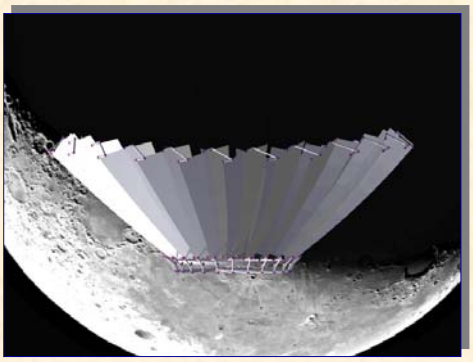
Introduction to SAMCEF MECANO



Outline

- Introduction
- Generalized coordinates
- Kinematic constraints
- Time integration
- Description and paramerization of finite rotations

Acknowledgements
Michel Géradin (ULg)
Doan D. Bao (GoodYear)
Alberto Cardona (Intec, Argentina)

Our Goal

Fixed deformable structures → **Finite Element Method**

Articulated rigid systems → **Multibody Dynamics**

Finite Element Method ? ↔ **Multibody Dynamics**

how to analyze in a general manner flexible articulated structures

Strategic choices

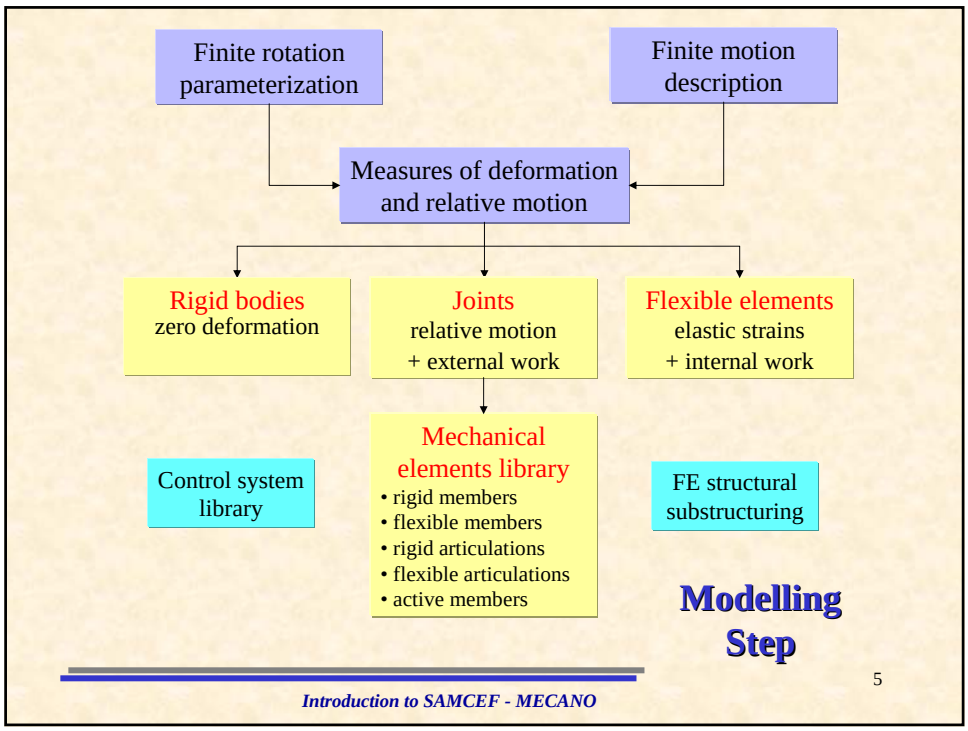
Kinematic description
Absolute

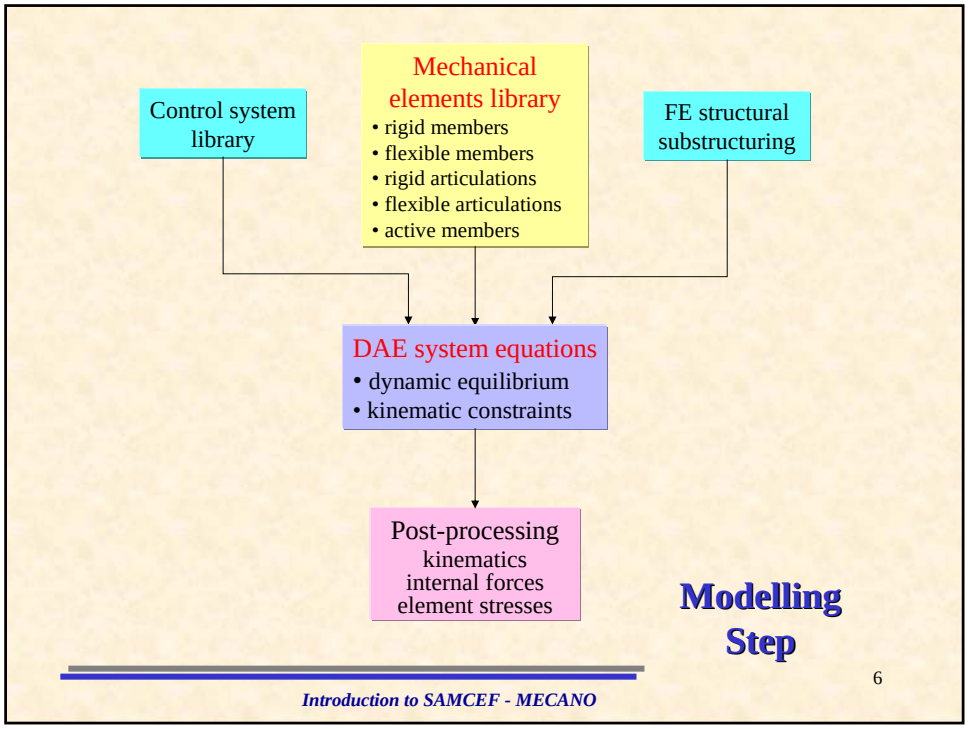
Structural behavior
Rigid/flexible

Software architecture

Motion equations formulation
Numerical

Time integration
Implicit


Finite rotation parameterization
Finite motion description
Measures of deformation and relative motion
Rigid bodies
zero deformation
Joints
relative motion
+ external work
Flexible elements
elastic strains
+ internal work
Mechanical elements library
