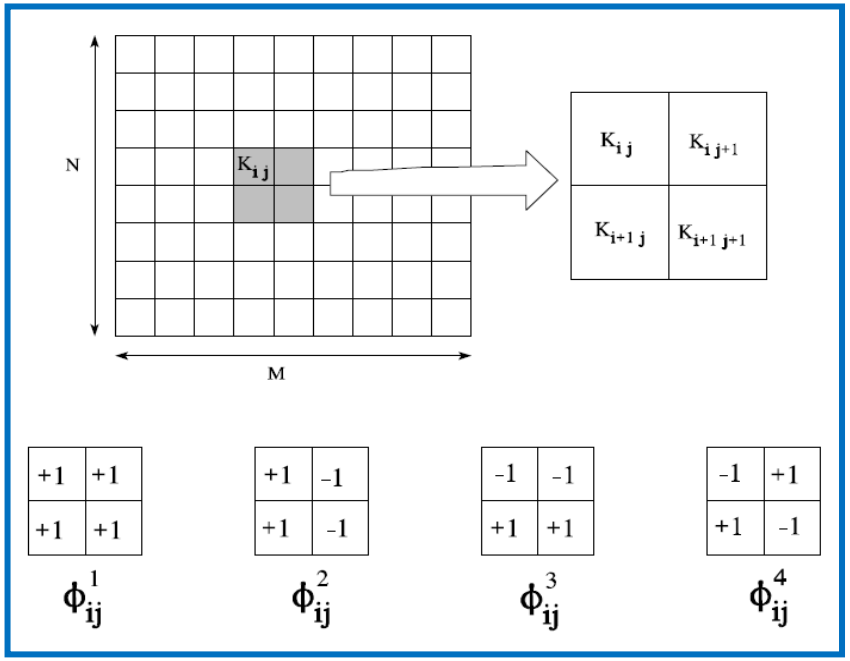


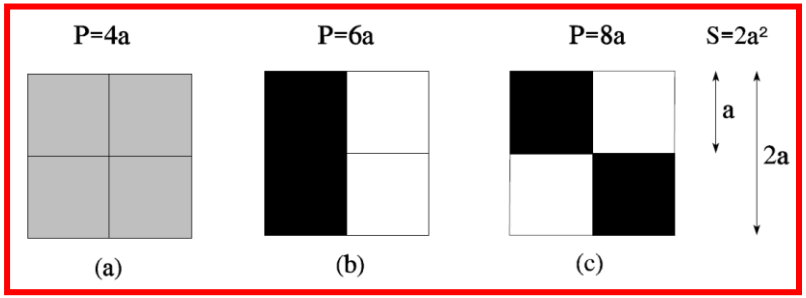
- Mesh dependency: the solution depends on the computing mesh.
  - New members appears when refining the mesh
  - Number of holes and structural features is modified when changing the mesh.
  - Stability (and meaning) of solutions?

Checkerboard patterns

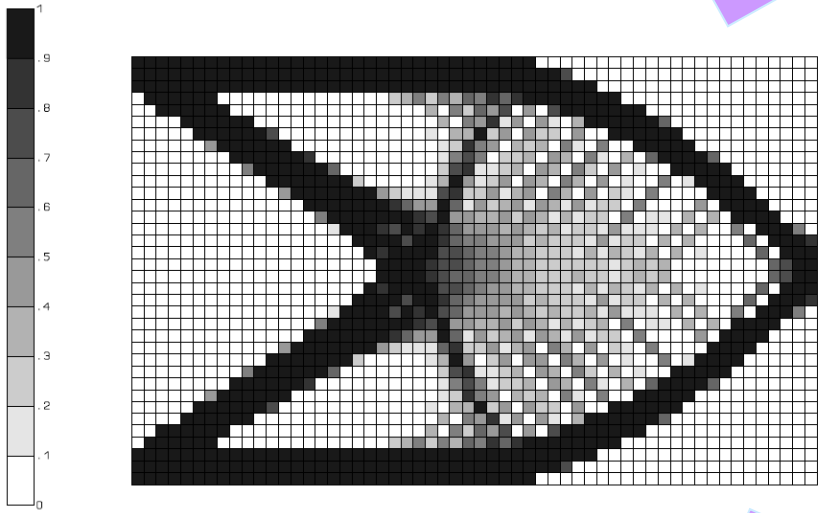
- Babuska Brezzi conditions of discretization schemes
- Checkerboard free numerical schemes
  - High order FE elements
  - Filtering density field solutions ➔ lower order density fields
  - Perimeter constraint



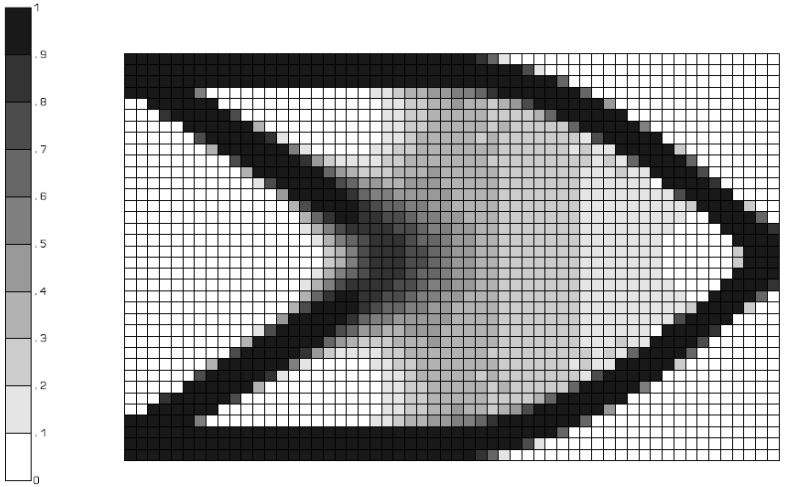




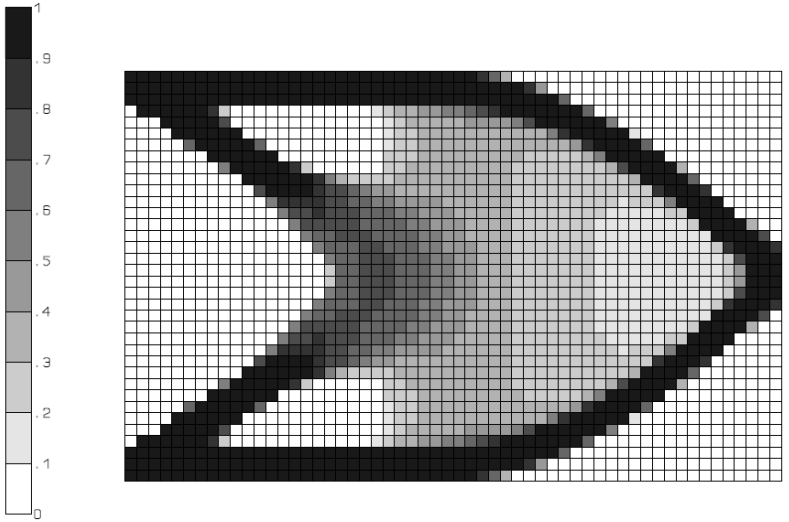
Checkerboard patterns



Solution with checkerboards
SIMP with p=2
FE u: degree 1 / Density : constant

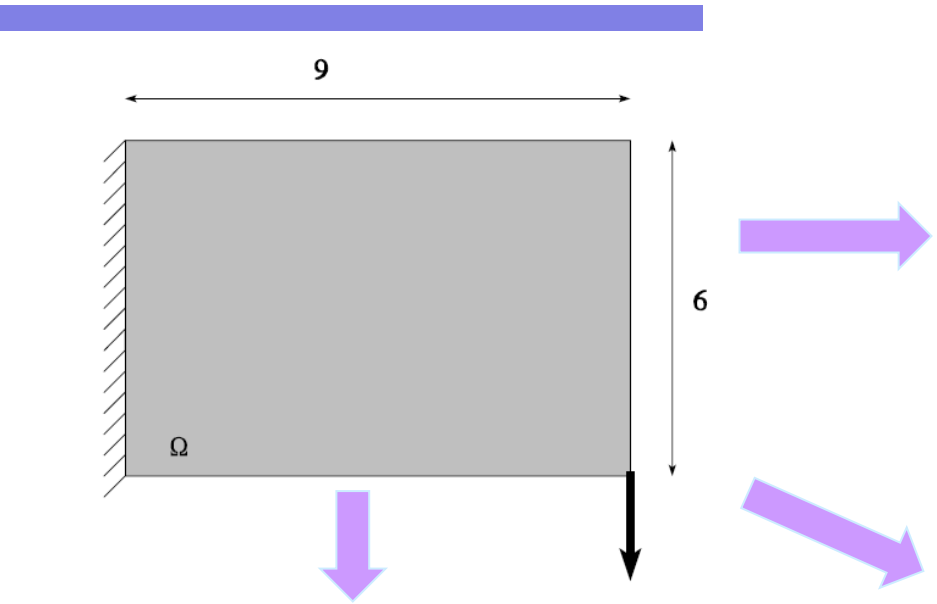


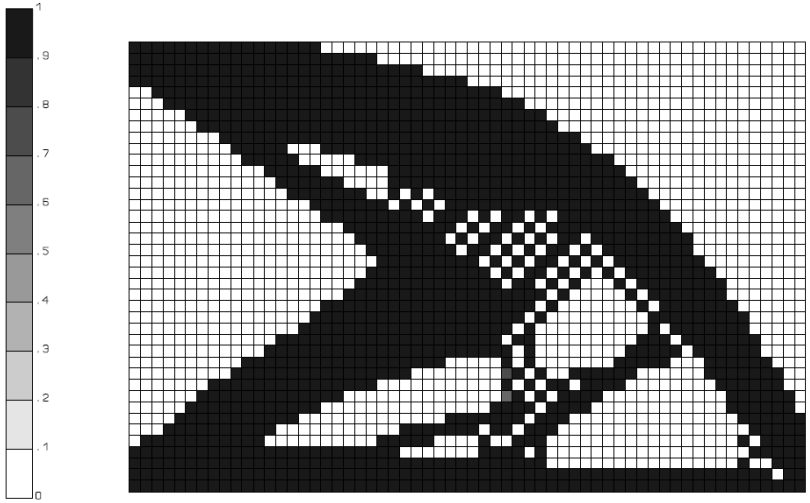
FE u: degree 2 / Density : constant



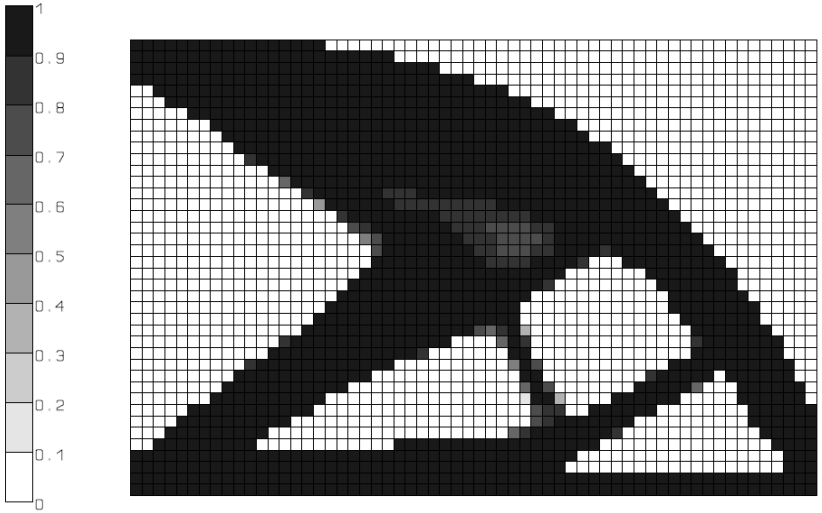
With perimeter constraint

Checkerboard patterns

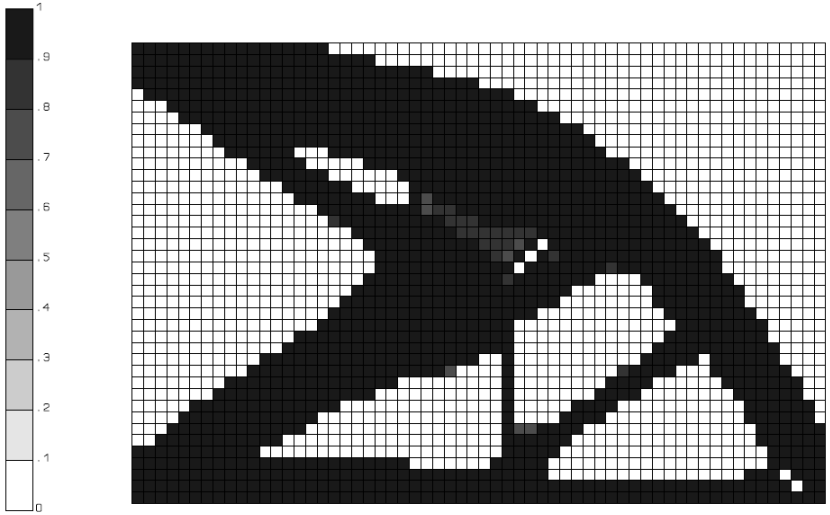




SIMP with p=3. F u: degree 1

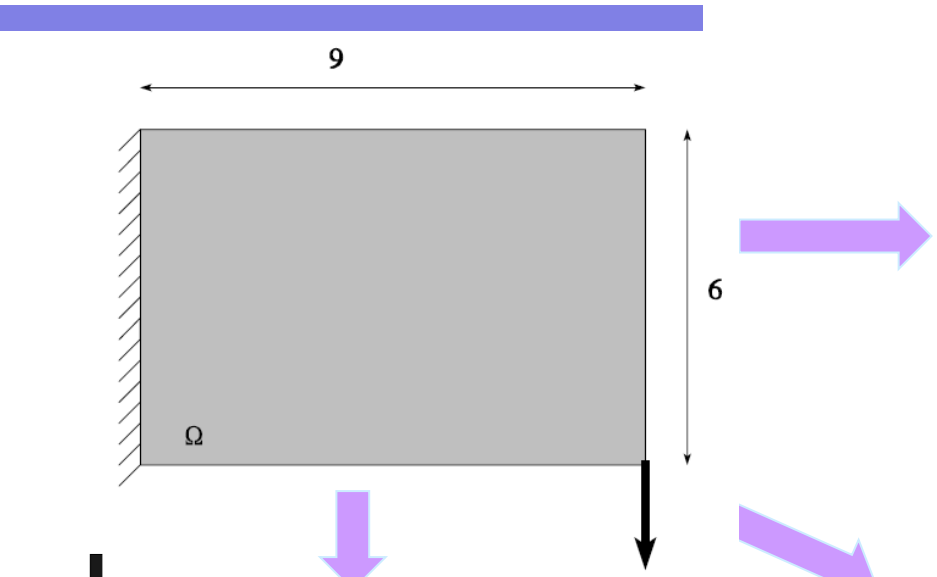


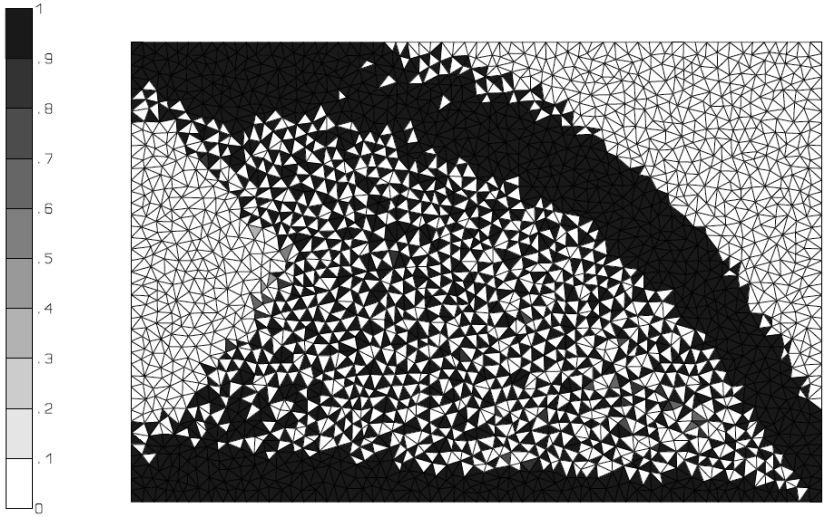
Perimeter < 60



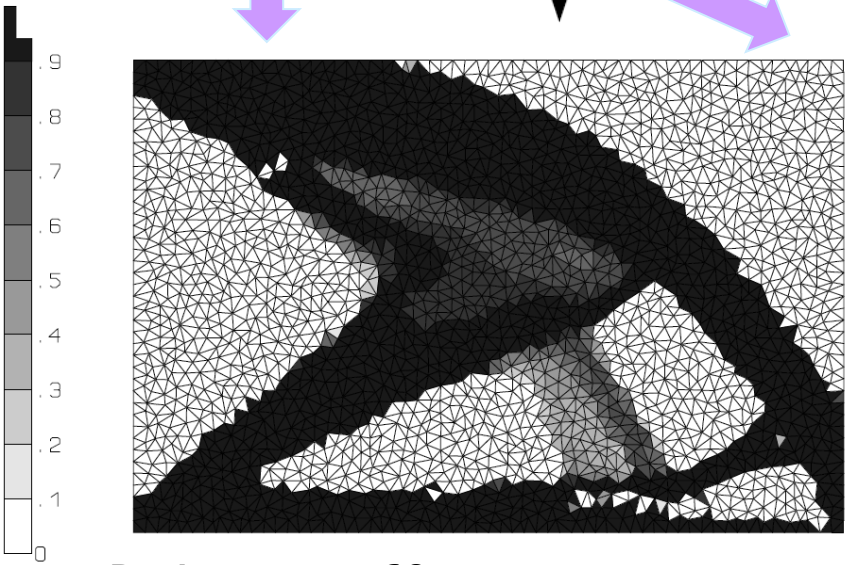
FE u: degree 2

Checkerboard patterns

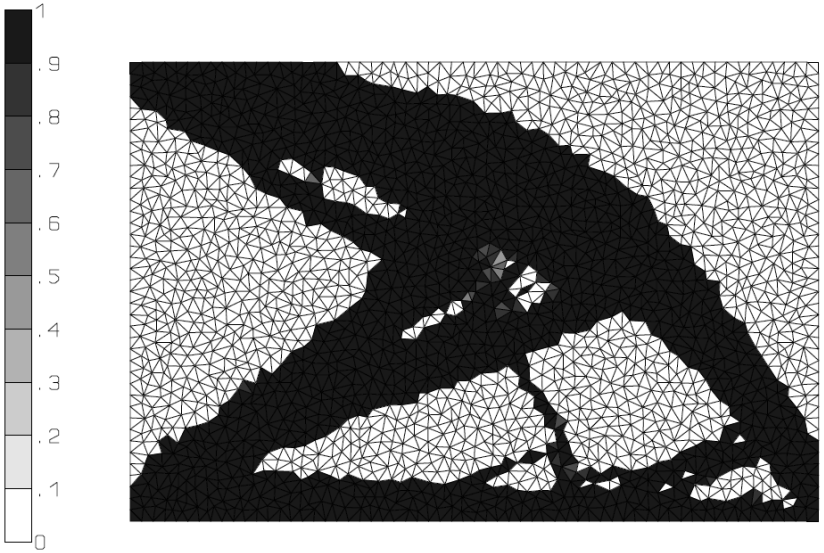




SIMP with p=3. FE u: degree 1



Perimeter < 60

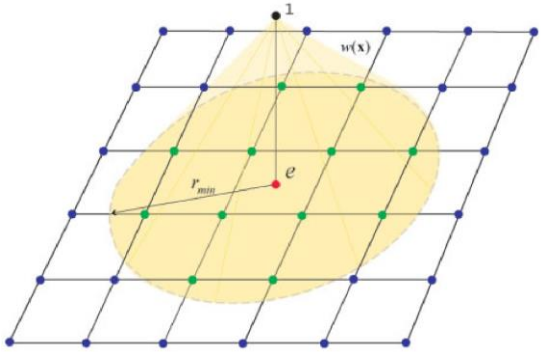


FE u: degree 2

Mesh dependency

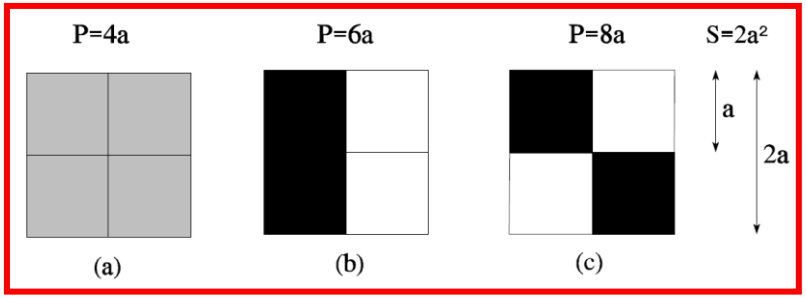
- Mesh independent solution: insure mesh independent filtering of lower size details
  - Low pass filter [Sigmund (1998)]

$$\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X)\, v_i\, x_i}{\sum_i w_i(X_i)\, v_i}$$

