Introduction to SAMCEF MECANO

Outline

- Introduction
- Generalized coordinates
- Kinematic constraints
- Time integration
- Description and parameterization of finite rotations

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**Our Goal**

- **Fixed deformable structures**
- **Articulated rigid systems**

**Finite Element Method**

- **Multibody Dynamics**

How to analyze in a general manner flexible articulated structures

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**Strategic choices**

- Kinematic description
  - Absolute

- Structural behavior
  - Rigid/flexible

- Software architecture

- Motion equations formulation
  - Numerical

- Time integration
  - Implicit

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*Introduction to SAMCEF - MECANO*
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Modelling Step

Finite rotation parameterization

Finite motion description

Measures of deformation and relative motion

Rigid bodies zero deformation

Joints relative motion + external work

Flexible elements elastic strains + internal work

Control system library

Mechanical elements library

- rigid members
- flexible members
- rigid articulations
- flexible articulations
- active members

FE structural substructuring

DAE system equations

- dynamic equilibrium
- kinematic constraints

Post-processing

kinematics internal forces element stresses
The MECANO software

Preprocessing
- topological description (FE)
- mechanical properties
- response specification

Initial Assembly

Static analysis
- equilibrium configuration
- static forces

Kinetostatic analysis
- trajectories
- quasi-static forces

Dynamic analysis
- trajectories
- dynamic forces

Linearized analysis
- eigenvalues
- stability

Preprocessing
- trajectories
- loads and stresses
- animation

The MECANO element library

Some rigid joint elements

Hinge joint

Prismatic joint

Spherical joint

Gear joint

Cylindrical joint

Rack-and-pinion

Kinematic constraints
Introduction to SAMCEF - MECANO

The MECANO element library

- Cam contact
- Wheel + tyre model
- Point-on-plane
- Cable + pulley

Some non-holonomic joints

The basic elastic elements

- Superelement
- Spring-damper-stop
- Nonlinear beam
- Bushing
The MECANO element library

Some nonlinear and control features

Hydraulic actuator

Nonlinear constraint

\[ \phi(u) = 0 \]

Nonlinear feedback

\[ F(u, \dot{u}) \]

Rail-wheel contact
**The MECANO element library**

Flexible slider

**The MECANO element library**

Symbolic element representation
Domains of application

- mechanical engineering
- automotive industry
- sport industry
- aeronautics
- space structures
- biomechanics
- ...

Dynamic car behavior

Adequacy of Mecano
- association of kinematic joints with finite elements
- open software through user element capability

M. Ebalard, J. Mercier (PSA Citroën)
Utilisation du logiciel Mecano pour le comportement routier à PSA

Examples of application
**Landing gear analysis**

Research conducted in framework of ELGAR consortium

**Functional phases**
- deployment/retraction
- ground impact
- rolling
- breaking
- taxiing

**Model components**
- mechanism
- damper (internal hydraulic system)
- tyre: contact, deformation, friction
MEA antenna: result animation

Simulation of Cluster satellite boom deployment
Generalized coordinates for mechanism analysis

- Generalized co-ordinates
  - description of various types
  - the 4-bar mechanism example
- The finite element method for articulated systems
  - kinematics
  - elasto-dynamics
  - differential-algebraic equations of motion
  - numerical integration by Newmark algorithm
  - the double pendulum example

Introduction

- Classical approach: minimal number of DOF.
- Lagrangian coordinates: best suited to robotics
  - open-tree structure + associated graph
  - loop closure conditions
  - lack of generality, difficult for flexible systems
- Cartesian coordinates: large scale simulation packages
  - inherent topology description
  - large sets of differential-algebraic equations (DAE)
  - still difficult for flexible systems
- Nonlinear Finite element coordinates:
  - inherent topology description
  - simplification of kinematic constraints
  - geometrically nonlinear effects naturally included
**The 4-bar mechanism example**

- Simplest planar, closed-loop mechanism
- one single variable: crank angle $\beta$