• rigid members
• flexible members
• rigid articulations
• flexible articulations
• active members
Control system library
FE structural substructuring
Modelling Step
5
Introduction to SAMCEF - MECANO


Control system library
Mechanical elements library
• rigid members
• flexible members
• rigid articulations
• flexible articulations
• active members
FE structural substructuring
DAE system equations
• dynamic equilibrium
• kinematic constraints
Post-processing
kinematics
internal forces
element stresses
Modelling Step
6
Introduction to SAMCEF - MECANO

The MECANO software

Preprocessing
- topological description (FE)
- mechanical properties
- response specification

Initial Assembly

Static analysis
equilibrium configuration
static forces

Kinetostatic analysis
trajectories
quasi-static forces

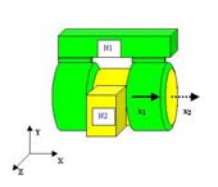
Dynamic analysis
trajectories
dynamic forces

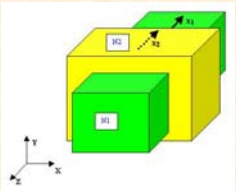
Linearized analysis
eigenvalues
stability

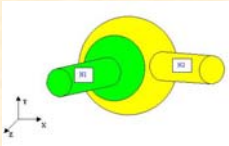
Preprocessing
- trajectories
- loads and stresses
- animation