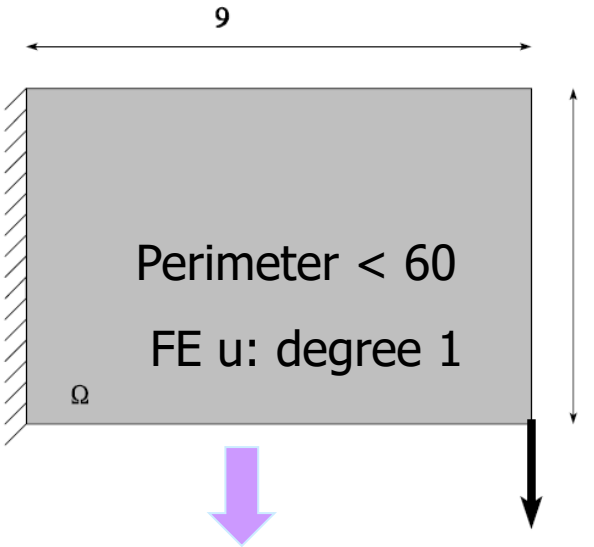


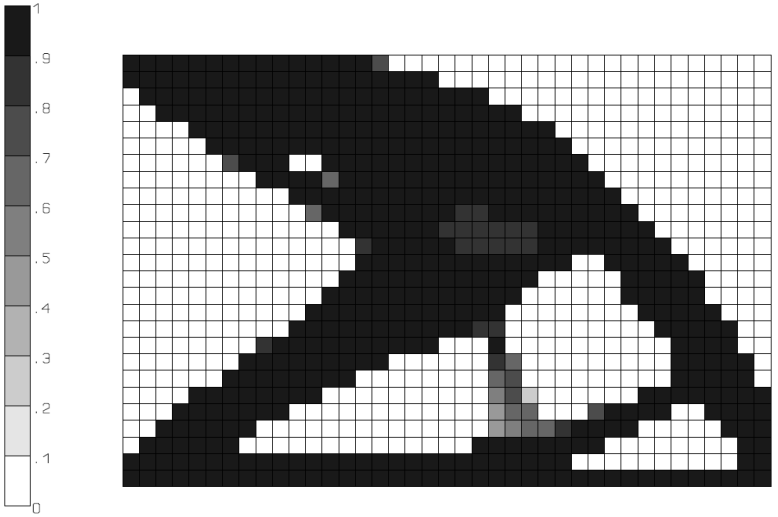
  - Perimeter constraint [Ambrosio & Butazzo (1993)]

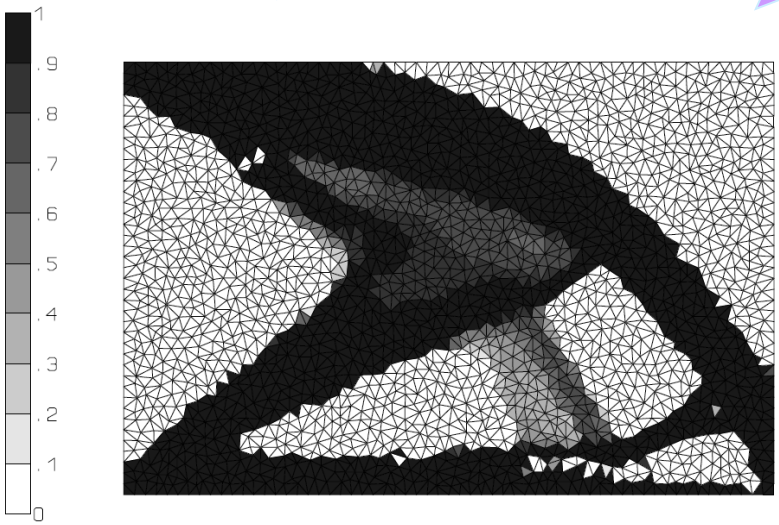
$$\begin{aligned} \min_{0 \le \rho(x) \le 1} \quad & \boldsymbol{g}^T \boldsymbol{q} \\ \text{s.t.} \quad & P(\rho(\boldsymbol{x})) \le \bar{P} \\ & V = \int_\Omega \rho(\boldsymbol{x})\, d\Omega \le \bar{V} \\ & g_j(\boldsymbol{x}) \le \bar{g}_j \qquad j = 1 \ldots m \end{aligned}$$

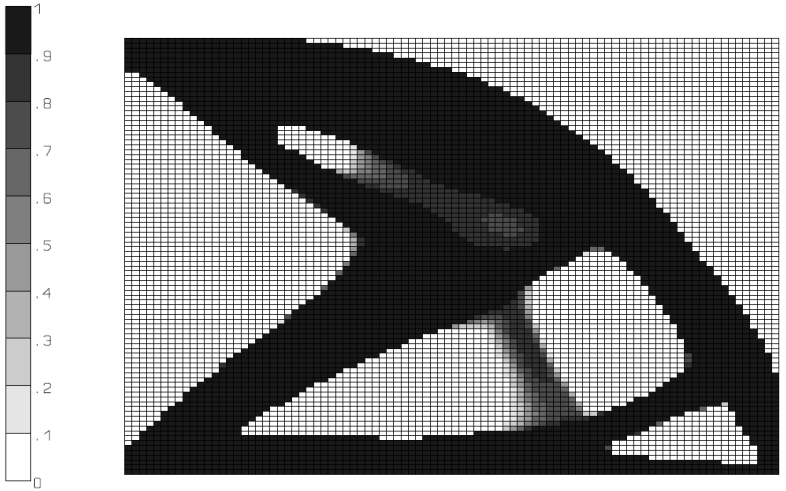


Mesh independency









PERIMETER METHOD

- Continuous version of perimeter measure

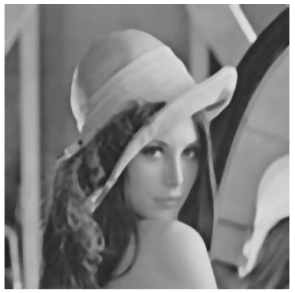
$$P(\rho) = \int_{\Omega} ||\nabla \rho|| d\Omega + \int_{\Gamma_j} |[\rho]|_j \; d\Gamma$$

  - With the gradient of the density field and the jump $[]_j$ of the density across discontinuity surfaces j

- The continuous approximation of the modulus of the gradient

$$P(\rho) \simeq \int_{\Omega} \sqrt{\nabla \rho^T \nabla \rho + \frac{\varepsilon^2}{h^2}} - \frac{\varepsilon}{h} \; d\Omega$$
$$+ \int_{\Gamma_j} \sqrt{[\rho]_j^2 + \varepsilon^2} - \varepsilon \; d\Gamma$$

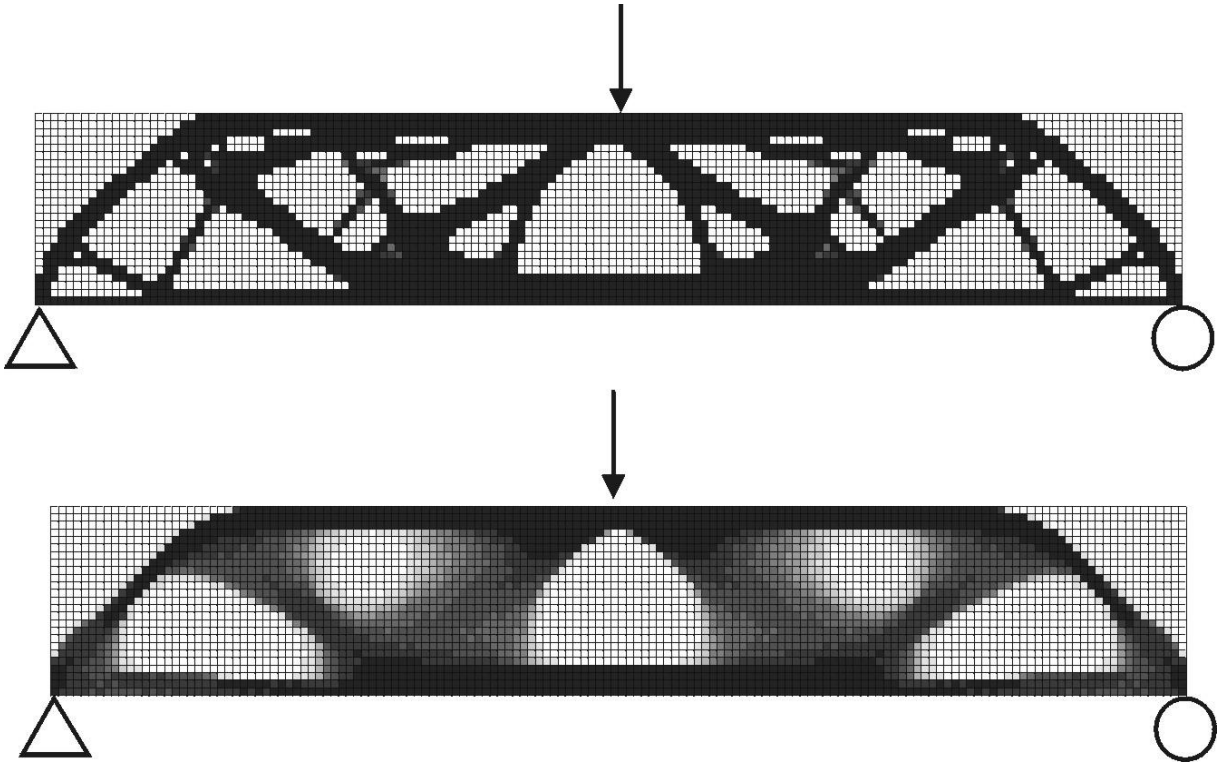
x



$\|\nabla x\|$

PERIMETER METHOD

- The structural complexity of the structure is controlled with a bound over the perimeter
- An efficient numerical strategy has been elaborated to cater with the difficult perimeter constraint (Zhang & Duysinx, 1998)



FILTERING MATERIAL DENSITIES

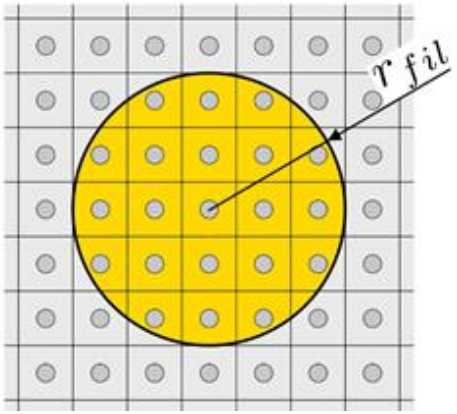
- To avoid mesh dependency and numerical instabilities like checkerboards patterns, one approach consists in restricting the design space of solutions by forbidding high frequency variations of the density field.

- Basic filtering by Bruns and Tortorelli (2001), proven by Bourdin (2001)

$$\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X)\, v_i\, x_i}{\sum_i w_i(X_i)\, v_i}$$

$$N_e = \{ i \mid ||\mathbf{X}_i - \mathbf{X}_e|| \leq R \}$$

$$w_i(\mathbf{X}_i) = \begin{cases} R - ||\mathbf{X}_i - \mathbf{X}_e|| & i \in N_e \\ 0 & i \notin N_e \end{cases}$$



FILTERING MATERIAL DENSITIES

- Other weighting functions
  - Gaussian

$$w(\mathbf{X}_i) = \exp\left[-\frac{1}{2}\left(\frac{||\mathbf{X}_i - \mathbf{X}_e||}{\sigma_d}\right)^2\right]$$

  - Constant

$$w(\mathbf{X}_i) = 1$$

- Density filter is equivalent to solving a Helmotz equation Lazarov et al.

$$-R_{\min}\,\nabla^2\bar{\delta} \;+\; \bar{\delta} \;=\; \delta$$

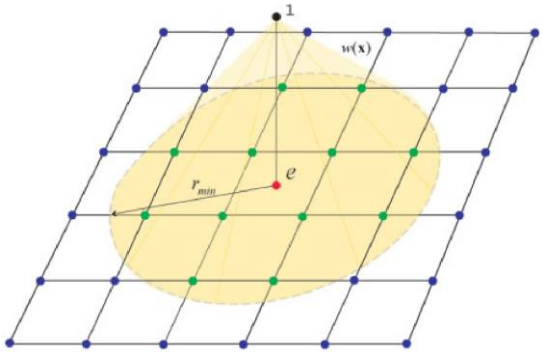
- With the following Neuman boundary conditions

$$\frac{\partial\bar{\delta}}{\partial n} \;=\; 0$$

FILTERING MATERIAL DENSITIES

- Historically Ole Sigmund (1994, 1997) introduced a filter of the sensitivities

$$\frac{\partial \tilde{f}}{\partial x_e} = \frac{\sum_{i \in N_e} w(\mathbf{X}_i) x_i \frac{\partial f}{\partial x_i}}{x_e \sum_{i \in N_e} w(\mathbf{X}_i)}$$



- with

$$w(\mathbf{X}_i) = R - ||\mathbf{X}_i - \mathbf{X}_e||$$

- For non uniform meshes, Sigmund proposed to use

$$\frac{\partial \tilde{f}}{\partial x_e} = \frac{\sum_{i \in N_e} w(\mathbf{X}_i) \frac{x_i}{v_i} \frac{\partial f}{\partial x_i}}{\frac{x_e}{v_e} \sum_{i \in N_e} w(\mathbf{X}_i)}$$

- The smoothed sensitivities correspond to the sensitivities of a smoothed version of the objective function (as well as the constraints)

FILTERING MATERIAL DENSITIES

ρ

$\tilde{\rho}(r_{fil} = 8)$

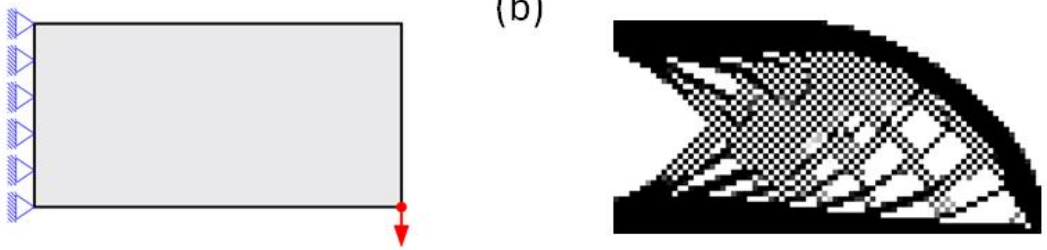
$\tilde{\rho}(r_{fil} = 15)$

FILTERING MATERIAL DENSITIES

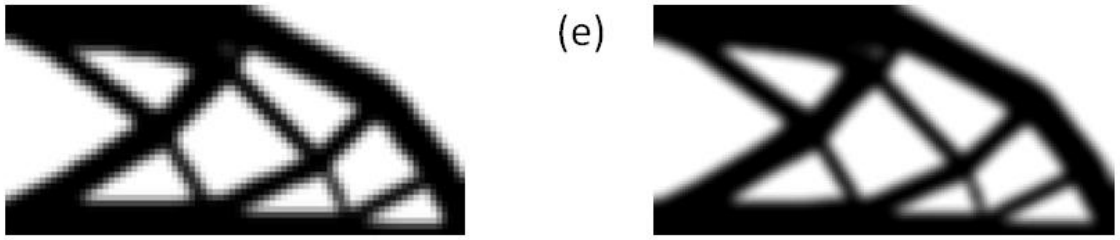
ρ $\tilde{\rho}(r_{fil} = 8)$ $\tilde{\rho}(r_{fil} = 15)$

$E_i = \rho_i^p E_0$

a) (b)



(d) (e)



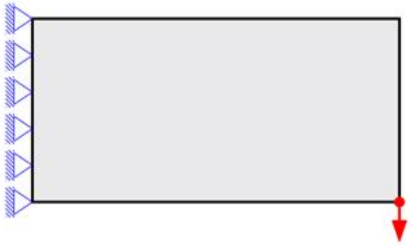
$E_i = \tilde{\rho}_i^p E_0$ $E_i = \tilde{\rho}_i^p E_0$

FILTERING MATERIAL DENSITIES

ρ $\tilde{\rho}(r_{fil} = 8)$ $\tilde{\rho}(r_{fil} = 15)$

$$E_i = \rho_i^p E_0$$

a)

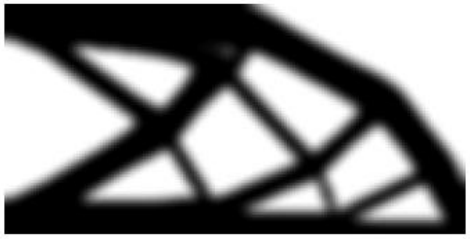


(b)

(d)

$$E_i = \tilde{\rho}_i^p E_0$$

(e)

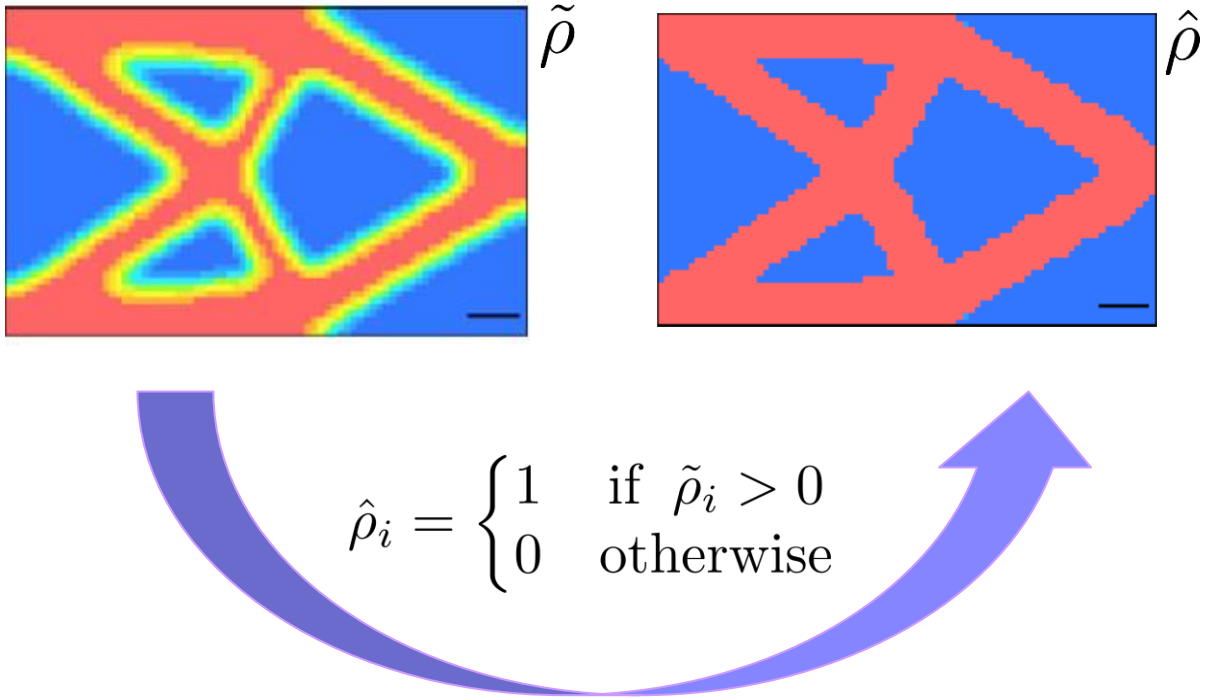


$$E_i = \tilde{\rho}_i^p E_0$$

- ~~Mesh dependent~~
- ~~Checkerboard~~
- Non-Discrete Solut.