**Gruebler’s formula**

$m$ : number of joints  
$N$: number of bodies  
$p$ : number of DOF removed by joint $i$

$$F = 3(N - 1) - \sum_{i=1}^{m} p_i$$

$$F = 1$$

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**Minimal coordinates**

Expression in terms of minimal coordinate $\beta$

$$\theta_j = f(\beta) \quad j = 2, \ldots, 4$$

From geometry

$$\cos \theta_2 = \frac{d^2 + s^2 + 2r \ell \cos \beta}{2ds - r^2 - \ell^2}$$

$$\ell - r \cos \beta \cos \theta_3 + r \sin \beta \sin \theta_3 = s - d \cos \theta_2$$

$$\theta_1 = 2\pi - \beta - \theta_2 - \theta_3$$

$$\sin(\theta_3 + \phi) = \frac{c}{\rho}$$
**Minimal coordinates: kinematic transformer**

Input-output relationship

Complex mechanisms as assembly of transformers

**Lagrangian coordinates**

3 generalized coordinates

\[ q^T = [\theta_1 \quad \theta_2 \quad \theta_3] \]

2 closure conditions

\[
\ell_1 \cos \theta_1 + \ell_2 \cos(\theta_1 + \theta_2) + \ell_3 \cos(\theta_1 + \theta_2 + \theta_3) - \ell_4 = 0
\]

\[
\ell_1 \sin \theta_1 + \ell_2 \sin(\theta_1 + \theta_2) + \ell_3 \sin(\theta_1 + \theta_2 + \theta_3) = 0
\]

Equations of motion: 3 differential equations linked by 2 constraints (DAE)
**Cartesian coordinates**

9 generalized coordinates \( \mathbf{q}^T = [x_1 \ y_1 \ \theta_1 \ x_2 \ y_2 \ \theta_2 \ x_3 \ y_3 \ \theta_3] \)

8 kinematic constraints

\[
\begin{align*}
  x_1 - \frac{\ell_1}{2} \cos \theta_1 &= 0 \\
  x_1 + \frac{\ell_1}{2} \cos \theta_1 - x_2 + \frac{\ell_2}{2} \cos \theta_2 &= 0 \\
  x_2 + \frac{\ell_2}{2} \cos \theta_1 - x_2 + \frac{\ell_3}{2} \cos \theta_3 &= 0 \\
  x_3 + \frac{\ell_3}{2} \cos \theta_3 &= \ell_4 \\
  y_1 - \frac{\ell_1}{2} \sin \theta_1 &= 0 \\
  y_1 + \frac{\ell_1}{2} \sin \theta_1 - y_2 + \frac{\ell_2}{2} \sin \theta_2 &= 0 \\
  y_2 + \frac{\ell_2}{2} \sin \theta_1 - y_3 + \frac{\ell_3}{2} \sin \theta_3 &= 0 \\
  y_3 + \frac{\ell_3}{2} \sin \theta_3 &= 0
\end{align*}
\]

Equations of motion: 9 differential equations linked by 8 constraints (DAE)

**Finite Element coordinates**

13 generalized coordinates:
- 12 finite element DOF \( \mathbf{q} \)
- 1 driving DOF \( \beta \)

Strong form of kinematic constraints

- 4 boundary nodal constraints
  \[
  x_1 = 0, \quad y_1 = 0 \\
  x_6 = 0, \quad y_6 = \ell_4
  \]
- 4 assembly nodal constraints
  \[
  x_2 = x_3, \quad y_2 = y_3 \\
  x_4 = x_5, \quad y_4 = y_5
  \]
- 3 zero strain constraints
  \[
  \epsilon_i = \frac{1}{2} \frac{\ell_i^2 - \ell_{i,0}^2}{\ell_{i,0}^2}
  \]
- 1 driving constraint (implicit)
  \[
  y_2 \cos \beta - x_2 \sin \beta = 0
  \]

\[\Phi_D(\mathbf{q}) = 0\]
Finite element description of a mechanical system

$q = \text{DOF at structural level}$
$q_e = \text{DOF at element level}$
$L_e = \text{DOF localization operator(boolean)}$
$\lambda = \text{lagrangian multipliers}$