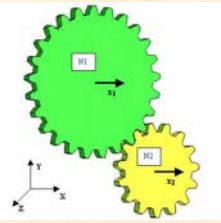
The MECANO element library

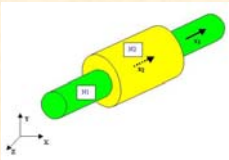
Some rigid joint elements

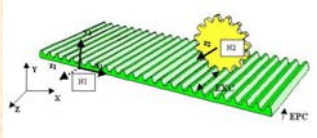

Hinge joint


Prismatic joint


Spherical joint

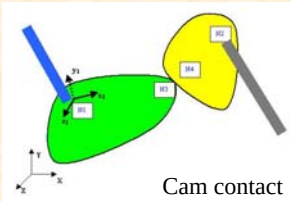

Gear joint


Cylindrical joint

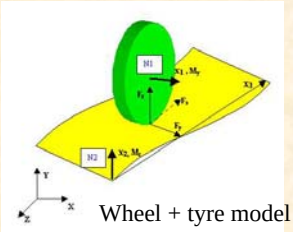

Rack-and-pinion

The MECANO element library

some non-holonomic joints

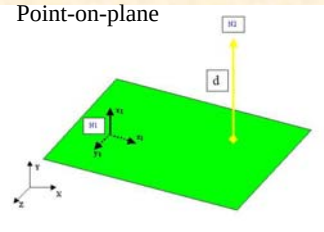


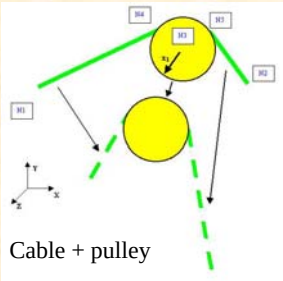
Cam contact



Wheel + tyre model

Point-on-plane

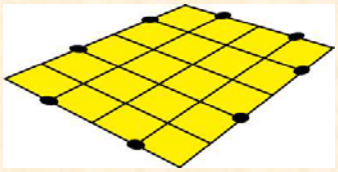




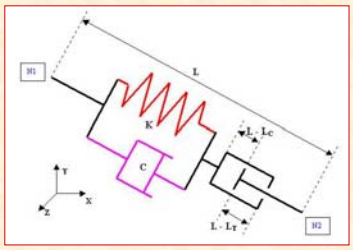
Cable + pulley

The MECANO element library

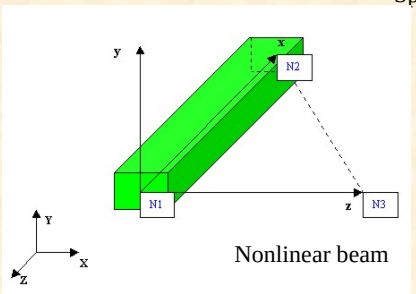
The basic elastic elements



Superelement

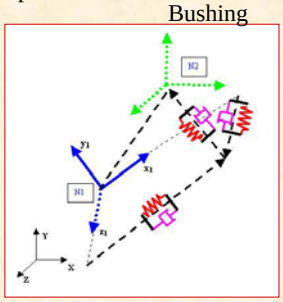


Spring-damper-stop



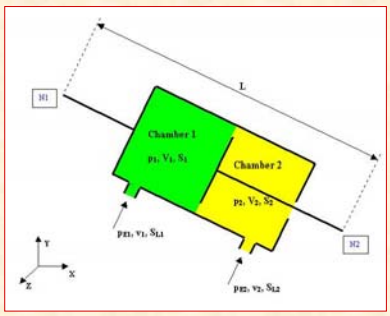
Nonlinear beam

Bushing