HEAVISIDE FILTER

- To obtain 0/1 solutions , Guest et al. (2014) modifies the density filter with a Heaviside function such that if $x_e>0$, the Heaviside gives a physical value of the density equal to '1' and if the $x_e=0$, the Heaviside gives a density '0'



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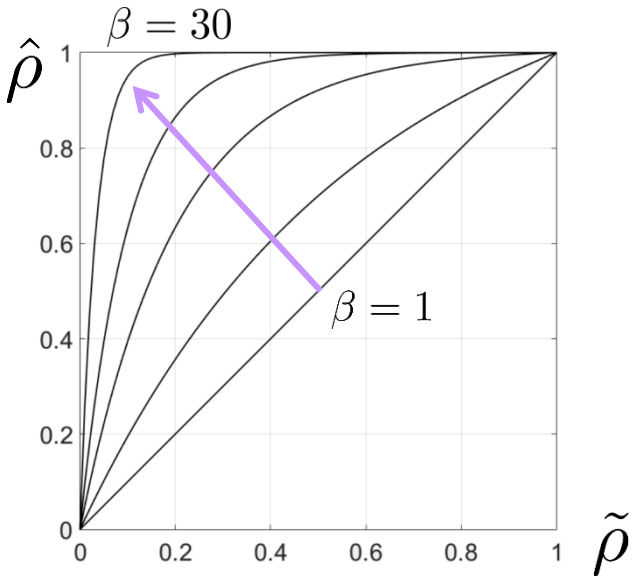
- Heaviside smooth approximation

$$\hat{\rho}_i = \begin{cases} 1 & \text{if } \tilde{\rho}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\rho}_i = 1 - e^{-\beta \tilde{\rho}_i} + \tilde{\rho}_i \, e^{-\beta}$$



  - For β→ 0, the filter gives the original filter
  - For β→ infinity, the function reproduces the max operator, that is the density becomes 1 if there is any element in the neighborhood that is non zero.

HEAVISIDE FILTER

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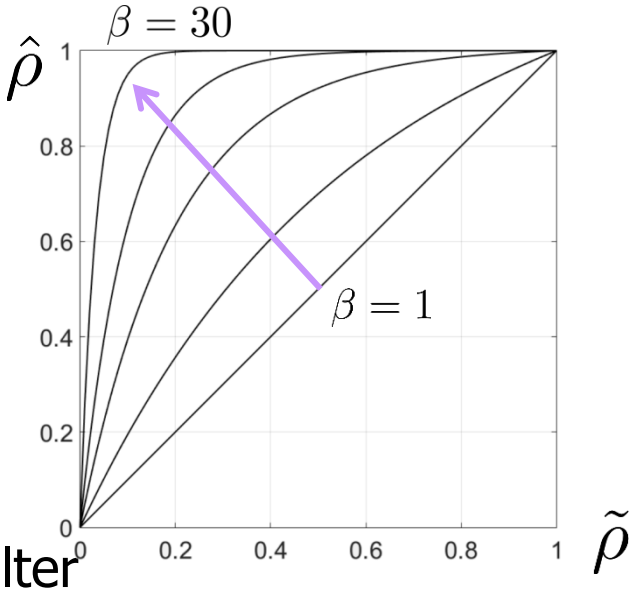

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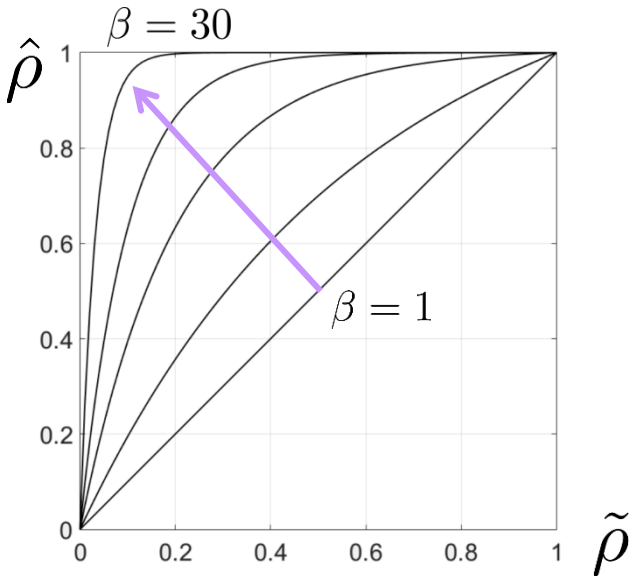
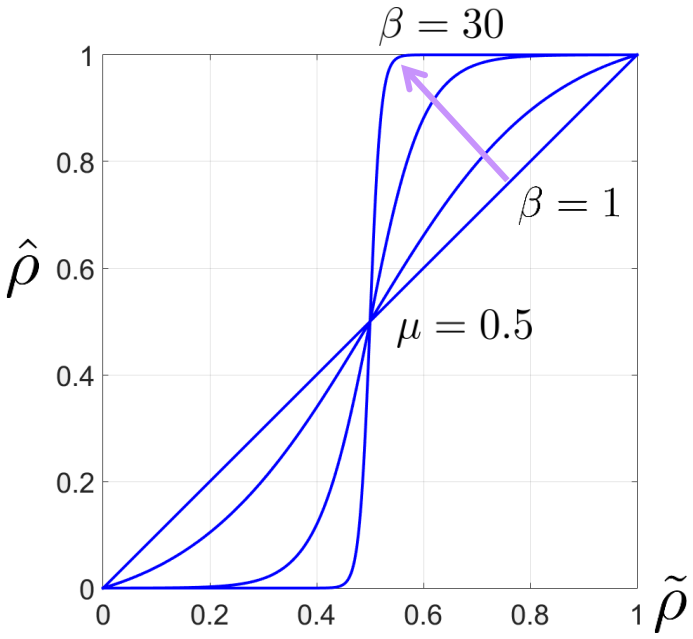
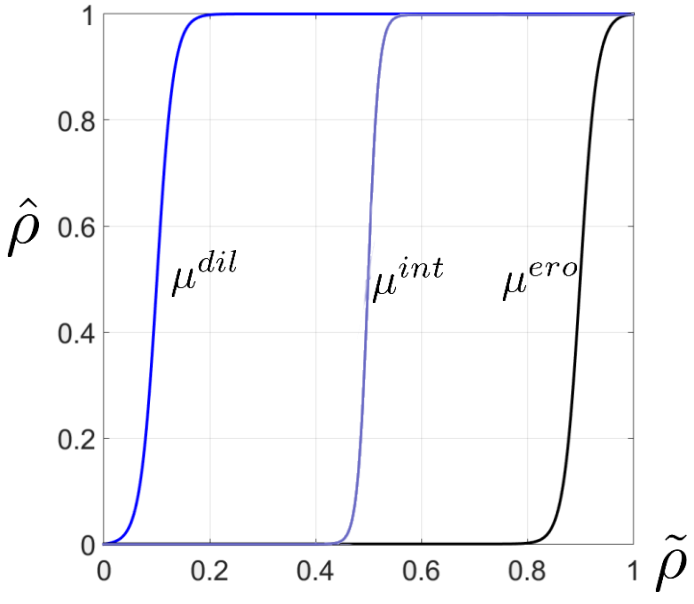


- ~~**Mesh dependent**~~
- ~~**Checkerboard**~~
- ~~**Non-Discrete Solution**~~
- **Need of continuation / Large number of iterations (>100)**

HEAVISIDE FILTER

- Heaviside function can be extended (Wang, Lazarov, Sigmund, 2011) to control minimum and maximum length scale



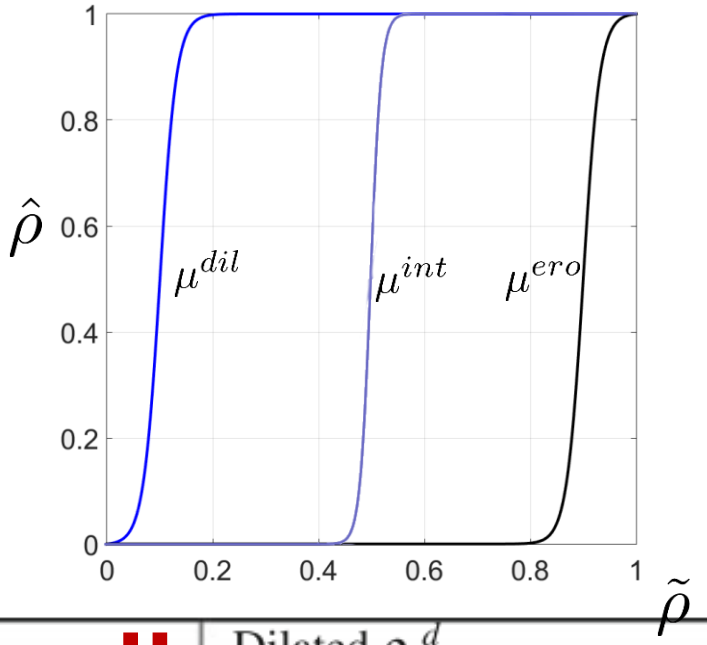


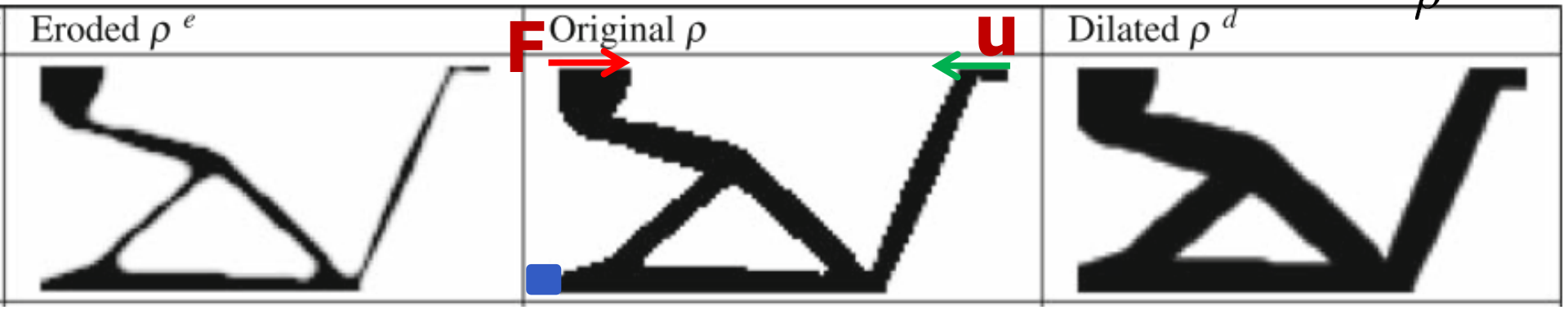
$$\hat{x}_e = \frac{\tanh(\beta\,\eta) \;+\; \tanh(\beta\,(\tilde{x}_e - \eta))}{\tanh(\beta\,\eta) + \tanh(\beta\,(1.0 - \eta)}$$

HEAVISIDE FILTER

- Heaviside function enables a control of manufacturing tolerant designs ➔ robust design

$$\hat{x}_e = \frac{\tanh(\beta\,\eta) \;+\; \tanh(\beta\,(\tilde{x}_e - \eta))}{\tanh(\beta\,\eta) + \tanh(\beta\,(1.0 - \eta)}$$

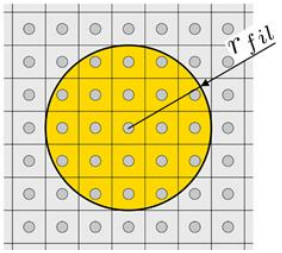




THE THREE FIELD APPROACH

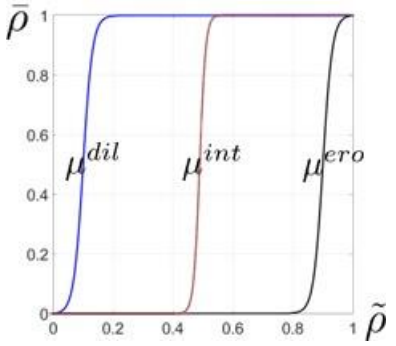
- Combining density filtering and Heaviside filter give rise to the so called three field topology optimization scheme proposed by Wang et al. (2011), one uses a design field, a filtered field and a physical field whose relations are defined though the following filter and thresholding processes
  - Filtering

$$\tilde{x}_e = \frac{\sum_{i \in N_e} w_i(X)\, v_i\, x_i}{\sum_i w_i(X_i)\, v_i}$$



  - Heaviside

$$\hat{x}_e = \frac{\tanh(\beta\,\eta) \;+\; \tanh(\beta\,(\tilde{x}_e - \eta))}{\tanh(\beta\,\eta) + \tanh(\beta\,(1.0 - \eta)}$$



SENSITIVITY ANALYSIS

SENSITIVITY ANALYSIS

- Study of the derivatives of the structure under linear static analysis when discretized by finite elements.
- The study is carried out for one load case, but it can be easily extended to multiple load cases.

- Equilibrium equation of the discretized structure:

$$\mathbf{K}\,\mathbf{q} = \mathbf{g}$$

  - **q** generalized displacement of the structure
  - **K** stiffness matrix of the structure discretized into F.E.
  - **g** generalized load vector consistent with the F.E. discretization

SENSITIVITY ANALYSIS

- Let **x** be the vector of design variables in number n.

- The differentiation of the equilibrium equation yields the sensitivity of the generalized displacements:

$$\frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q} + \mathbf{K}\frac{\partial \mathbf{q}}{\partial x_i} = \frac{\partial \mathbf{g}}{\partial x_i} \quad \Rightarrow \quad \boxed{K\frac{\partial q}{\partial x_i} = \{\frac{\partial g}{\partial x_i} - \frac{\partial K}{\partial x_i}\,q\}}$$

- The right-hand side term is called pseudo load vector

$$\tilde{g}_i = \frac{\partial g}{\partial x_i} - \frac{\partial K}{\partial x_i}\,q$$

- Physical interpretation of the pseudo load (Irons): load that is necessary to re-establish the equilibrium when perturbating the design.

SENSITIVITY ANALYSIS

- A central issue is the calculation of the derivatives of the stiffness matrix and of the load vector.

- In some cases the structure of the stiffness matrix makes it easy to have the sensitivity of the matrix with respect to the design variable

- In topology optimization using SIMP model:

$$E = x^p \, E^0$$

- The stiffness matrix

$$\mathbf{K}_e = x_e^p \, \bar{\mathbf{K}}_e$$

- And its derivatives

$$\frac{\partial \mathbf{K}_e}{\partial x_e} = p \, x_e^{p-1} \, \bar{\mathbf{K}}_e$$

SENSITIVITY OF COMPLIANCE

- The compliance is defined as the work of the applied load.

$$C = g^T q$$

- It is equal to the twice the deformation energy

$$C = g^T q = q^T K q$$

- The derivative of the compliance constraint gives:

$$\frac{dC}{dx_i} = \frac{\partial g^T}{\partial x_i} q + g^T \frac{\partial q}{\partial x_i}$$

- Introducing the value of the derivatives of the generalized displacements:

$$\frac{\partial \mathbf{q}}{\partial x_i} = \mathbf{K}^{-1} \{ \frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q} \}$$

$$\frac{dC}{dx_i} = \frac{\partial \mathbf{g}^T}{\partial x_i} \mathbf{q} + \mathbf{g}^T \mathbf{K}^{-1} \{ \frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{q} \}$$

SENSITIVITY OF COMPLIANCE

- The expression of the sensitivity of the compliance writes

$$\frac{dC}{dx_i} = \frac{\partial \mathbf{g}^T}{\partial x_i}\,\mathbf{q} + \mathbf{g}^T\,\mathbf{K}^{-1}\,\{\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q}\}$$

$$= \frac{\partial \mathbf{g}^T}{\partial x_i}\,\mathbf{q} + \mathbf{q}^T\,\{\frac{\partial \mathbf{g}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}\,\mathbf{q}\}$$

$$\frac{dC}{dx_i} = -q^T\,\frac{\partial \boldsymbol{K}}{\partial x_i}\,q + 2\,\frac{\partial g^T}{\partial x_i}\,q$$

- Generally the load vector derivative is zero (case of no body load), it comes:

$$\frac{dC}{dx_i} = -q^T\,\frac{\partial \boldsymbol{K}}{\partial x_i}\,q$$

NUMERICAL SOLUTION OF TOPOLOGY PROBLEMS USING GRADIENT BASED MATH PROGRAMMING

NUMERICAL SOLUTION OF TOPOLOGY OPTIMIZATION PROBLEMS

- Optimal material distribution = very large scale problem
  - Large number of design variables: 1 000 $\rightarrow$ 100 000
  - Number of restrictions:
    - 1 $\rightarrow$ 10 (for stiffness problems)
    - 1 000 $\rightarrow$ 10 000 (for strength problem with local constraints)

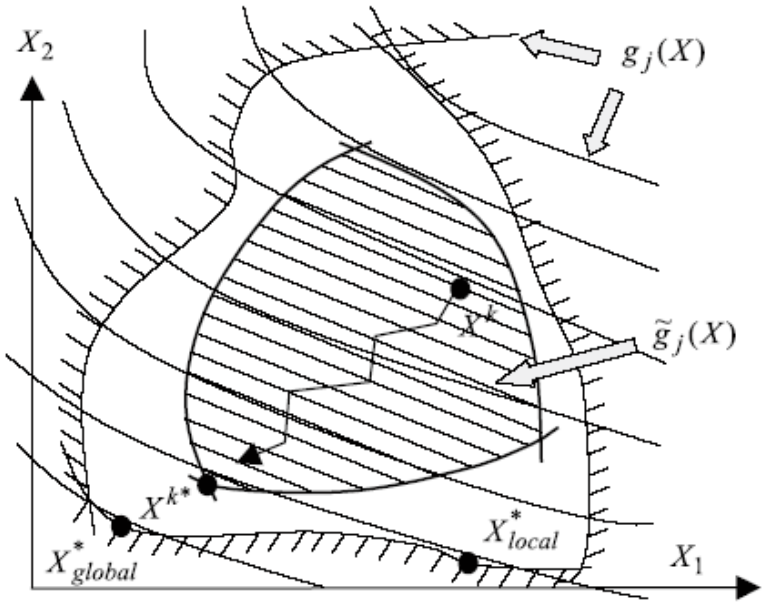
- Solution approach based on the sequential programming approach and mathematical programming
  - Sequence of convex separable problems based on structural approximations
  - Efficient solution of sub problems based on dual maximization

- Major reduction of solution time of optimization problem
- Generalization of problems that can be solved

SEQUENTIAL CONVEX PROGRAMMING APPROACH

Direct solution of the original optimisation problem which is generally non-linear, implicit in the design variables