- Boolean constraints: localization of DOF
- Implicit constraints: algebraic treatment

System topology results implicitly from Boolean assembly and constraint description

$q_e = L_e q$
$\Phi(q_e, \dot{q}_e, t) = 0$

Classification of kinematic constraints

- **Holonomic constraints**
  - ex: prismatic joint
  - Restrict the number of DOF of the system
    $\Phi(q, t) = 0$

- **Non-holonomic constraints**
  - ex: rolling motion
  - Restrict the system behavior
    $\Psi(q, \dot{q}, t) = 0$

- **Unilateral constraints**
  $\Psi(q, \dot{q}, t) \geq 0$
**Classical mechanical pairs**

- All other mechanical pairs are higher pairs

- 3 lower pairs
  - surface contact
  - motion reversibility

Other modes of classification:
- number of DOF (1 for all lower pairs)
- mode of closure

**The algebraic constrained problem**

\[
\begin{align*}
\min_q & \quad \mathcal{F}(q) \\
\text{subject to} & \quad \Phi(q) = 0
\end{align*}
\]

Possible methods of solution
- constraint elimination
- Lagrange multipliers
- penalty function
- augmented Lagrangian
- perturbed Lagrangian

Formulated in terms of

- Residual vector: \( r(q) = \frac{\partial \mathcal{F}}{\partial q} \)
- Jacobian matrix: \( B \) such that \( B_{mj} = \frac{\partial \Phi_m}{\partial q_j} \)
- Hessian matrix: \( H \) such that \( H_{ij} = \frac{\partial^2 \mathcal{F}}{\partial q_i \partial q_j} \)
**Constraint elimination method**

- First variation of constraints
  \[ \delta \Phi = B \delta q = 0 \]

- Partitioning into independent and dependent variables
  \[ B_I \delta q_I + B_D \delta q_D = 0 \]

- Elimination of dependent variables
  \[ \delta q = R \delta q_I \]
  where
  \[ R = \begin{bmatrix} I \\ -B_D^{-1} B_I \end{bmatrix} \]
  reduction matrix

- Projected residual equation
  \[ R^T \frac{\partial F}{\partial q} = R^T r(q) = 0 \]
  • exact verification of constraints
  • complex and computationally expensive

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**Augmented Lagrangian method**

Augmented functional
\[ F_p^*(q, \lambda) = F(q) + (k \lambda^T \Phi + \frac{p}{2} \Phi^T \Phi) \]

Second-order solution method: by Newton-Raphson
\[ q = q^* + \Delta q \]
\[ \lambda = \lambda^* + \Delta \lambda \]

\[ \begin{bmatrix} H + pB^T B & kB^T \\ kB & 0 \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -r^* - B^T(k \lambda^* + p \Phi) \\ k \Phi \end{bmatrix} \]

- provides exact solution regardless of penalty and scaling factors
- quadratic term \( pB^T B \) reinforces positive character of \( H \)
- scaling factor required for pivoting
Constrained dynamic problems

Hamilton's principle

\[ \int_{t_1}^{t_2} \left[ \delta (L - k \lambda^T \Phi - p \Phi^T \Phi ) + \delta W \right] dt = 0 \]

Matrix form of constrained motion equations

\[ M \ddot{q} + B^T (p \Phi + k \lambda) = g(q, \dot{q}, t) \]
\[ \Phi(q, t) = 0 \]

- Differential - Algebraic nature (DAE)
- vanishing of penalty term at equilibrium → exact
- scaling of constraints for numerical conditioning
- multipliers \( \lambda \) = forces needed to close constraints
- all remaining forces (internal, external, complementary inertia) in RHS