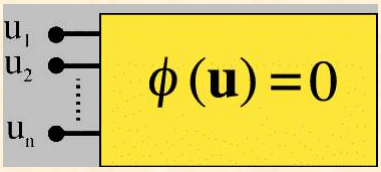


The MECANO element library

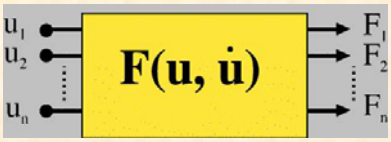
Some nonlinear and control features



Hydraulic actuator

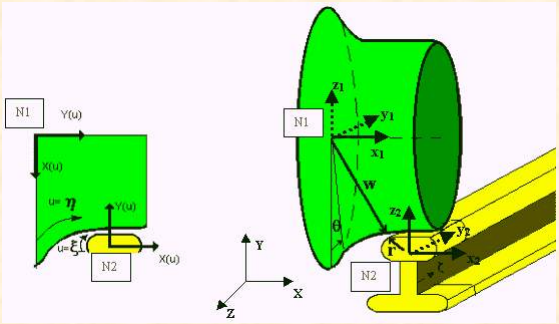


Nonlinear constraint



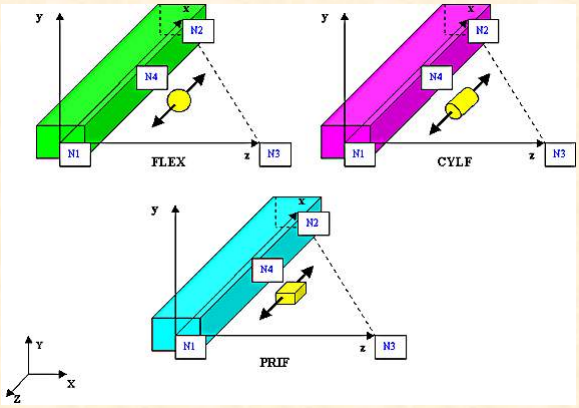
Nonlinear feedback

The MECANO element library



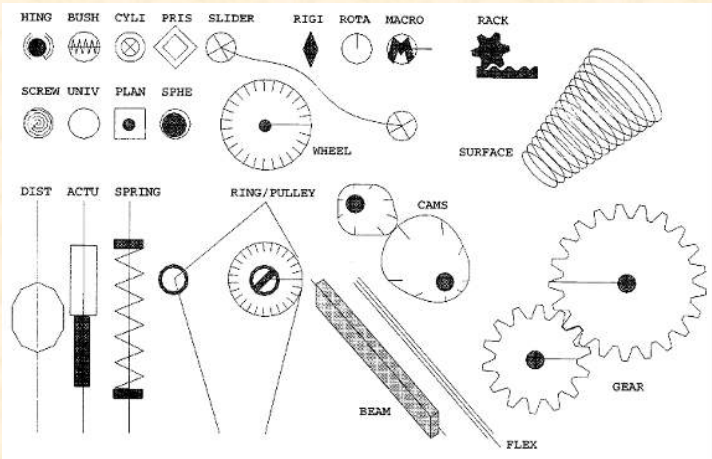
Rail-wheel contact

The MECANO element library



Flexible slider

The MECANO element library



Symbolic element representation

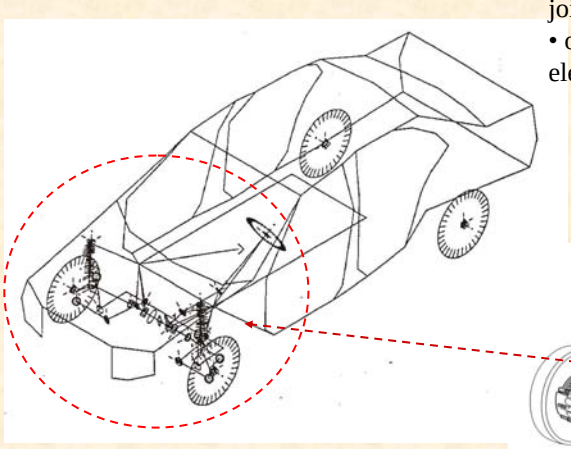
Domains of application

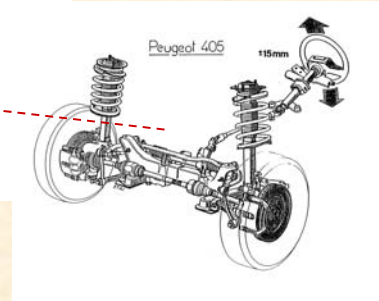
- mechanical engineering
- automotive industry
- sport industry
- aeronautics
- space structures
- biomechanics
- ...

Dynamic car behavior

Adequacy of Mecano

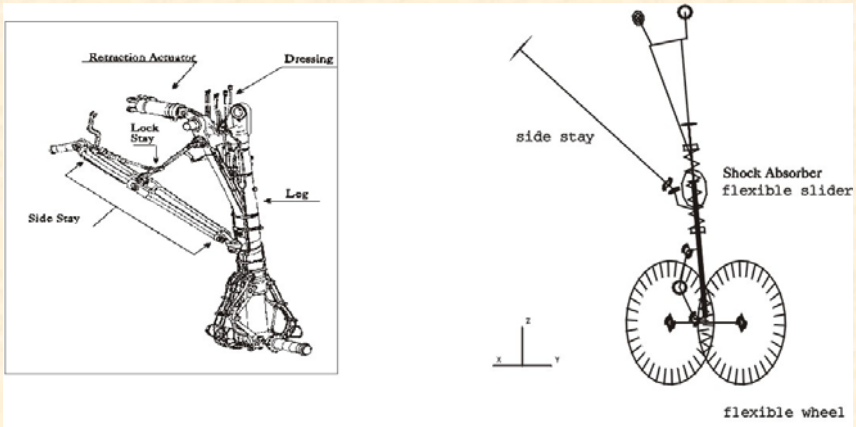
- association of kinematic joints with finite elements
- open software through user element capability