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & g_0(\boldsymbol{x}) \\ \text{s.t.} \quad & g_j(\boldsymbol{x}) \leq \bar{g}_j \qquad j = 1 \ldots m \end{aligned}$$



is replaced by a sequence of optimisation sub-problems

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \tilde{g}_0(\boldsymbol{x}) \\ \text{s.t.} \quad & \tilde{g}_j(\boldsymbol{x}) \leq \bar{g}_j \qquad j = 1 \ldots m \end{aligned}$$

by using approximations of the responses and using powerful mathematical programming algorithms

SEQUENTIAL CONVEX PROGRAMMING APPROACH

- Two basic concepts:
  - Structural approximations replace the implicit problem by an explicit optimisation sub-problem using convex, separable, conservative approximations; e.g. CONLIN, MMA
  - Solution of the convex sub-problems: efficient solution using dual methods algorithms or SQP method.
- Advantages of SCP:
  - Optimised design reached in a reduced number of iterations: typically 100 F.E. analyses in topology optimization
  - Efficiency, robustness, generality, and flexibility, small computation time
  - Large scale problems in terms of number of design constraints and variables

Linear approximation and Sequential Linear Programming

- Linear approximation = first order Taylor expansion around x^0:

$$\tilde{g}_j(\boldsymbol{x}) \;=\; g_j(\boldsymbol{x^o}) + \sum_{i=1}^{n} \frac{\partial g(\boldsymbol{x^o})}{\partial x_i}\,(x_i - x_i^0)$$

- When linear approximation is applied to each function of the problem, one transforms the problem into a *sequence of linear programming problems (SLP)*:

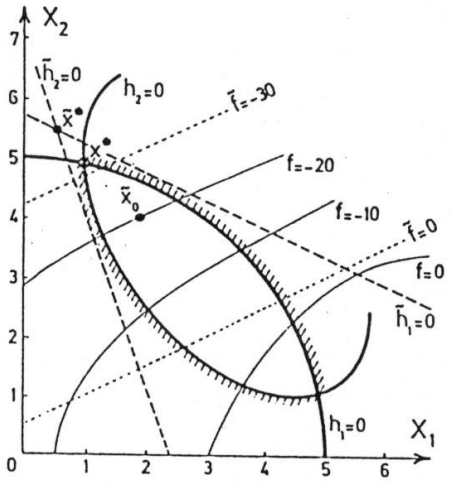
$$\begin{aligned} \min_{x_i} \quad & g_0(\boldsymbol{x^o}) + \nabla g_0(\boldsymbol{x^o})^T(\boldsymbol{x} - \boldsymbol{x^o}) \\ \text{s.t.} \quad & g_j(\boldsymbol{x^o}) + \nabla g_j(\boldsymbol{x^o})^T(\boldsymbol{x} - \boldsymbol{x^o}) \;\leq\; 0 \\ & \underline{x}_i \;\leq\; x_i \;\leq\; \bar{x}_i \end{aligned}$$

SEQUENTIAL LINEAR PROGRAMMING METHOD

- The current design point is $x^{(k)}$. Using the first order Taylor expansion of f(x), $h_j(x)$, we can get a linear approximation of the NL problem in $x^{(k)}$:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}^{(k)}) + (\boldsymbol{x} - \boldsymbol{x}^{(k)})^T \nabla f(\boldsymbol{x}^{(k)}) \\ \text{s.t.} \quad & h_j(\boldsymbol{x}^{(k)}) + (\boldsymbol{x} - \boldsymbol{x}^{(k)})^T \nabla h_j(\boldsymbol{x}^{(k)}) \leq 0 \quad j = 1 \ldots q \\ & \underline{x}_i \leq x_i \leq \overline{x}_i \quad i = 1, \ldots, n \end{aligned}$$

- Solving this LP problem, we get a new point in $x^{(k+1)}$ and start again.



PROBLEM FUNCTIONS
TRUE
$f(x) = 4x_1 - x_2^2 - 12$
$h_1(x) = 25 - x_1^2 - x_2^2$
$h_2(x) = 10x_1 - x_1^2 + 10x_2 - x_2^2 - 34$
LINEARIZED AT $\tilde{X}_o = (2,4)$
$\tilde{f}(x) = 4x_1 - 8x_2 + 4$
$\tilde{h}_1(x) = 45 - 4x_1 - 8x_2$
$\tilde{h}_2(x) = -14 + 6x_1 + 2x_2$

DOESN'T WORK!

SEQUENTIAL LINEAR PROGRAMMING METHOD

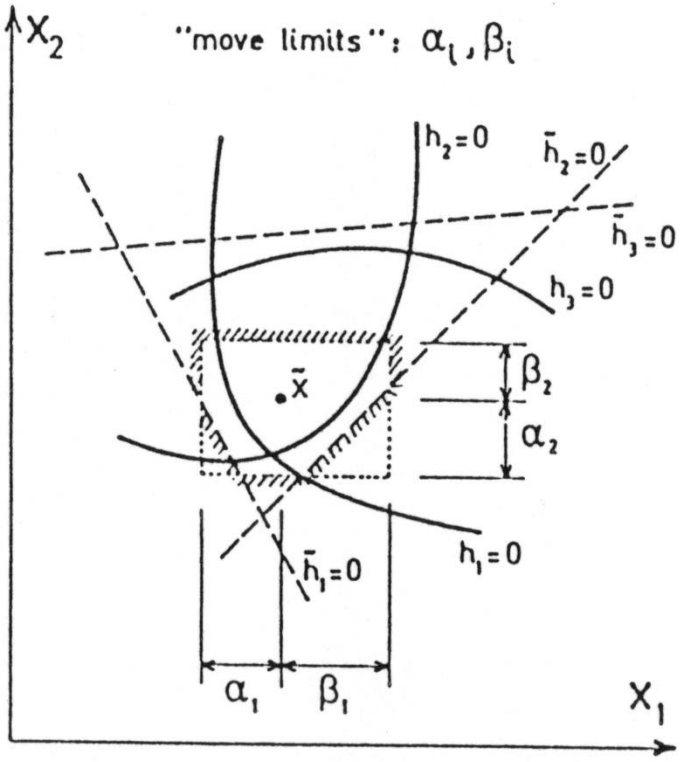
MOVE LIMIT STRATEGY

- Introduce a box constraint around the current design point to limit the variation domain of the design variables

$$\hat{x}_i - \alpha_i \leq x_i \leq \hat{x}_i + \beta_i$$

- Of course take the most restrictive constraints with the

$$\max\{\underline{x}_i, \hat{x}_i - \alpha_i\} \leq x_i \leq \min\{\overline{x}_i, \hat{x}_i + \beta_i\}$$



STRUCTURAL APPROXIMATIONS

- Convex Linearisation (CONLIN)

$$\tilde{g}_j(\boldsymbol{x}) = g_j(\boldsymbol{x^o}) + \sum_{+} \frac{\partial g_j}{\partial x_i}(x_i - x_i^0) - \sum_{-} (x_i^0)^2 \frac{\partial g_j}{\partial x_i}(\frac{1}{x_i} - \frac{1}{x_i^0})$$

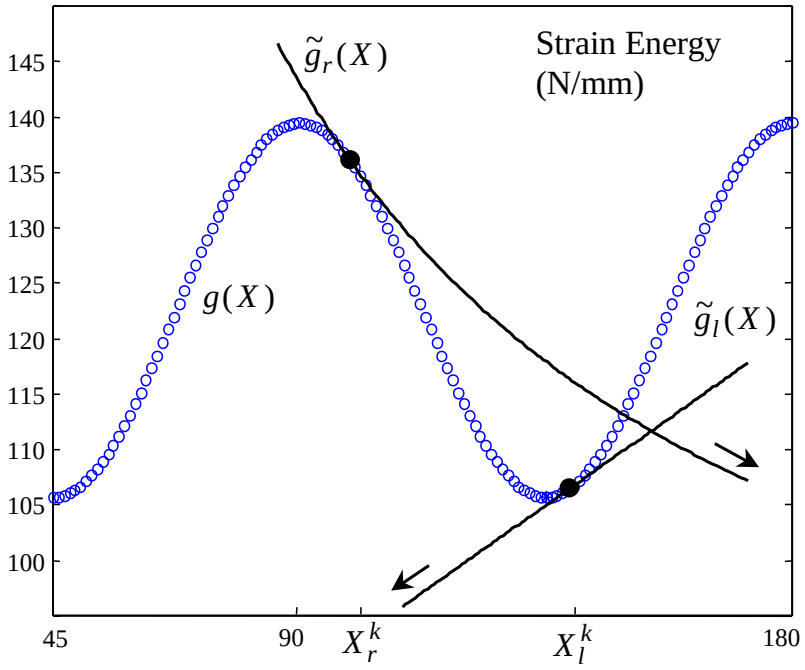
- Method of Moving Asymptotes (MMA)

$$\tilde{g}_j(\boldsymbol{x}) = r_j^0 + \sum_{i=1}^{n} \frac{p_{ij}}{U_i - x_i} + \sum_{i=1}^{n} \frac{q_{ij}}{x_i - L_i}$$

$$p_{ij} = \max\{0, (U_i - x_i^0)^2 \frac{\partial g_j}{\partial x_i}\}$$

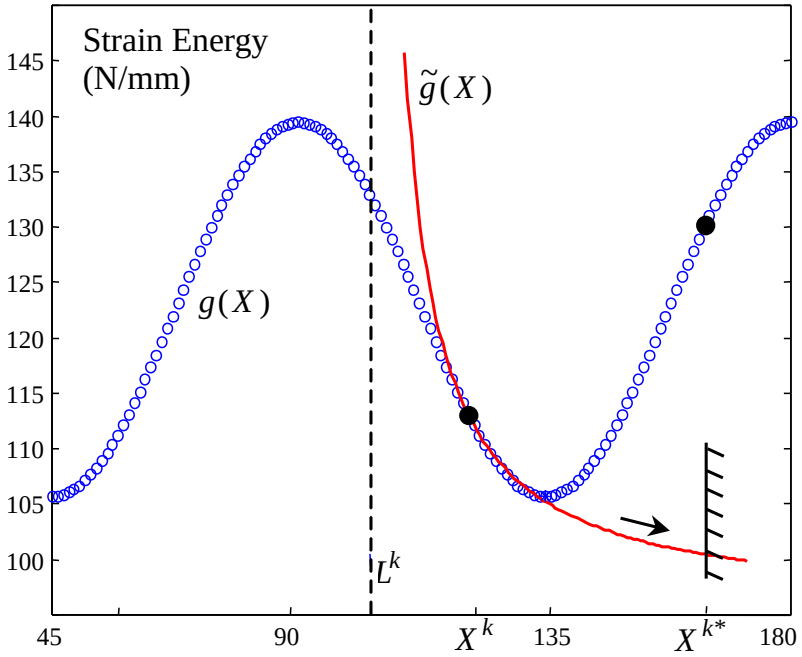
$$q_{ij} = \max\{0, -(x_i^0 - L_i)^2 \frac{\partial g_j}{\partial x_i}\}$$

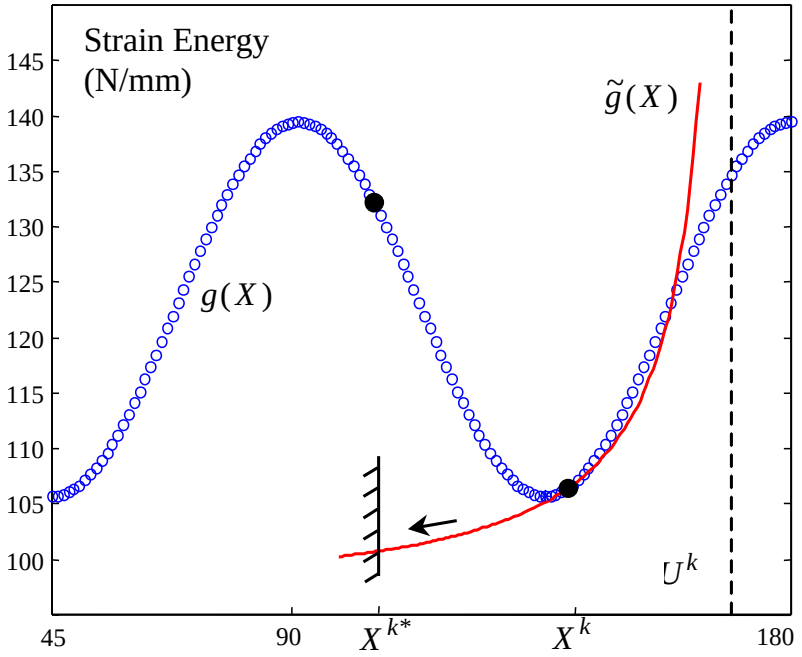
CONLIN approximation



Approximation of the strain energy in a two plies symmetric laminate subject to shear load and torsion (Bruyneel and Fleury, 2000)

MMA approximation





Approximation of the strain energy in a two plies symmetric laminate subject to shear load and torsion (Bruyneel and Fleury, 2000)

DUAL METHODS

- Primal problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g_j(\boldsymbol{x}) \leq 0 \qquad j = 1, \ldots, m \end{aligned}$$

- Lagrange function:

$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \sum_{j=1}^{m} \lambda_j g_j(\boldsymbol{x})$$

- If the problem is convex…

$$\min_{\mathbf{x}} \max_{\lambda \geq 0} L(\mathbf{x}, \lambda) \quad \Longleftrightarrow \quad \max_{\lambda \geq 0} \min_{\mathbf{x}} L(\mathbf{x}, \lambda)$$

DUAL METHODS

- Dual problem

$$\begin{array}{lll} \max\limits_{\lambda_j} & \ell(\lambda) & \\ \text{s.t.} & \lambda_j \geq 0 & j = 1, \ldots, m \end{array}$$

  - with

$$\ell(\lambda) = \min_{\boldsymbol{x} \in X} \quad L(\boldsymbol{x}, \lambda)$$

- Solve Lagrangian problem

$$\boldsymbol{x} = \boldsymbol{x}(\lambda) = \arg \min_{\boldsymbol{x}} L(\boldsymbol{x}, \lambda)$$

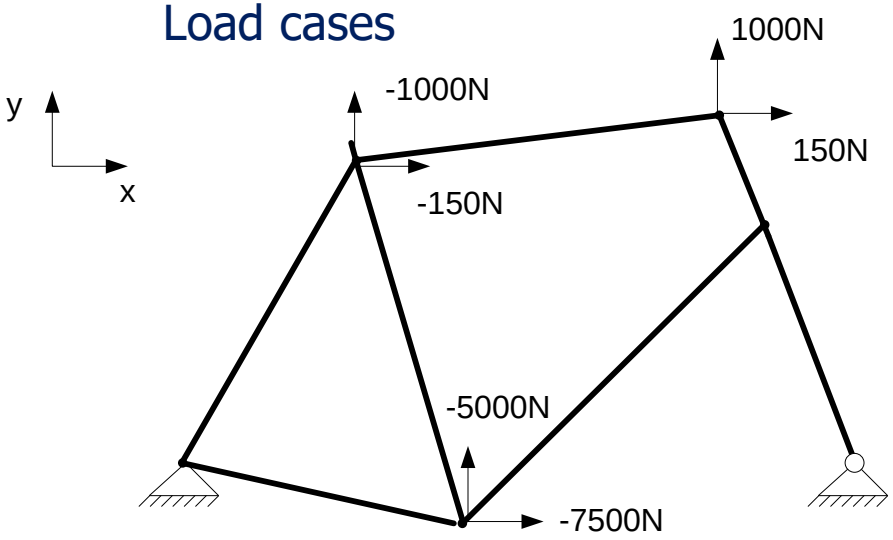
- Lagrangian problem

$$\begin{aligned} \ell(\lambda) &= L(\boldsymbol{x}(\lambda), \lambda) \\ &= f(\boldsymbol{x}(\lambda)) + \sum_{j=1}^{m} \lambda_j \, g_j(\boldsymbol{x}(\lambda)) \end{aligned}$$