Constrained equations of motion

\[ M \ddot{q} + B^T (p \Phi + k \lambda) = g(q, \dot{q}, t) \]
\[ \Phi(q, t) = 0 \]

Methods of solution

- Constraint reduction
- Constraint regularization \( \rightarrow m + n \) ODE + constraint stabilization
- second-order solution \( \rightarrow \) linearization

But: specific problem of numerical stability of time integration

Introduction to SAMCEF - MECANO  Kinematic constraints
**Linearization of motion equations**

\[
\begin{bmatrix}
M & 0 & \Delta q \\
0 & 0 & \Delta \lambda \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta \dot{q} \\
\Delta \dot{\lambda} \\
0
\end{bmatrix}
+ \begin{bmatrix}
C & 0 & \Delta \ddot{q} \\
0 & 0 & \Delta \ddot{\lambda}
\end{bmatrix}
+ \begin{bmatrix}
K & kB^T \\
0 & kB
\end{bmatrix}
\begin{bmatrix}
\Delta q \\
\Delta \lambda
\end{bmatrix}
= \begin{bmatrix}
r^* \\
\Phi
\end{bmatrix} + O(\Delta^2)
\]

Residual vector of equilibrium

\[r = g(q, \dot{q}, t) - M \ddot{q} - B^T (p \Phi + k \lambda)\]

Tangent damping matrix

\[C^t = -\frac{\partial g}{\partial \dot{q}}\]

Tangent stiffness matrix

\[K^t = -\frac{\partial g}{\partial q} + \frac{\partial (M \ddot{q})}{\partial q} + \frac{\partial (B^T (p \Phi + k \lambda))}{\partial q}\]

Approx.

\[K^t \approx \frac{\partial g}{\partial q} + pB^T B + k \frac{\partial (B^T \lambda)}{\partial q}\]

- plays central role for convergence in iteration
- quality of approximation needed depends on penalty factor

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**Integration of motion equations: Implicit Method**

**Motivations**
- filtering of high frequencies (elasticity)
- independence of algorithm stability on time step size
- one-step method  simplicity of software
- experience in structural dynamics.

**Newmark algorithm**

**Application to multibody systems**
- appropriate integration of rotation motion
- stability in presence of constraints (DAE)
**Implicit integration**

Differential - algebraic equations

\[
\begin{align*}
    r(q, \dot{q}, \ddot{q}, \lambda) &= M\ddot{q} + B^T \lambda - g(q, \dot{q}, t) = 0 \\
    \Phi(q, t) &= 0
\end{align*}
\]

Correction of solution at time \( t \)

\[
(q^*, \dot{q}^*, \ddot{q}^*, \lambda^*) \quad \Rightarrow \\
\begin{align*}
    q &= q^* + \Delta q \\
    \dot{q} &= \dot{q}^* + \Delta \dot{q} \\
    \ddot{q} &= \ddot{q}^* + \Delta \ddot{q} \\
    \lambda &= \lambda^* + \Delta \lambda
\end{align*}
\]

Linearized equations of motion

\[
\begin{bmatrix}
    M & 0 & \Delta \ddot{q} \\
    0 & 0 & \Delta \ddot{\lambda}
\end{bmatrix}
+ 
\begin{bmatrix}
    C_T & \Delta \dot{q} \\
    0 & 0
\end{bmatrix}
+ 
\begin{bmatrix}
    K_T & B^T \\
    B & 0
\end{bmatrix}
\begin{bmatrix}
    \Delta q \\
    \Delta \lambda
\end{bmatrix}
= 
\begin{bmatrix}
    r^* \\
    -\Phi^*
\end{bmatrix}
\]

**Newmark formula**

- Interpolation of displacements and velocities

\[
\begin{align*}
    q_{n+1} &= q_n + (1 - \gamma)h\dot{q}_n + \gamma h\ddot{q}_{n+1} \\
    \dot{q}_{n+1} &= q_n + h\dot{q}_n + (\frac{1}{2} - \beta)h^2\ddot{q}_n + \gamma h^2\ddot{q}_{n+1}
\end{align*}
\]