M. Ebalard, J. Mercier (PSA Citroën)
Utilisation du logiciel Mecano pour le comportement routier à PSA

Landing gear analysis



(Dowty Aerospace Landing Gear Ltd. design)

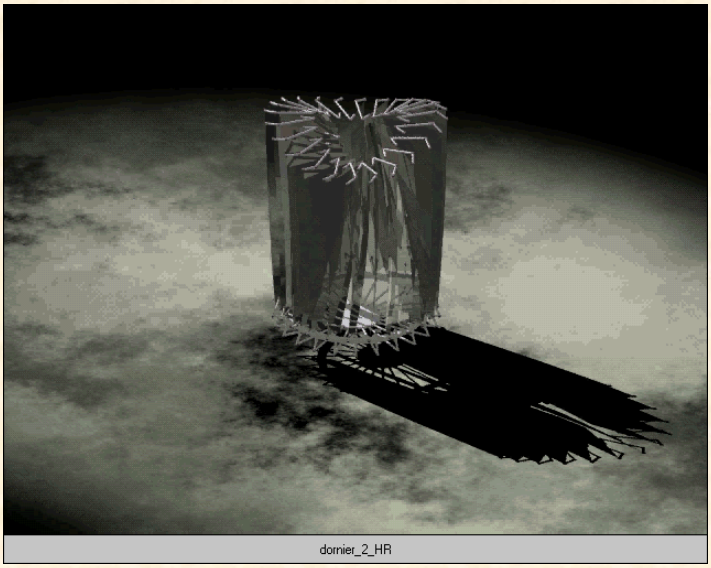
Research conducted in framework of ELGAR consortium

Landing gear analysis

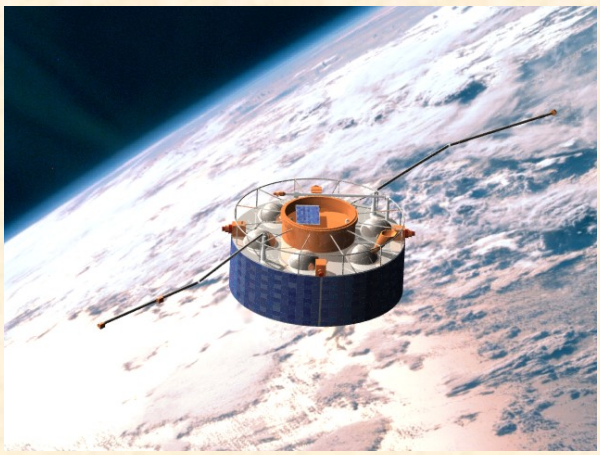
Functional phases
- deployment/retraction
- ground impact
- rolling
- breaking
- taxiing

Model components
- mechanism
- damper (internal hydraulic system)
- tyre: contact, deformation, friction

MEA antenna: result animation



Simulation of Cluster satellite boom deployment



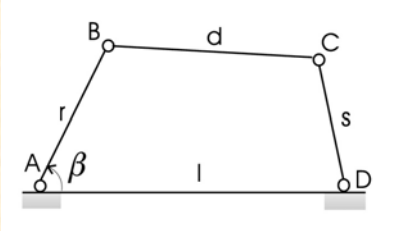
Generalized coordinates for mechanism analysis

- Generalized co-ordinates
  - description of various types
  - the 4-bar mechanism example
- The finite element method for articulated systems
  - kinematics
  - elasto-dynamics
    - differential-algebraic equations of motion
    - numerical integration by Newmark algorithm
    - the double pendulum example

Introduction

- Classical approach: minimal number of DOF.
- *Lagrangian coordinates:* best suited to robotics
  - open-tree structure + associated graph
  - loop closure conditions
  - lack of generality, difficult for flexible systems
- *Cartesian coordinates:* large scale simulation packages
  - inherent topology description
  - large sets of differential-algebraic equations (DAE)
  - still difficult for flexible systems
- *Nonlinear Finite element coordinates:*
  - inherent topology description
  - simplification of kinematic constraints
  - geometrically nonlinear effects naturally included