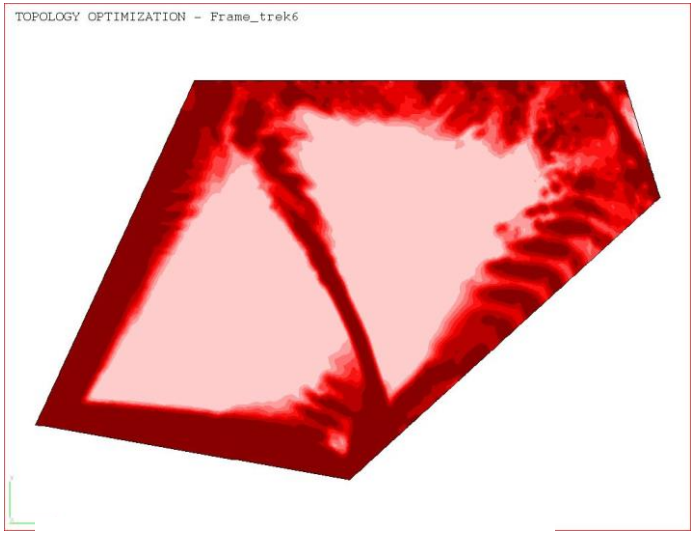
NUMERICAL APPLICATIONS OF COMPLIANCE MINIMIZATION BASED TOPOLOGY OPTIMIZATION

Optimization of a maximum stiffness bicycle frame

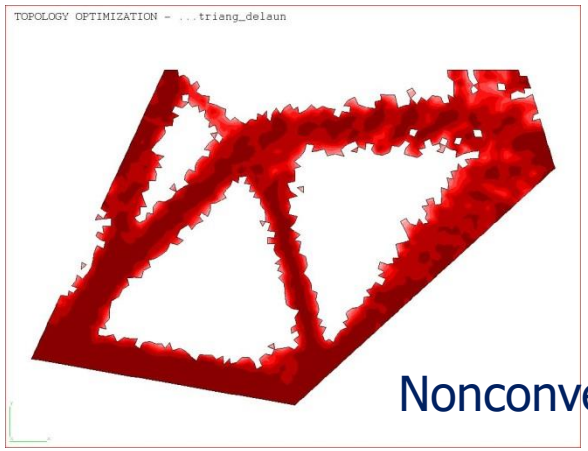
Load cases



Conventional design

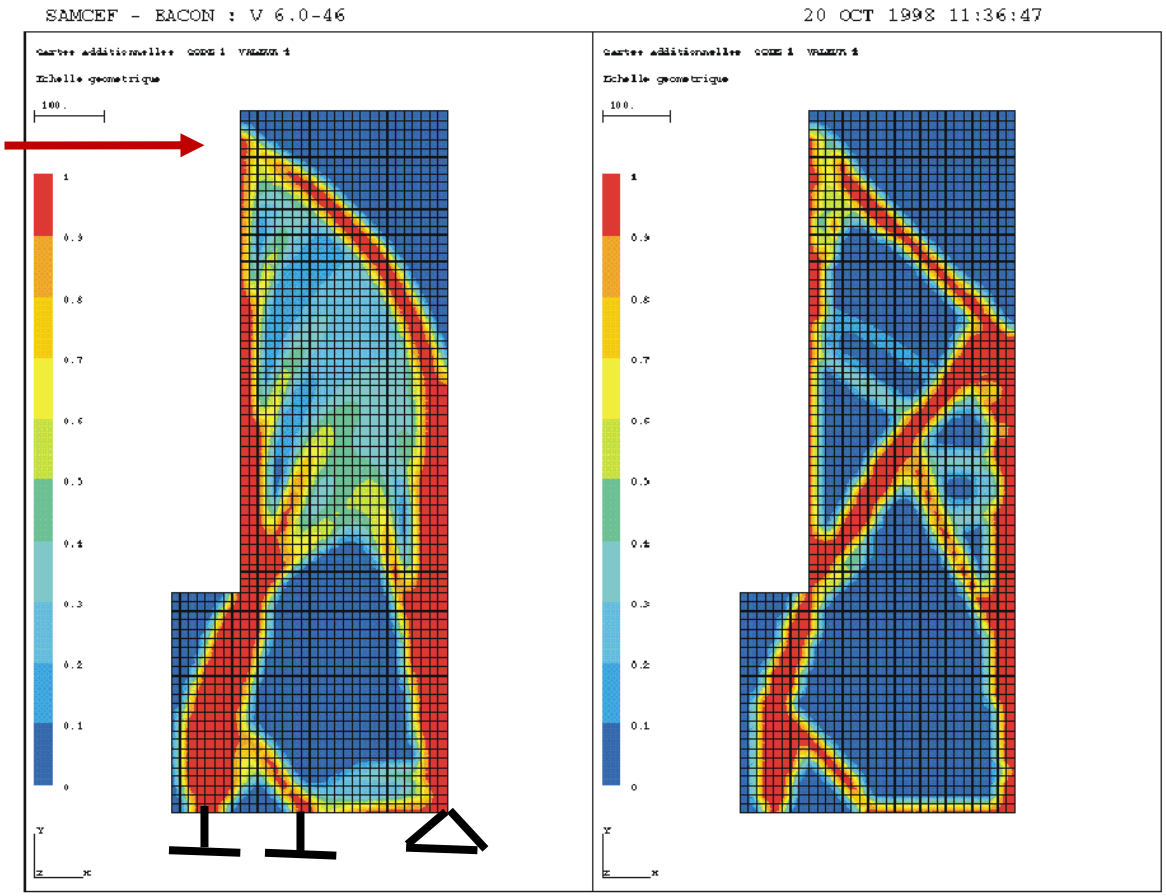


Optimum topology

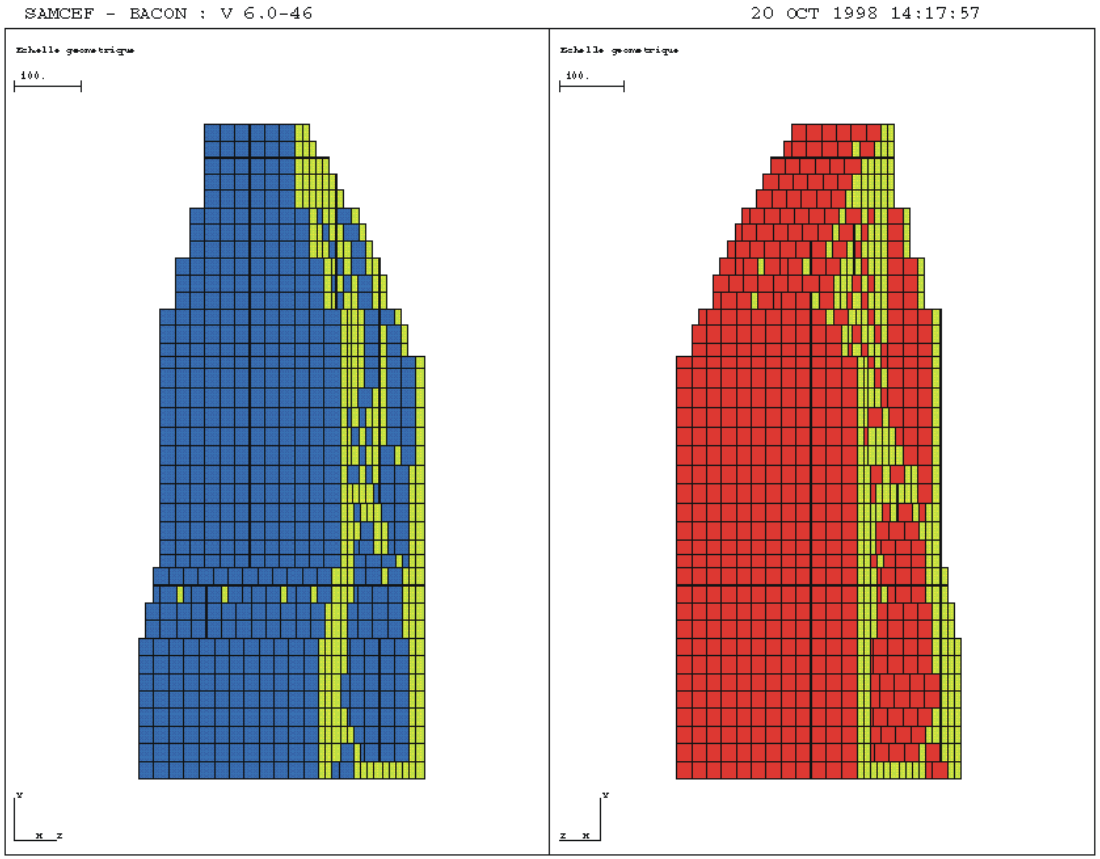


Nonconventional design

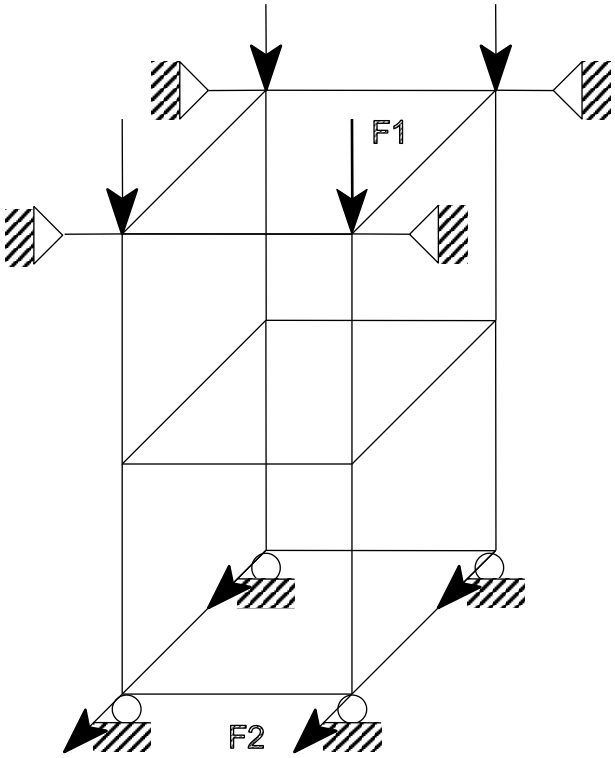
Design of a Crash Barrier Pillar (SOLLAC)

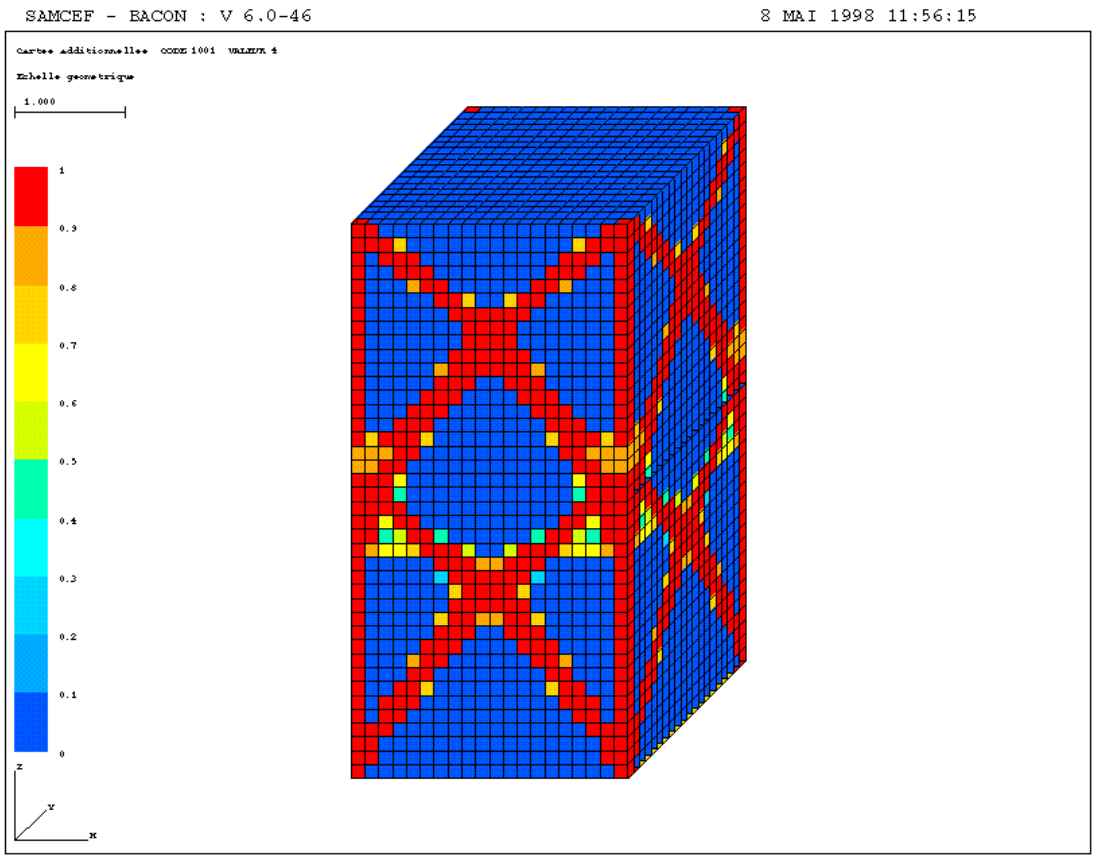


Design of a Crash Barrier Pillar (SOLLAC)

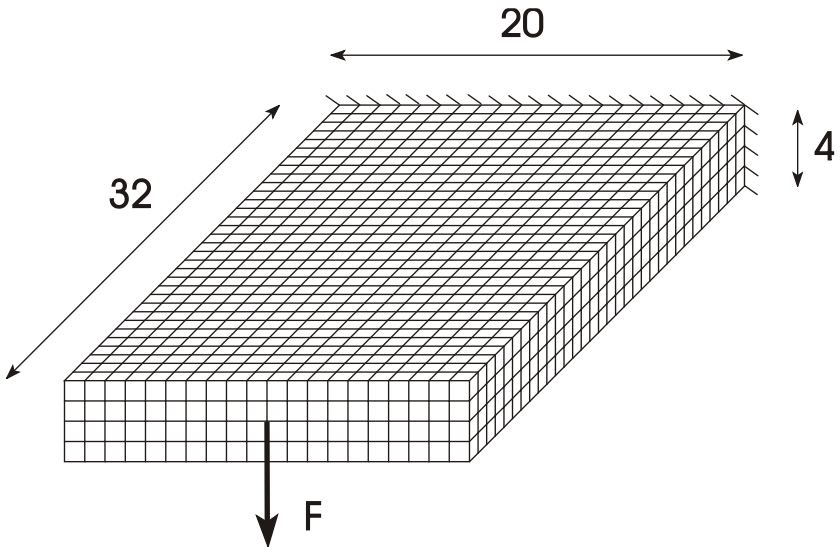


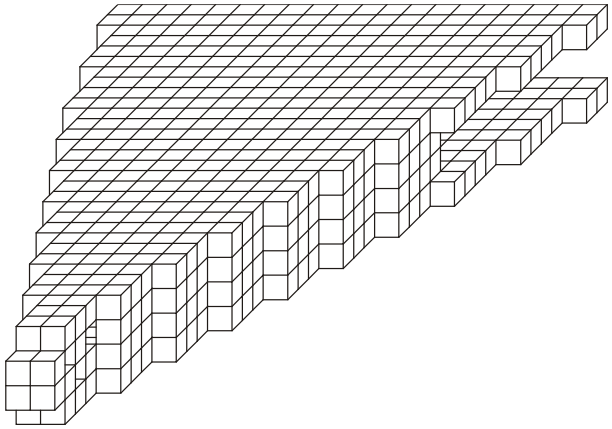
Topology Optimization of a Parasismic Building (DOMECO)





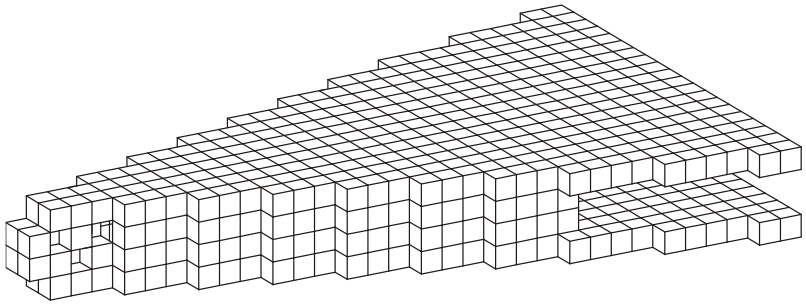
3D cantilever beam problem





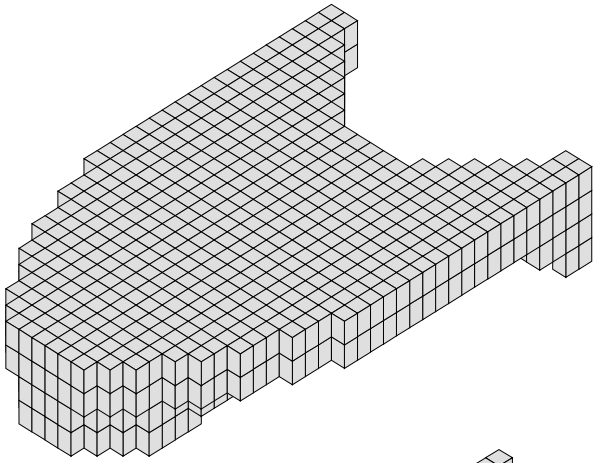
E=100 N/m², ν=0.3

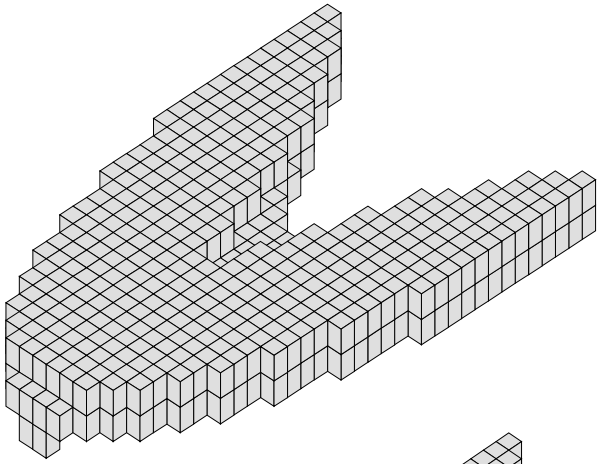
20 x 32 x 4 = 2560 F.E.s

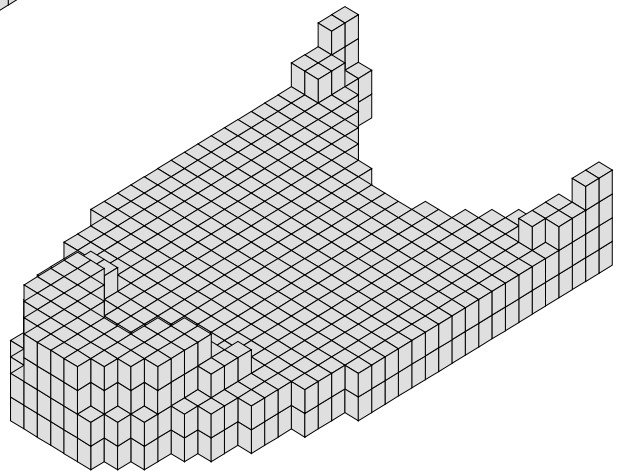


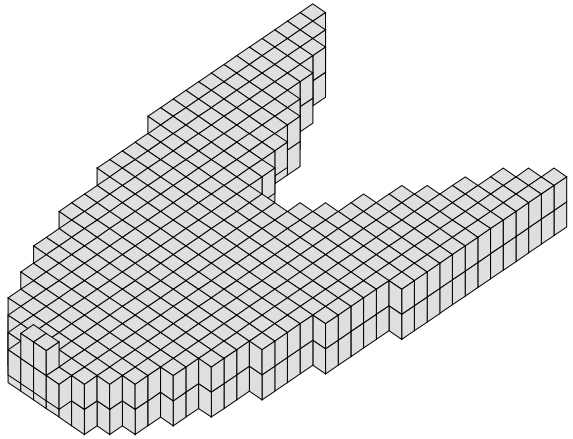
No perimeter constraint

3D cantilever beam problem









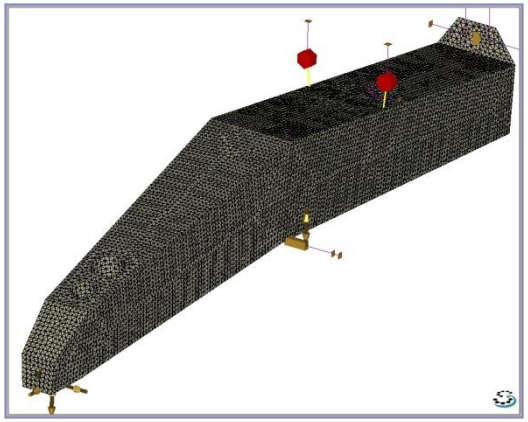
Perimeter = 1400

Perimeter = 1000

An industrial application: Airbus engine pylon

- Application
  - carried out by SAMTECH and ordered by AIRBUS
- Engine pylon
  = structure fixing engines to the wing

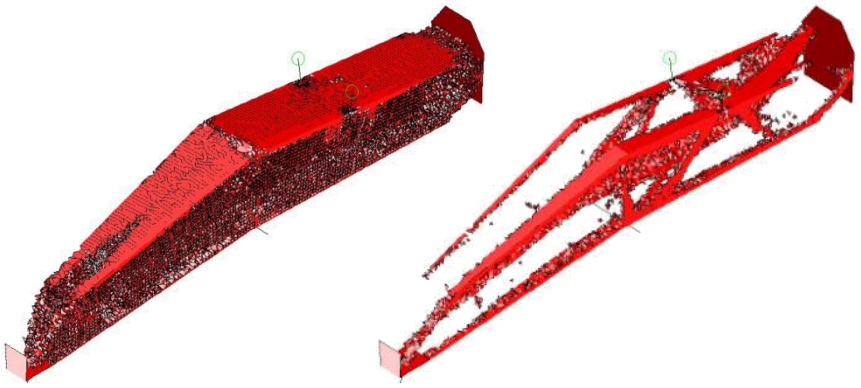
- Initial Model
  - CATIA V5 import → Samcef Model
  - BC's: through shell and beam FE
  - 10 load cases:
    - GUSTS
    - FBO (Fan blade out)
    - WUL (Without undercarriage landing)



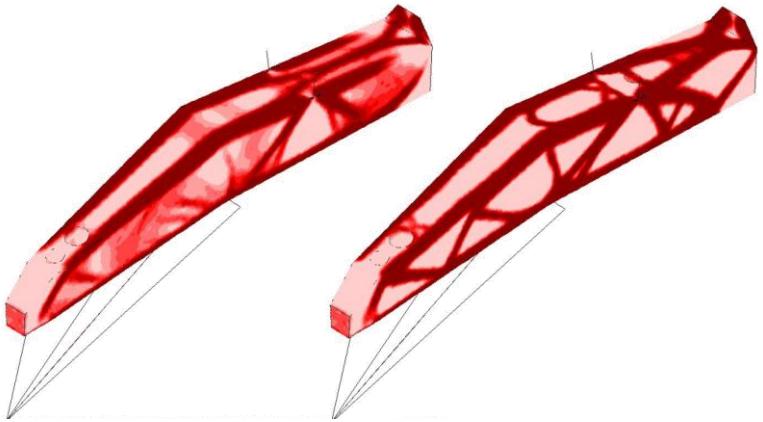
Over 250.000 tetraedral FE

An industrial application: Airbus engine pylon

- Target mass: 10%
- Additional constraints:
  - Engine CoG position
- Optimization parameters
  - Sensitivity filtering: (Sigmund's filter)
  - Symmetry (left right) condition
  - Penalty factor
- CONLIN optimizer: special version for topology optimization