- Free parameters: average constant acceleration

\[
\gamma = \frac{1}{2} \quad \text{and} \quad \beta = \frac{1}{4}
\]

- Introduction of numerical damping

\[
\gamma = \frac{1}{2} + \alpha \quad \text{and} \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2 \quad \alpha > 0
\]
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The double pendulum

- Description by Cartesian coordinates (redundant)
  \[ \mathbf{q}^T = [x_1 \ y_1 \ \theta_1 \ x_2 \ y_2 \ \theta_2] \]

- Kinetic and potential energies
  \[ \mathbf{K} = \frac{1}{2}m_1(x_1^2 + y_1^2) + \frac{1}{2}m_2(x_2^2 + y_2^2) \]
  \[ \mathbf{P} = m_1gy_1 + m_2gy_2 \]

- Kinematic constraints

- Constraint matrix
  \[ B = \begin{bmatrix}
  1 & 0 & -\ell_1 \cos \theta_1 & 0 & 0 \\
  0 & 1 & -\ell_1 \sin \theta_1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  -1 & 0 & 0 & 1 & -\ell_2 \cos \theta_2 \\
  0 & -1 & 0 & 0 & 1 & -\ell_3 \sin \theta_1 \\
  \end{bmatrix} \]

- External force vector and mass matrix are constant
- Tangent stiffness contains only geometric terms

\[ \mathbf{K}_T = \text{diag} \begin{bmatrix} 0 & 0 & \ell_1 \sin \theta_1 \ -\ell_1 \cos \theta_1 & 0 & 0 & \ell_2 \sin \theta_2 \ -\ell_2 \cos \theta_2 \end{bmatrix} \]
Newmark integration: undamped \((\gamma = \frac{1}{2}, \beta = \frac{1}{4})\)

- normal evolution of angular displacements and velocities
- but: numerical instability detected after 2 sec.

Weak instability: generated by kinematic constraints

Numerical damping
Newmark integration with damping

\[ (\gamma = \frac{1}{2} + \alpha, \beta = \frac{1}{4}(\frac{1}{2} + \gamma)^2, \alpha = 0.015) \]

Normal aspect of response over 5 sec. period, but ...

Unacceptable amount of energy loss

Search for better compromise between accuracy and stability
**Kinematics of finite motion**

- Spherical motion
  - Eigenvalue analysis of rotation operator
  - Euler theorem
  - Explicit expressions of rotation operator
  - Expression in terms of linear invariants
  - Exponential map

- General motion of rigid body
- Velocity analysis of spherical motion
- Explicit expression of angular velocities
  - Rotation parameterization
  - Cartesian rotation vector

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**Spherical motion**

Initial position: material configuration $X = [X_1, X_2, X_3]^T$

Current position: material configuration $x = [x_1, x_2, x_3]^T$

Base vectors of initial configuration $[\tilde{e}_1, \tilde{e}_2, \tilde{e}_3]$

Base vectors of current configuration $[\check{e}_1, \check{e}_2, \check{e}_3]$

Spatial coordinates $x = [x_1, x_2, x_3]^T$

Material coordinates $X = [X_1, X_2, X_3]^T$

Linear transformation $x = RX$

Spherical motion such that:
- length of position vector unaffected by pure rotation
- relative angle between 2 directions remain constant

Proper orthogonal $R^T = R^{-1}\det(R) = 1$
**Euler theorem**

If a rigid body undergoes a motion leaving fixed one of its points, then a set of points of the body lying on a line that passes through that point, remains fixed as well.