The 4-bar mechanism example



- Simplest planar, closed-loop mechanism
- one single variable: crank angle β

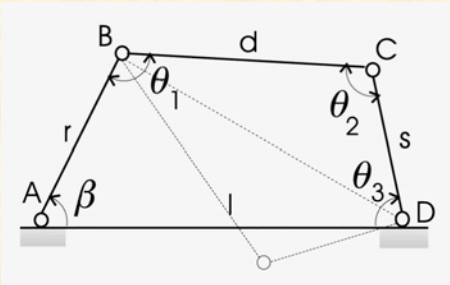
Gruebler's formula

m : number of joints
N: number of bodies
p : number of DOF removed by joint i

$$F = 3(N-1) - \sum_{i=1}^{m} p_i$$

$F = 1$

Minimal coordinates



Expression in terms of minimal coordinate β

$$\theta_j = f(\beta) \qquad j = 2, \ldots 4$$

From geometry

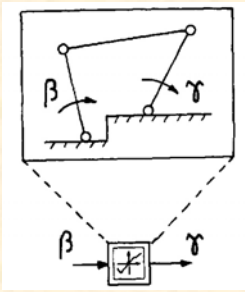
$$\cos\theta_2 = \frac{d^2 + s^2 + 2r\ell\cos\beta}{2ds - r^2 - \ell^2}$$

$$(\ell - r\cos\beta)\cos\theta_3 + r\sin\beta\sin\theta_3 = s - d\cos\theta_2$$

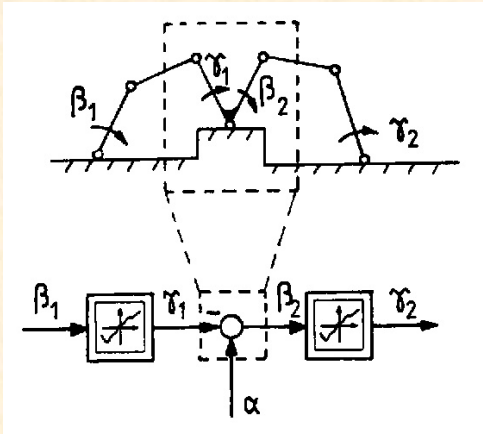
$$\sin(\theta_3 + \phi) = \frac{c}{\rho}$$

$$\theta_1 = 2\pi - \beta - \theta_2 - \theta_3$$

Minimal coordinates: kinematic transformer



Input-output relationship

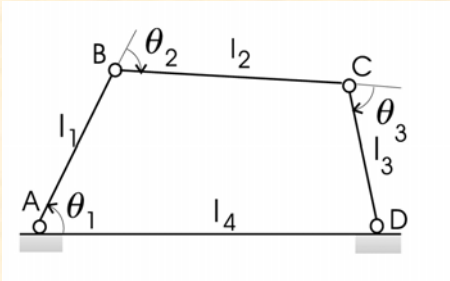


Complex mechanisms as assembly of transformers

Lagrangian coordinates

3 generalized coordinates

$$\boldsymbol{q}^T = [\theta_1 \quad \theta_2 \quad \theta_3]$$



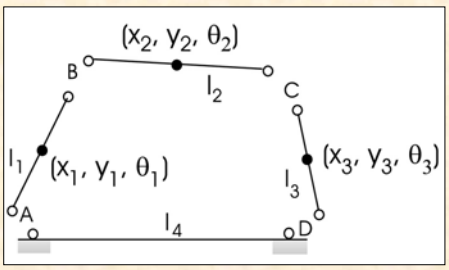
2 closure conditions

$$\ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2) + \ell_3 \cos(\theta_1+\theta_2+\theta_3) - \ell_4 = 0$$
$$\ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1+\theta_2) + \ell_3 \sin(\theta_1+\theta_2+\theta_3) = 0$$

⇨ **Equations of motion: 3 differential equations linked by 2 constraints (DAE)**

Cartesian coordinates

9 generalized coordinates $\boldsymbol{q}^T = [\, x_1 \quad y_1 \quad \theta_1 \quad x_2 \quad y_2 \quad \theta_2 \quad x_3 \quad y_3 \quad \theta_3 \,]$



Equations of motion: 9 differential equations linked by 8 constraints (DAE)