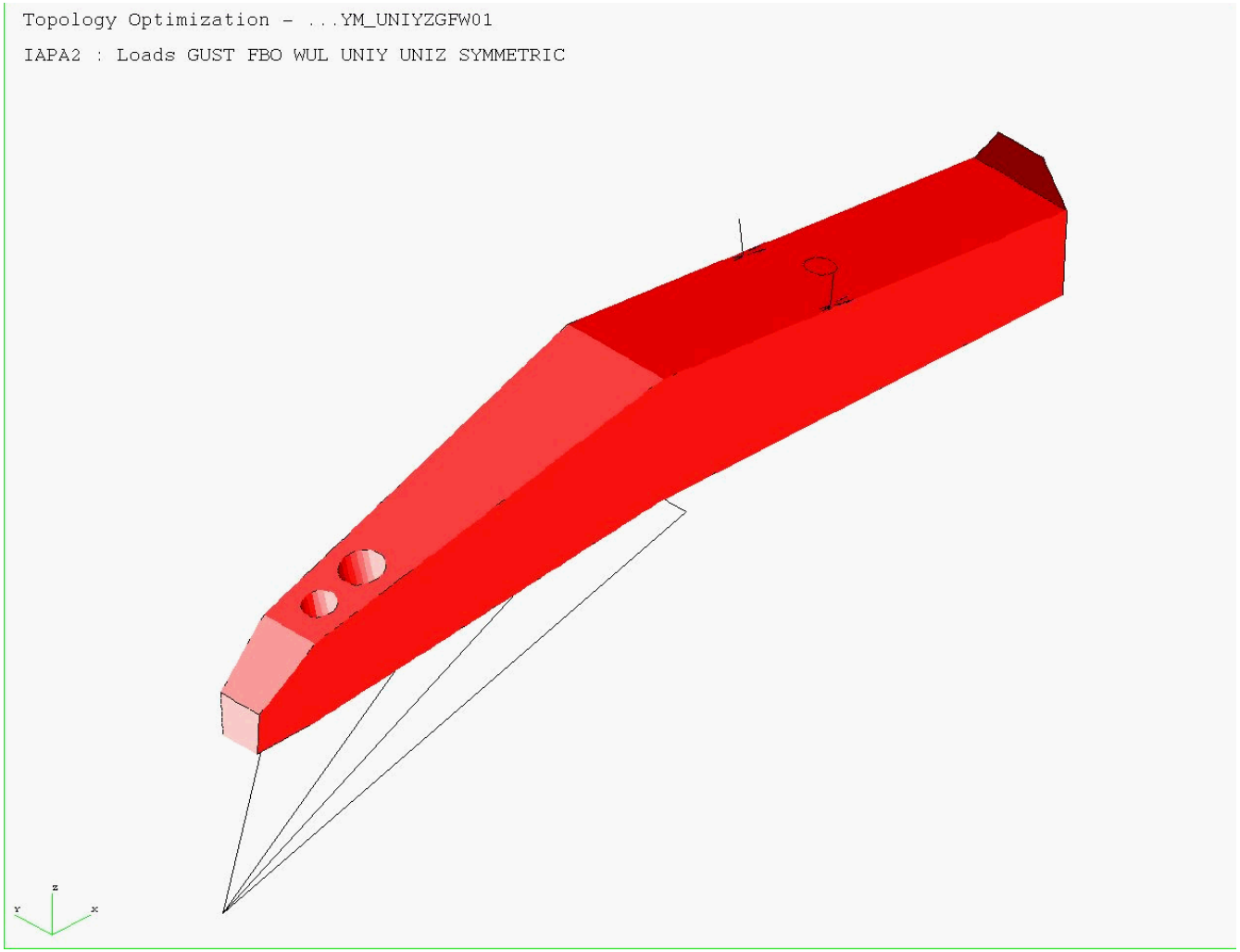


Sensitivities filtering



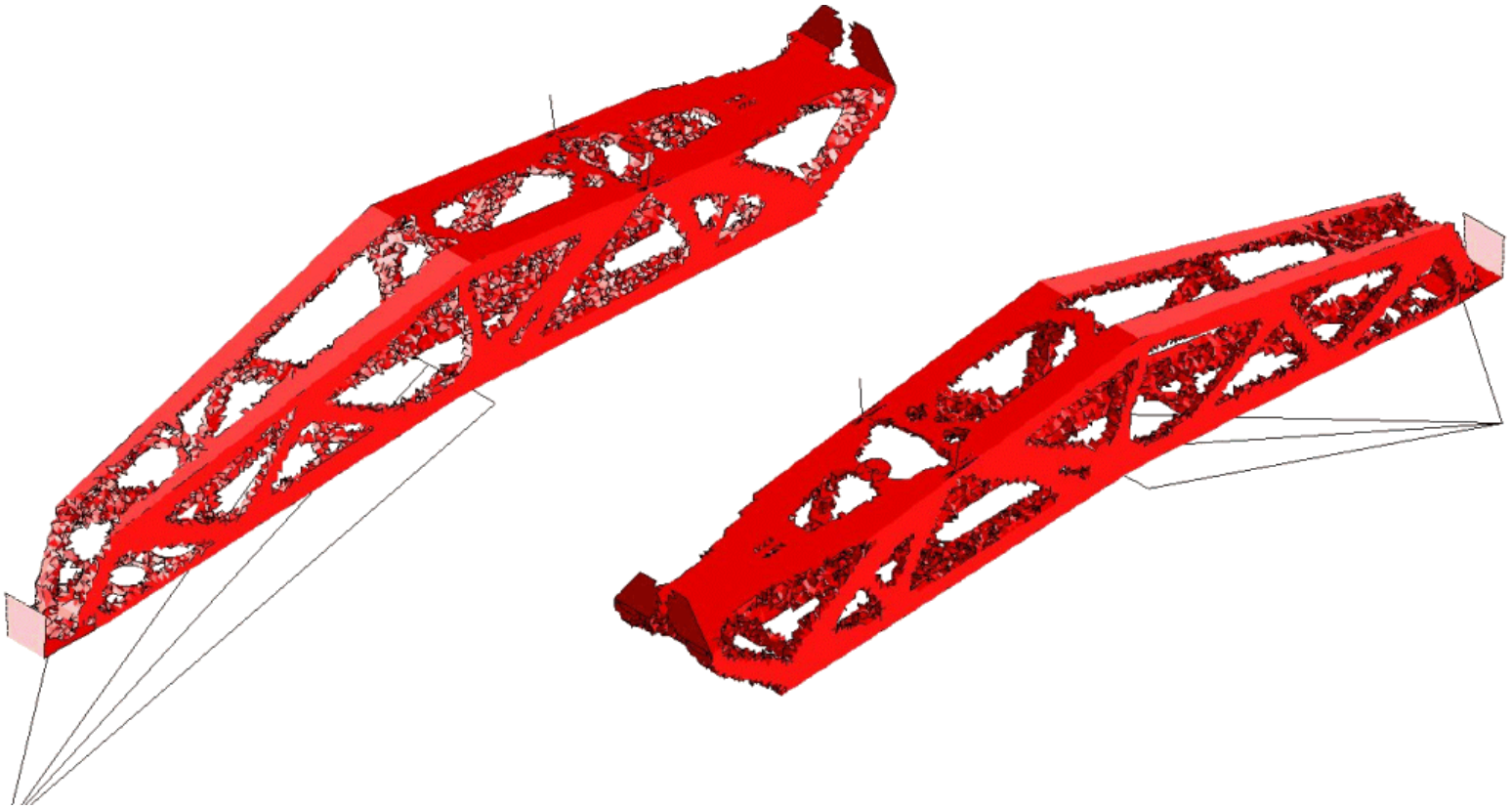
Penalty factor from 2 to 4

Airbus engine pylon



With courtesy by Samtech and Airbus Industries

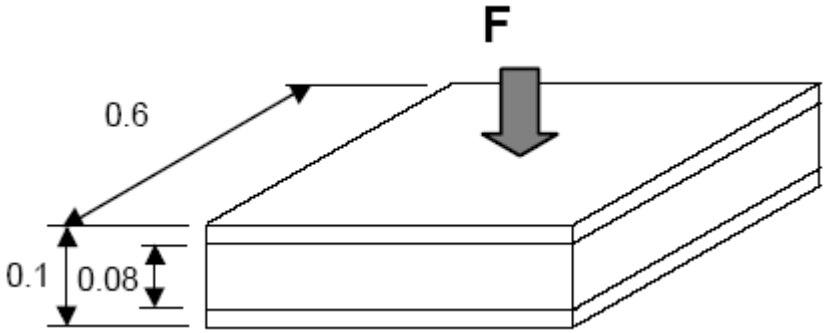
Airbus engine pylon



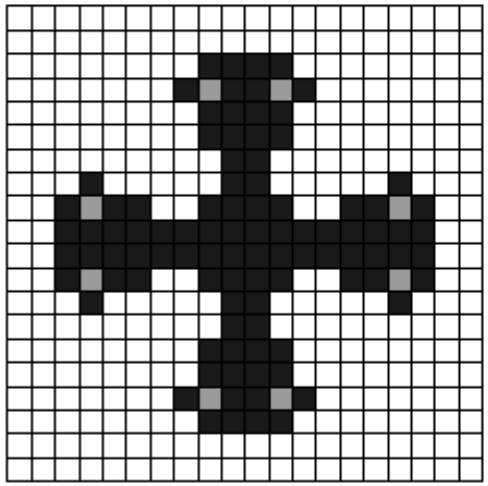
With courtesy by Samtech and Airbus Industries

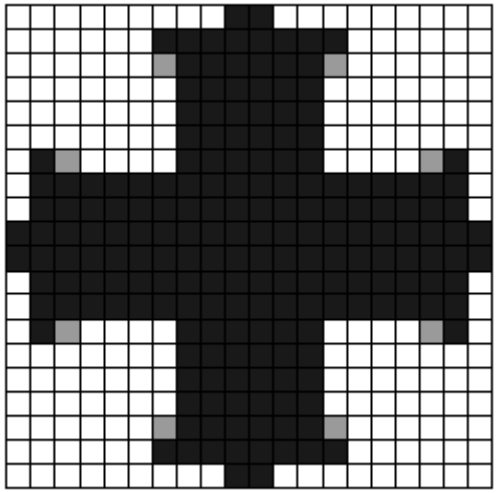
Sandwich panel optimization

- Geometry of the sandwich panel reinforcement problem



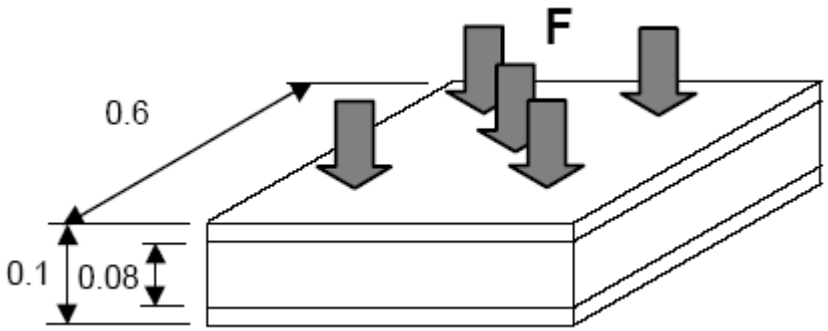
- Optimal topolog



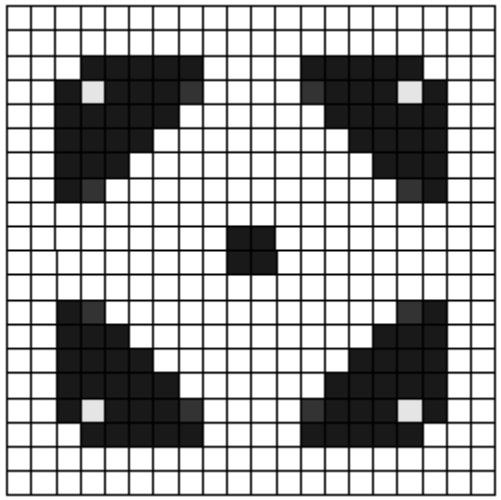


Sandwich panel optimization

- Geometry of the sandwich panel reinforcement problem



- Optimal topology



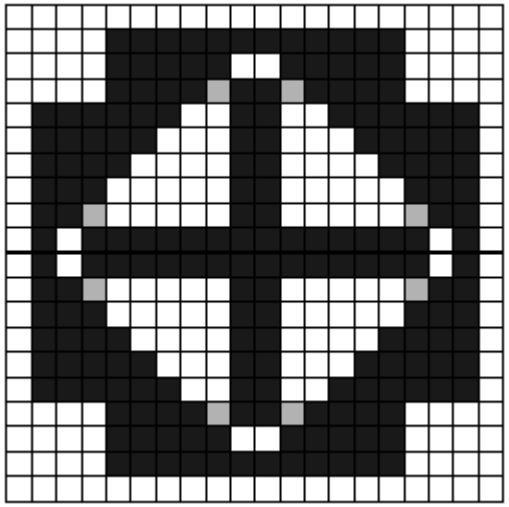
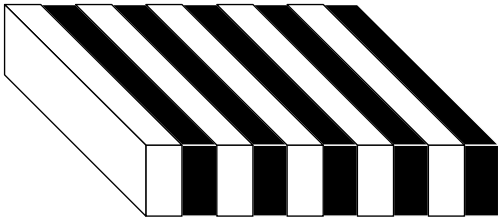
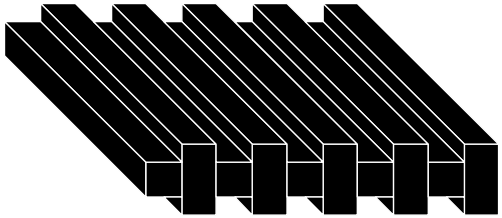


PLATE AND SANDWICH PLATE MODELS

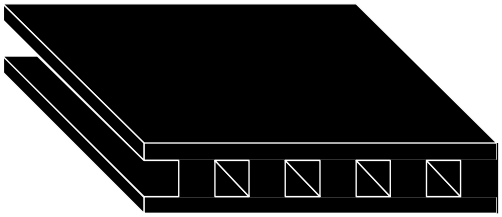


(a)



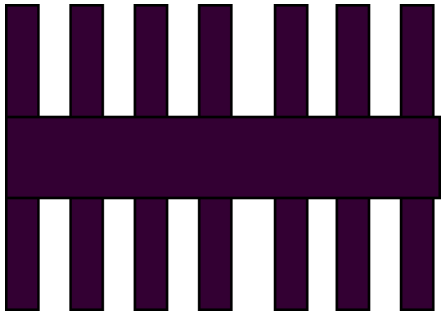
(b)

(c)

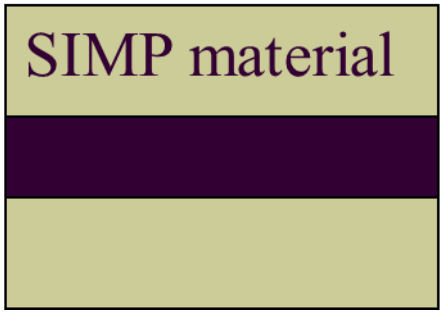


(d)

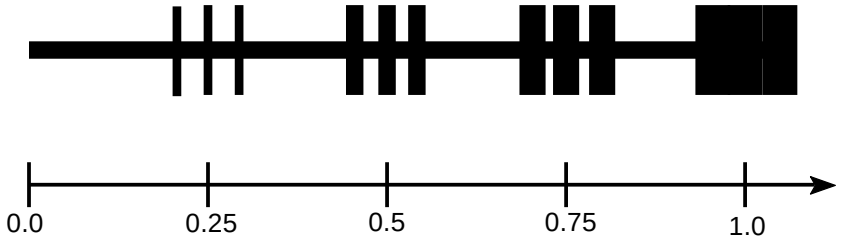
USING SIMP MODEL FOR TOPOLOGY OPTIMISATION OF PLATES AND SHELLS



is replaced by:

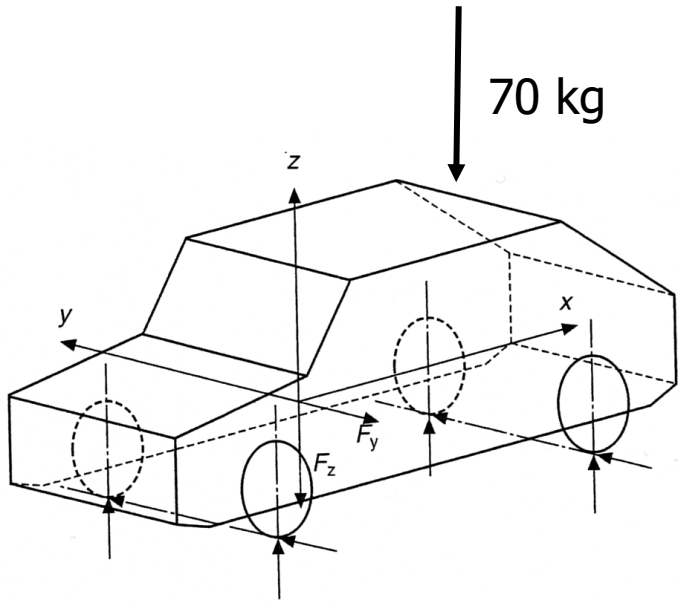


PHYSICAL MEANING OF DENSITY VARIABLE:



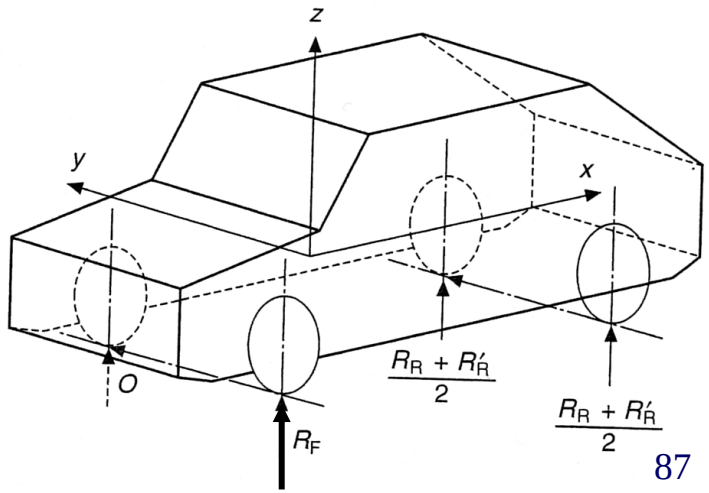
PROTOTYPE CAR BODY OPTIMIZATION

- Load case 1: bending
  - Self weight
  - Components (20 kg)
  - Pilot (50 kg)
  - Roll over load (70 kg on top of roll cage)



(Figures from Happian-Smith)