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**Explicit expressions of rotation operator**

Orthonormality implies 6 constraints

\[ \mathbf{R} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \quad \Rightarrow \quad r_i^T r_j = \delta_{ij} \]

3 independent rotation parameters

\[ \mathbf{R} = \mathbf{R}(\alpha_1, \alpha_2, \alpha_3) \]

Outer product expression

\[ \mathbf{R} = \sum_i e_i E_i^T \]

Inner product expression

\[ \mathbf{R} = \begin{bmatrix} E_1^T e_1 & E_1^T e_2 & E_1^T e_3 \\ E_2^T e_1 & E_2^T e_2 & E_2^T e_3 \\ E_3^T e_1 & E_3^T e_2 & E_3^T e_3 \end{bmatrix} \]
Expression in terms of linear invariants

Rotation operator

\[ R = I \cos \phi + (1 - \cos \phi) \mathbf{n} \mathbf{n}^T + \mathbf{n} \sin \phi \]

Invariants

\[
\begin{align*}
\text{tr}(R) &= \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 \cos \phi \\
\text{vect}(R) &= n \sin \phi 
\end{align*}
\]

Exponential map

• Differentiate with respect to \( \phi \)

\[ x = RX \]

• verify that

\[ \frac{dR}{d\phi} \mathbf{R}^T = \tilde{n} \]

• differential motion

\[ \frac{dx}{d\phi} - \tilde{n}x = 0 \quad \text{with} \quad x(0) = X \]

• integration

\[ x = \exp(\tilde{n} \phi) \cdot X \quad \Rightarrow \quad R = \exp(\tilde{n} \phi) \]
General motion of rigid body

Combination of translation and rotation

\[ \mathbf{x}_P = \mathbf{x}_0 + \boldsymbol{R} \mathbf{x}_P \]

spatial motion of origin
material coordinates of point \( P \)

Velocity analysis of spherical motion

Spatial description

Differentiate

\[ \mathbf{x}_P = \mathbf{R} \mathbf{x}_P \]

with \( \mathbf{R}^T = \mathbf{R}^{-1} \)

spatial velocities

\[ \mathbf{v}_P = \dot{\mathbf{x}}_P = \dot{\mathbf{R}} \mathbf{x}_P \]

material velocities

\[ \mathbf{v}_P = \mathbf{R}^T \mathbf{v}_P \]

skew-symmetry

\[ \dot{\mathbf{R}} \mathbf{R}^T + (\dot{\mathbf{R}} \mathbf{R}^T)^T = 0 \]

Spatial angular velocities

\[ \mathbf{\omega} = \text{vect}(\dot{\mathbf{R}} \mathbf{R}^T) \]

Material angular velocities

\[ \dot{\mathbf{\Omega}} = \text{vect}(\dot{\mathbf{R}} \dot{\mathbf{R}}^T) \]
**Cartesian rotation vector**

defined as  \[ \mathbf{\Psi} = n \phi \]

Rotation operator

Trigonometric form

\[ R = I + \sin \left( \frac{\| \mathbf{\Psi} \|}{\| \mathbf{\Psi} \|^2} \right) \mathbf{\Psi} + \frac{1 - \cos \| \mathbf{\Psi} \|}{\| \mathbf{\Psi} \|^2} \mathbf{\Psi} \mathbf{\Psi} \]

Exponential map

\[ R = \exp(\mathbf{\Psi}) \]

Angular velocities

\[ \Omega = T(\mathbf{\Psi}) \mathbf{\Psi} \]
\[ \mathbf{\omega} = T^T(\mathbf{\Psi}) \mathbf{\Psi} \]

Limit properties

\[ \lim_{\| \mathbf{\Psi} \| \to 0} T(\mathbf{\Psi}) = \lim_{\| \mathbf{\Psi} \| \to 0} R(\mathbf{\Psi}) = I \]

Tangent operator

\[ T(\mathbf{\Psi}) = I + \frac{\cos \| \mathbf{\Psi} \| - 1}{\| \mathbf{\Psi} \|^2} \mathbf{\Psi} + \left( 1 - \sin \| \mathbf{\Psi} \| \right) \frac{\mathbf{\Psi} \mathbf{\Psi}}{\| \mathbf{\Psi} \|^2} \]

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**Finite Motion**