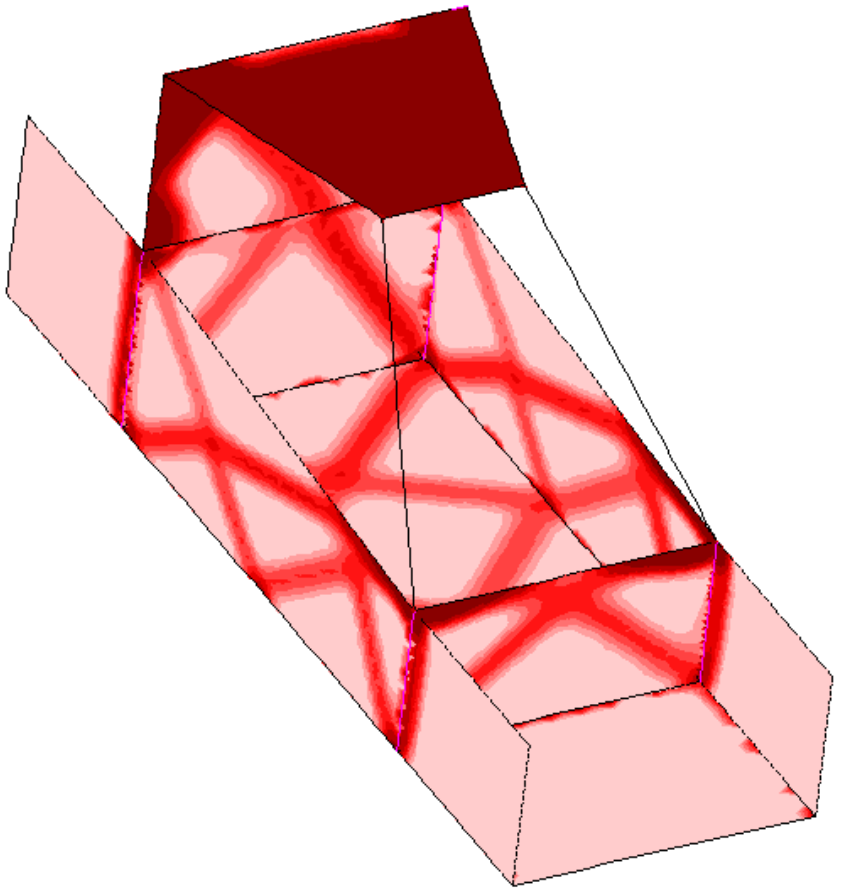
- Load case 2: torsion + bending = curb impact
  - Rear axle clamped
  - Right front wheel free supported
  - Left front wheel withstanding 3 times the weight of the axle



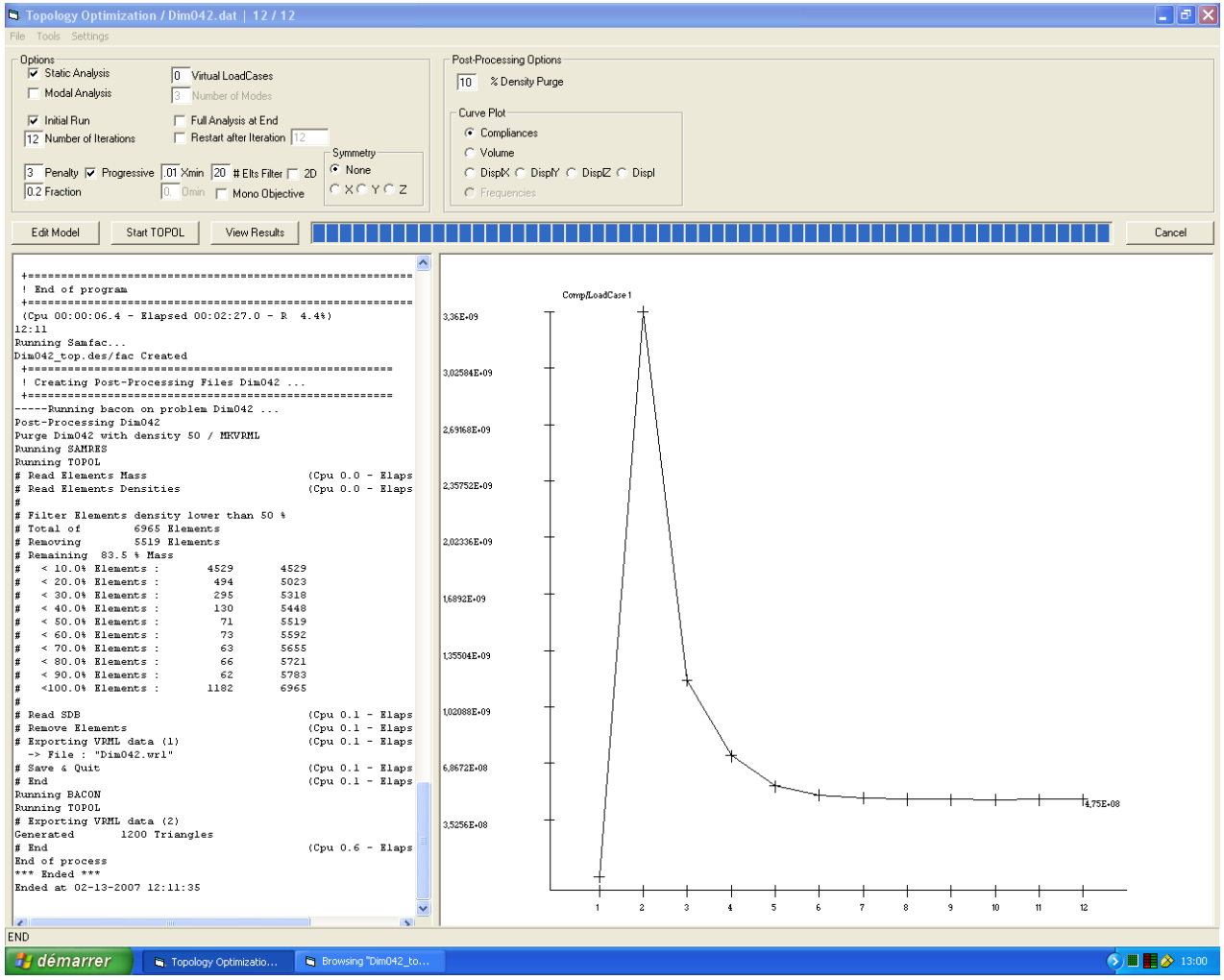
weight x 3= 2700 N

DESIGN OF A URBAN CONCEPT STRUCTURE

- Topology optimization of the truss structure
  - Target mass of 15 kg
  - Minimum compliance
  - Mostly determined by load case 2 (torsion)
  - SIMP material with p=3
  - Left / right symmetry of material distribution
  - Filtering

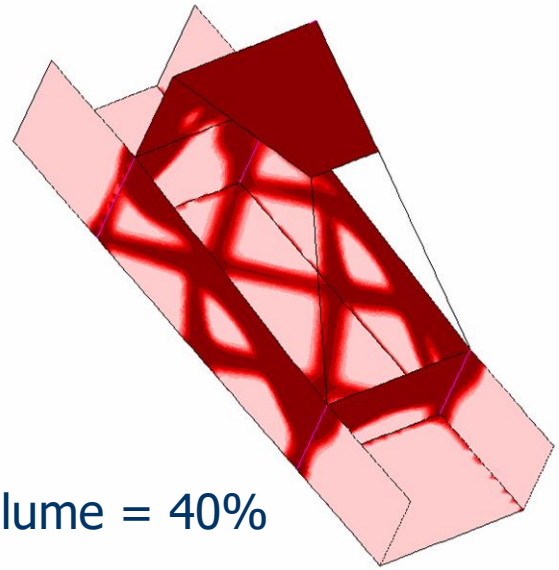


DESIGN OF AN URBAN CONCEPT STRUCTURE



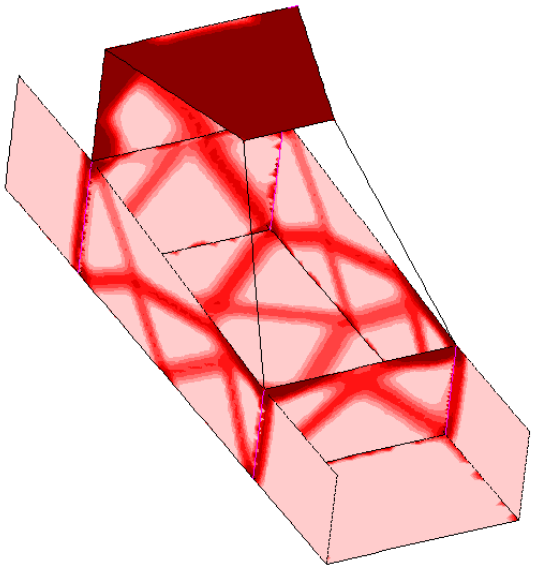
Convergence history

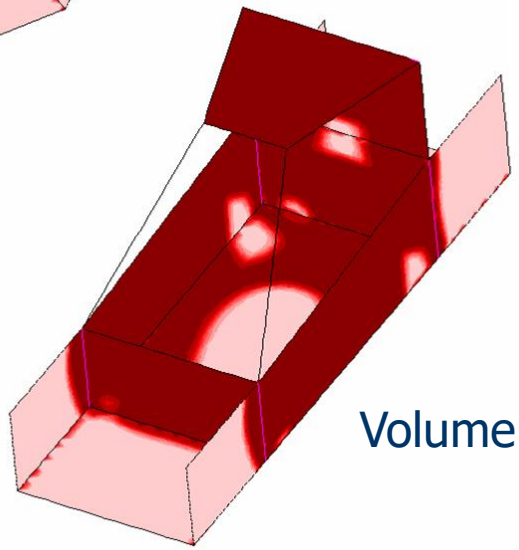
DESIGN OF AN URBAN CONCEPT STRUCTURE



Volume = 40%

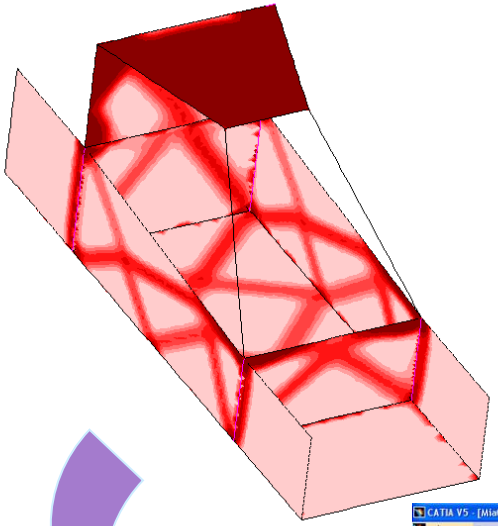
Volume = 20%

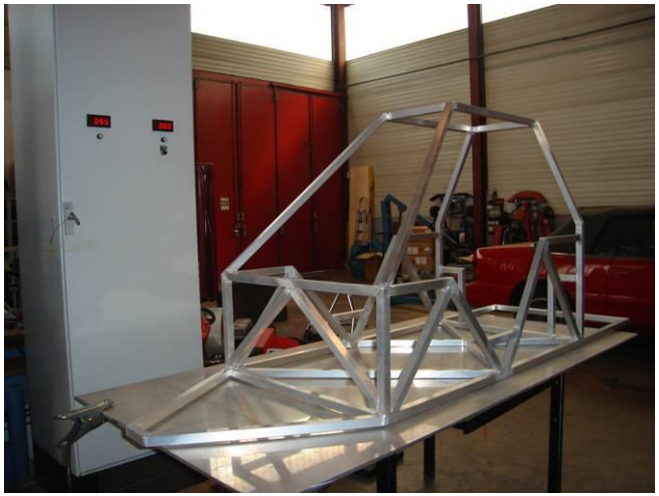


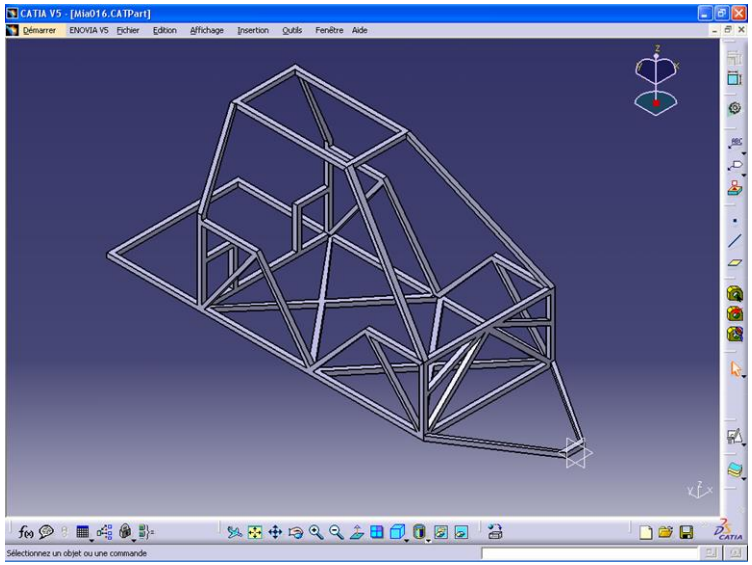


Volume = 60%

DESIGN OF AN URBAN CONCEPT STRUCTURE







VIBRATION PROBLEMS

NATURAL VIBRATION PROBLEMS

- Finite element discretization of the system

$$\mathbf{u}(X) = \mathbf{N}(X)\,\mathbf{q}$$
$$\epsilon(X) = \partial\mathbf{N}(X)\,\mathbf{q} = \mathbf{B}\mathbf{q}$$
$$\sigma = \mathbf{H}\,\epsilon = \mathbf{H}\mathbf{B}\,\mathbf{q} = \mathbf{T}\,\mathbf{q}$$

- Kinetic energy

$$\mathcal{K} = \frac{1}{2}\int_\Omega \rho\,\dot{\mathbf{u}}^T\dot{\mathbf{u}}d\Omega = \frac{1}{2}\dot{\mathbf{q}}^T\,\mathbf{M}\,\dot{\mathbf{q}} \qquad \mathbf{M} = \int_\Omega \mathbf{N}^T\mathbf{N}\rho\,d\Omega$$

- Strain energy

$$\mathcal{U} = \frac{1}{2}\int_\Omega \sigma^T\epsilon\,d\Omega = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} \qquad \mathbf{K} = \int_\Omega \mathbf{B}^T\mathbf{H}\,\mathbf{B}\,d\Omega$$

NATURAL VIBRATION PROBLEMS

- Hamilton principle

$$\delta \int_{t_1}^{t_2} \mathcal{K} - \mathcal{U}\, dt = 0$$

- Dynamic equation of the system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{g}(t)$$

- Free vibrations : assume periodic solutions

$$\mathbf{q}(t) = \mathbf{q}\, e^{i\omega t}$$

- Nontrivial solutions are solutions of the eigenvalue problem

$$-\omega^2\, \mathbf{M}\, \mathbf{q} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

NATURAL VIBRATION PROBLEMS

- Eigenvalue problem

$$\left(\mathbf{K} - \omega^2\,\mathbf{M}\right)\,\mathbf{q} = \mathbf{0}$$

- Rayleigh ratio

$$\omega^2_{min} = \min_{\mathbf{q}}\ \frac{\mathbf{q}^T\mathbf{K}\mathbf{q}}{\mathbf{q}^T\mathbf{M}\mathbf{q}}$$

DESIGN PROBLEM FORMULATION

- Fundamental topology optimization problem of vibrating structures

$$\begin{aligned} \min_{\mathbf{x}} \quad & \omega_{min}^2 \\ \text{s.t.} \quad & V = \int_\Omega \rho \; d\Omega \leq \bar{V} \\ & (\mathbf{K} - \omega_{min}^2 \mathbf{M})\mathbf{q} = \mathbf{0} \end{aligned}$$

- To avoid mode crossing, it is better to select several eigenvalues and to maximize the minimum of the first NF frequencies

$$\begin{aligned} \max_{\mathbf{x}} \quad & \min_{k=1\ldots NF} \omega_k^2 \\ \text{s.t.} \quad & V = \int_\Omega \rho \; d\Omega \leq \bar{V} \\ & (\mathbf{K} - \omega_k^2 \mathbf{M})\mathbf{q}_k = \mathbf{0} \end{aligned}$$

DESIGN PROBLEM FORMULATION

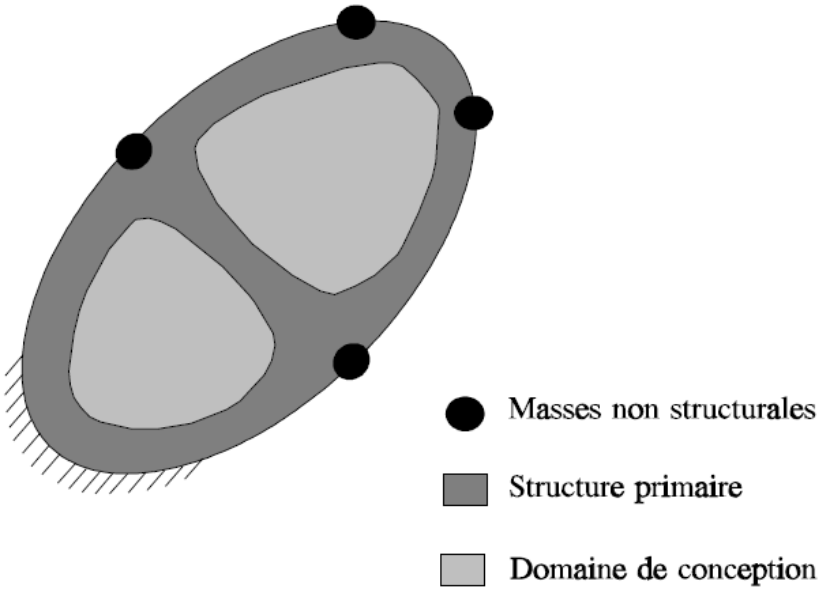
- However, the fact that both mass and stiffness depends on the density design variables, the trivial solution 0=0 is feasible.
- Therefore, best eigenfrequency design problem is formulated as a reinforcement problem, i.e. there exist some non design mass or stiffness

$$\min_{\mathbf{x}} \quad \omega^2_{min}$$

$$\text{s.t.} \quad V = \int_\Omega \rho \, d\Omega \leq \bar{V}$$

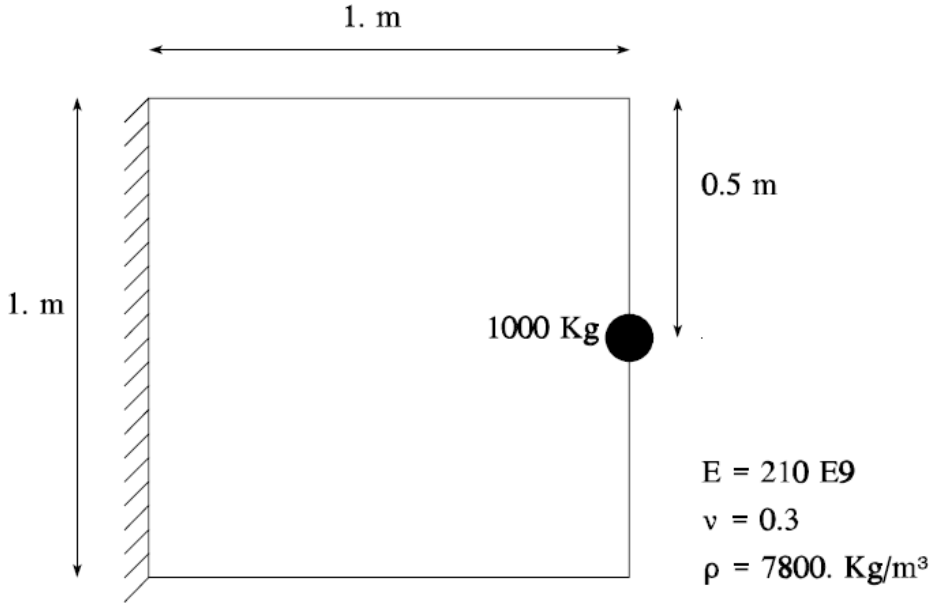
$$([\mathbf{K} + \mathbf{K}_0] - \omega^2_{min}[\mathbf{M} + \mathbf{M}_0])\mathbf{q} = \mathbf{0}$$

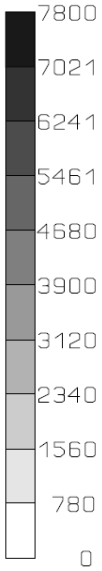
$$\omega^2_{min} = \min_{\mathbf{q}} \frac{\mathbf{q}^T[\mathbf{K} + \mathbf{K}_0]\mathbf{q}}{\mathbf{q}^T[\mathbf{M} + \mathbf{M}_0]\mathbf{q}}$$

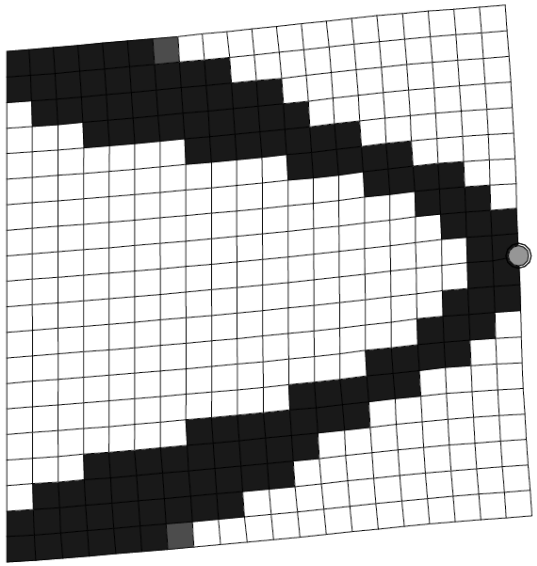


REINFORCEMENT DESIGN PROBLEM

- Topology optimization of sheet of steel:
  - Basic sheet: t=1mm
  - Reinforcement sheet t=1mm







REINFORCEMENT DESIGN PROBLEM

- Topology optimization of sheet of steel:
  - Basic sheet: t=1mm
  - Reinforcement sheet t=1mm

Mode propre	Fréquences propres avant optimisation (Hz)	Fréquences propres après optimisation (Hz)
1	26,89	52,11
2	43,89	88,26
3	540,03	820,98
4	817,92	1394,22
5	860,40	1583,39

SENSITIVITY OF EIGENVALUE PROBLEMS

- Eigenvalue problem
  - **K** stiffness matrix, **M** mass matrix
  - **q** the eigenmode vector
  - And ω the eigenfrequency

$$\boxed{\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{q} = \mathbf{0}}$$

- The magnitude of the modes is arbitrary, so they are normalized according to a given matrix W (generally the mass matrix M)

$$\boxed{\mathbf{q}^T \mathbf{W} \mathbf{q} = 1}$$

- At first let's consider the simplified approach: we assume that all eigenvalues are distinct and ordered from the smallest to the largest:

$$0 < \omega^{(1)} < \omega^{(2)} < \ldots < \omega^{(N)}$$

$$0\,, \quad \mathbf{q}^{(1)}\,, \quad \mathbf{q}^{(2)}\,, \quad \ldots \quad \mathbf{q}^{(N)}$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- Let's differentiate the eigenvalue equation

$$\left(\mathbf{K}-\omega_{(k)}^2\mathbf{M}\right)\frac{\partial \mathbf{q}^{(k)}}{\partial x_i} = \frac{\partial \omega_{(k)}^2}{\partial x_i}\mathbf{M}\mathbf{q}^{(k)} - \left(\frac{\partial \mathbf{K}}{\partial x_i}-\omega_{(k)}^2\frac{\partial \mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)}$$

- Differentiating the normalization equation gives

$$\mathbf{q}^{(k)\,T}\mathbf{W}\frac{\partial \mathbf{q}^{(k)}}{\partial x_i} = -\frac{1}{2}\mathbf{q}^{(k)T}\frac{\partial \mathbf{W}}{\partial x_i}\mathbf{q}^{(k)}$$

- To obtain the derivatives of the eigenvalue $\lambda^{(k)}$, one has to premultiply the first equation by the eigenmode $q^{(k)}$

$$\mathbf{q}^{(k)\,T}\left[\left(\mathbf{K}-\omega_{(k)}^2\mathbf{M}\right)\frac{\partial \mathbf{q}^{(k)}}{\partial x_i} - \frac{\partial \omega_{(k)}^2}{\partial x_i}\mathbf{M}\mathbf{q}^{(k)} + \left(\frac{\partial \mathbf{K}}{\partial x_i}-\omega_{(k)}^2\frac{\partial \mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)} = \mathbf{0}\right]$$

SENSITIVITY OF EIGENVALUE PROBLEMS

- Since $q^{(k)}$ is an eigenmode

$$\mathbf{q}^{(k)T}\left(\mathbf{K}-\omega_{(k)}^2\mathbf{M}\right)=\mathbf{0}$$

- And one gets

$$-\left(\mathbf{q}^{(k)T}\mathbf{M}\mathbf{q}^{(k)}\right)\frac{\partial\omega_{(k)}^2}{\partial x_i}+\mathbf{q}^{(k)\,T}\left(\frac{\partial\mathbf{K}}{\partial x_i}-\omega_{(k)}^2\frac{\partial\mathbf{M}}{\partial x_i}\right)\mathbf{q}^{(k)}=\mathbf{0}$$

- With the scaling factor

$$m^{(k)}=\mathbf{q}^{(k)T}\mathbf{M}\mathbf{q}^{(k)}$$

- We finally obtain the final expression of the sensitivity of the eigen values:

$$\boxed{\frac{\partial\omega_{(k)}^2}{\partial x_i}=\frac{1}{m^{(k)}}\left(\mathbf{q}^{(k)T}\frac{\partial\mathbf{K}}{\partial x_i}\mathbf{q}^{(k)}-\omega_{(k)}^2\;\mathbf{q}^{(k)T}\frac{\partial\mathbf{M}}{\partial x_i}\mathbf{q}^{(k)}\right)}$$

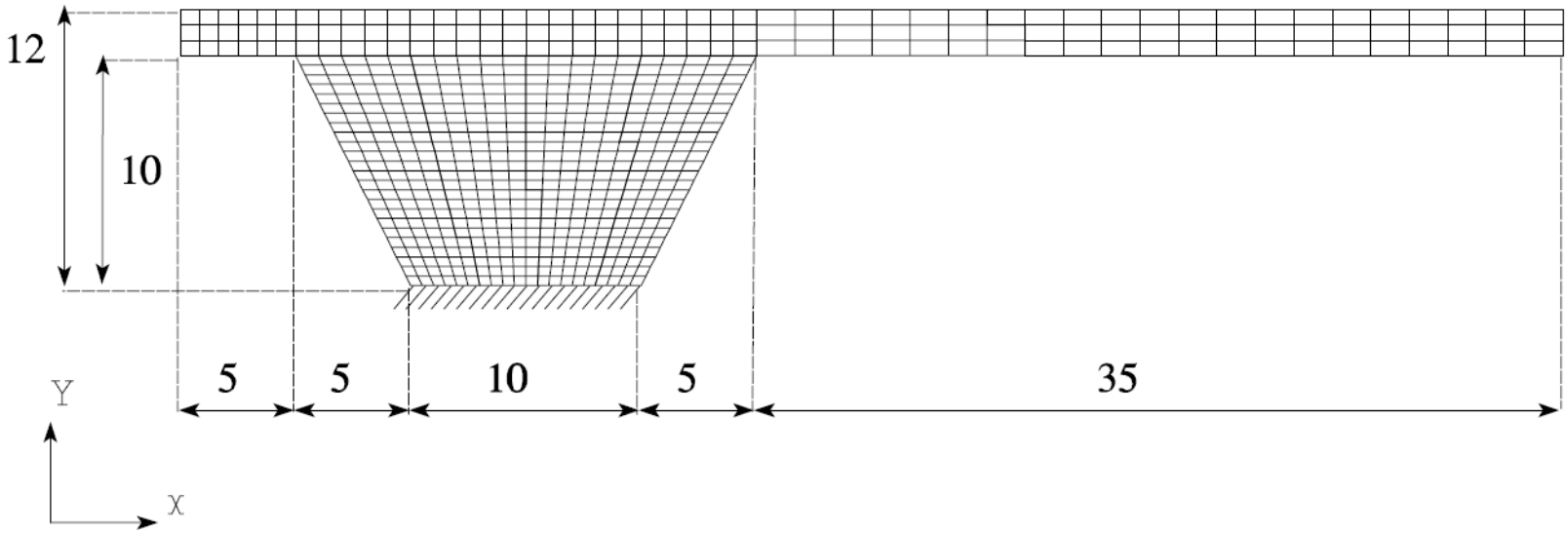
DIFFICULTIES IN VIBRATION PROBLEMS

- Topology optimization of vibrating structures presents a major difficulty:
  - Appearance of dummy eigenmodes, i.e. local modes with zero frequency

- Require special strategy
  - Modification of material interpolation ➔ modified SIMP or RAMP
  - Filtering the dummy modes

DIFFICULTIES IN VIBRATION PROBLEMS

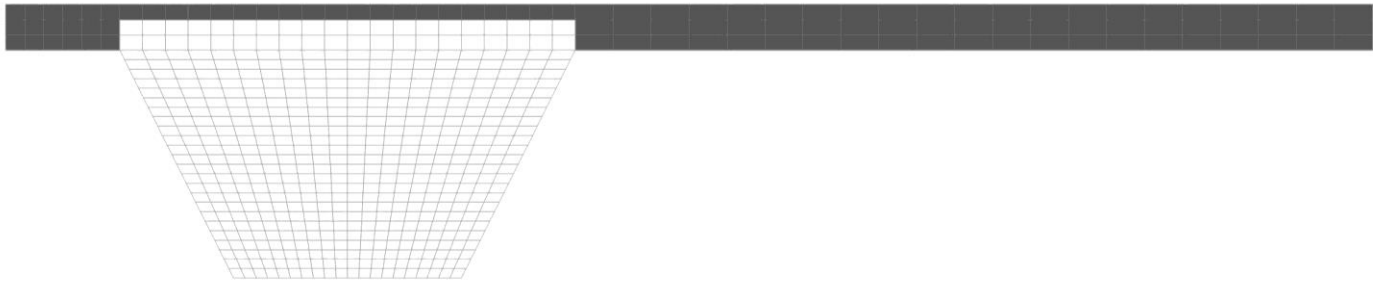
- Illustration using a numerical example



- T=1e-3 m
- Interpolation law: SIMP p=3
- Max volume= 80%
- Design domain = Support only

DIFFICULTIES IN VIBRATION PROBLEMS

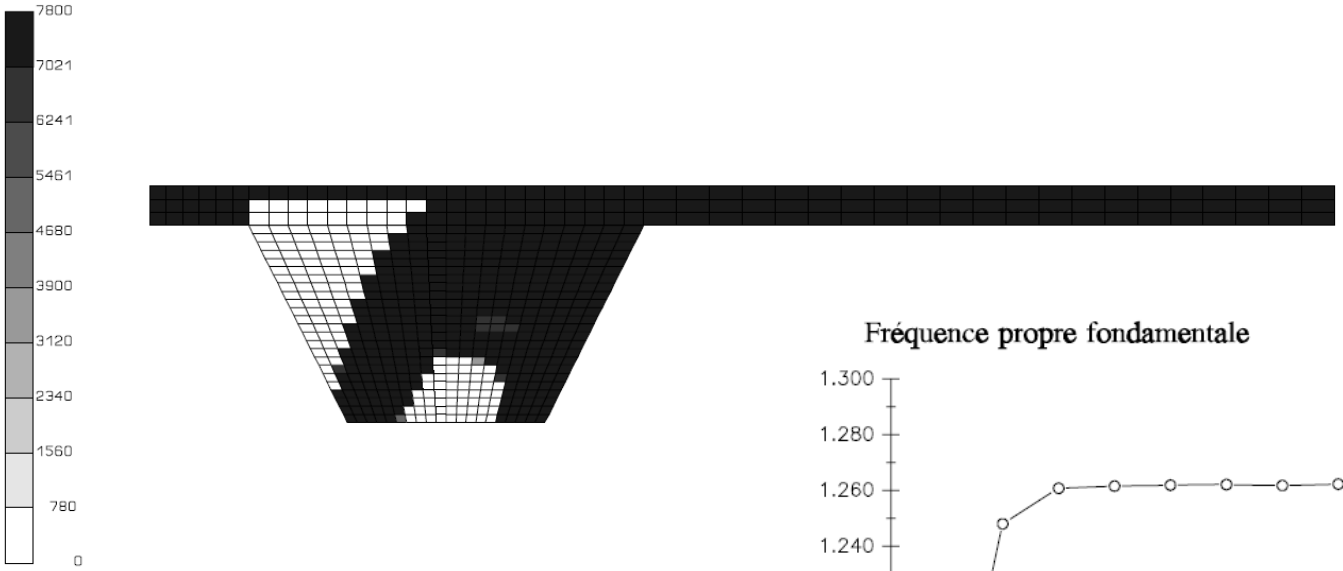
- Illustration using a numerical example

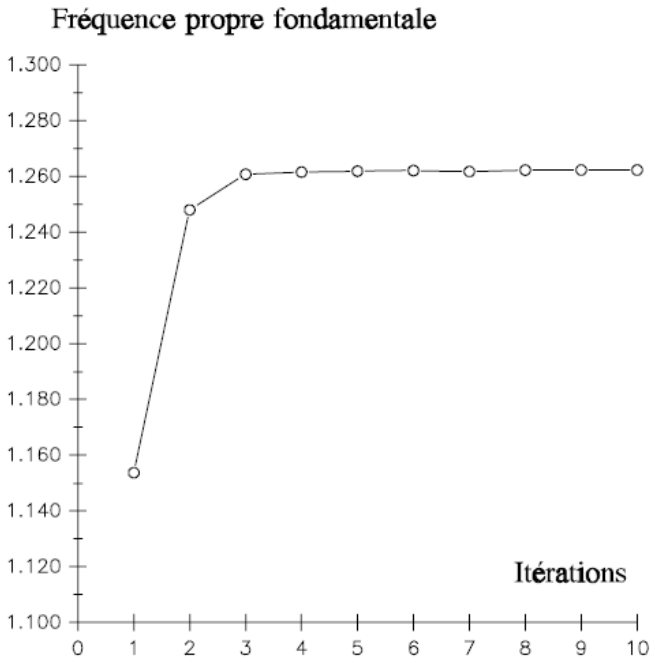


- Material : Steel E=210 Gpa, ν=0.3, ρ=7800 kg/m^3
- T=1e-3 m
- Interpolation law: SIMP p=3
- Max volume= 80%
- Design domain = Support only

DIFFICULTIES IN VIBRATION PROBLEMS

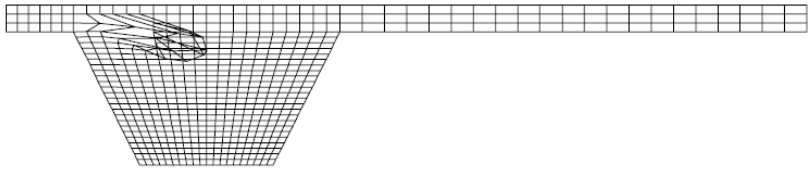
- Convergence curve after 10 iterations



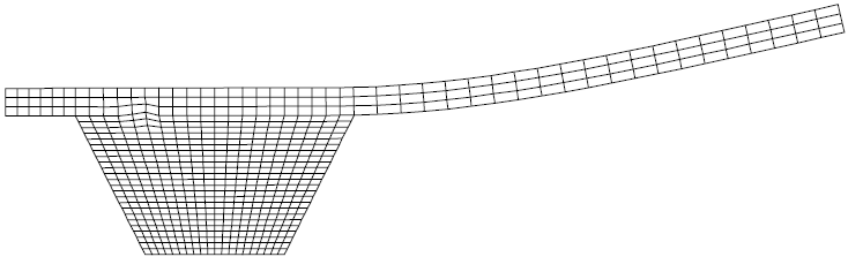


DIFFICULTIES IN VIBRATION PROBLEMS

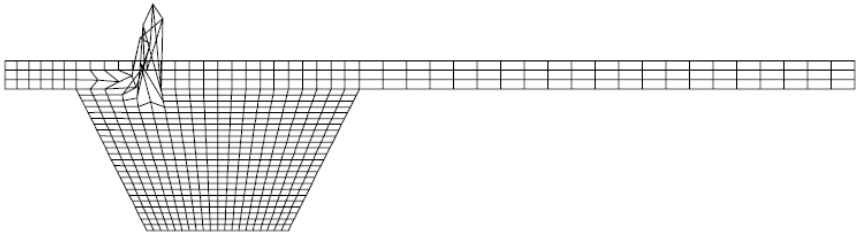
- Convergence broken after 4 iterations when SIMP p=4



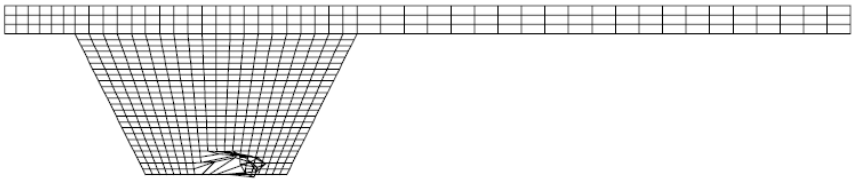
f1 = 0.9802 Hz



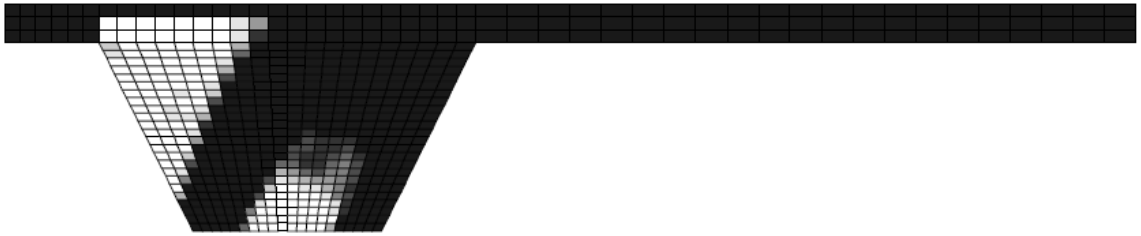
f3 = 1.2618



f2 = 1.2247 Hz



f4 = 1.2984



DIFFICULTIES IN VIBRATION PROBLEMS

- Low frequency modes are present in low density regions

$$\omega^2_{min} = \min_{\mathbf{q}} \frac{\mathbf{q}^T\mathbf{K}\mathbf{q}}{\mathbf{q}^T\mathbf{M}\mathbf{q}} = \frac{x^p\, E_0}{x\, \rho_0} \frac{\mathbf{q}^T\mathbf{K}_0\mathbf{q}}{\mathbf{q}^T\mathbf{M}_0\mathbf{q}}$$

$$= x^{p-1} \frac{E_0}{\rho_0} \frac{\mathbf{q}^T\mathbf{K}_0\mathbf{q}}{\mathbf{q}^T\mathbf{M}_0\mathbf{q}}$$

$$\omega^2_{min} = x^{p-1} \frac{E_0}{\rho_0} \frac{\mathbf{q}^T\mathbf{K}_0\mathbf{q}}{\mathbf{q}^T\mathbf{M}_0\mathbf{q}} \to 0 \quad \text{if} \quad x \to 0$$

- Modify SIMP to give a lower bound in low density to the ratio E/ρ

$$E = \min(E_{min}, x^p\, E_0)$$

  - Modified SIMP

$$E = E_{min} + x^p\, (E_0 - E_{min})$$

DIFFICULTIES IN VIBRATION PROBLEMS

- Ignore / filter eigenmodes which are not significant
  - Selection criteria: generalized mass ➔ local character of the mode

$$\mu^{(k)} = \mathbf{q}^{(k)T}\mathbf{M}\mathbf{q}^{(k)}$$

Mode	Fréquence propre (Hz)	Energie cinétique μ (m² kg)	Rapport à l'énergie cinétique maximale
1	0,9821	1,907E-1	1,3810E-3
2	1,2271	1,3823E-1	9,6902E-4
3	1,2618	1,4265E+2	1,0
4	1,2814	9,8634E-1	6,9144E-4
5	1,3264	1,6051E-1	1,0946E-3