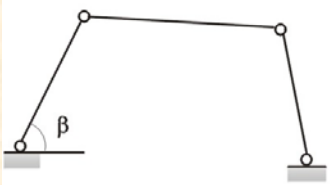
8 kinematic constraints

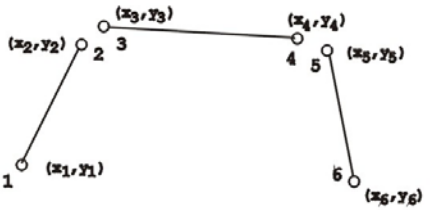
$$
\begin{array}{ll}
x_1 - \frac{\ell_1}{2}\cos\theta_1 = 0 & y_1 - \frac{\ell_1}{2}\sin\theta_1 = 0 \\
x_1 + \frac{\ell_1}{2}\cos\theta_1 - x_2 + \frac{\ell_2}{2}\cos\theta_2 = 0 & y_1 + \frac{\ell_1}{2}\sin\theta_1 - y_2 + \frac{\ell_2}{2}\sin\theta_2 = 0 \\
x_2 + \frac{\ell_2}{2}\cos\theta_1 - x_2 + \frac{\ell_3}{2}\cos\theta_3 = 0 & y_2 + \frac{\ell_2}{2}\sin\theta_2 - y_3 + \frac{\ell_3}{2}\sin\theta_3 = 0 \\
x_3 + \frac{\ell_3}{2}\cos\theta_3 = \ell_4 & y_3 + \frac{\ell_3}{2}\sin\theta_3 = 0
\end{array}
$$

Finite Element coordinates

$\boldsymbol{q} = [\, x_1 \quad y_1 \quad \ldots \quad x_6 \quad y_6 \,]^T$

13 generalized coordinates:
- 12 finite element DOF $\boldsymbol{q}$
- 1 driving DOF $\boldsymbol{\beta}$





Strong form of kinematic constraints

- 4 boundary nodal constraints

$$
\begin{array}{ll}
x_1 = 0, & y_1 = 0 \\
x_6 = 0, & y_6 = \ell_4
\end{array}
$$

- 4 assembly nodal constraints

$$
\begin{array}{ll}
x_2 = x_3 & y_2 = y_3 \\
x_4 = x_5 & y_4 = y_5
\end{array}
$$

- 3 zero strain constraints

$$\epsilon_i = \frac{1}{2}\frac{\ell_i^2 - \ell_{i,0}^2}{\ell_{i,0}^2}$$

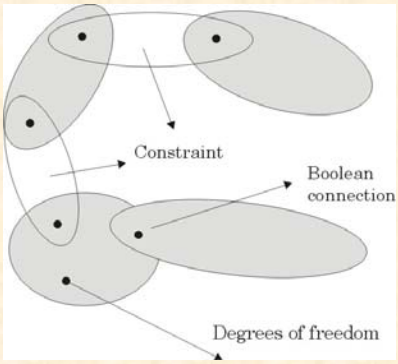
$\Rightarrow$ $\epsilon_1 = 0 \qquad \epsilon_2 = 0 \qquad \epsilon_3 = 0$

- 1 driving constraints (implicit)

$$y_2 \cos\beta - x_2 \sin\beta = 0$$

$\Rightarrow$ $\Phi_D(\boldsymbol{q}) = 0$

Finite element description of a mechanical system



$\boldsymbol{q}$ = DOF at structural level

$\boldsymbol{q}_e$ = DOF at element level

$\boldsymbol{L}_e$ = DOF localization operator(boolean)

$\boldsymbol{\lambda}$ = lagrangian multipliers

- Boolean constraints: localization of DOF
- Implicit constraints: algebraic treatment

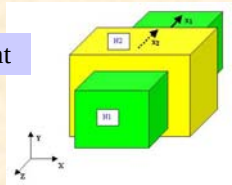
$$\boldsymbol{q}_e = \boldsymbol{L}_e \boldsymbol{q}$$

$$\boldsymbol{\Phi}(\boldsymbol{q}_e, \dot{\boldsymbol{q}}_e, t) = 0$$

System topology results implicitly from Boolean assembly and constraint description

Classification of kinematic constraints

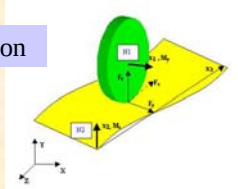
- **Holonomic constraints** ex: prismatic joint



  - Restrict the number of DOF of the system

$$\boldsymbol{\Phi}(\boldsymbol{q}, t) = 0$$

- **Non-holonomic constraints** ex: rolling motion



  - Restrict the system behavior

$$\boldsymbol{\Psi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0$$

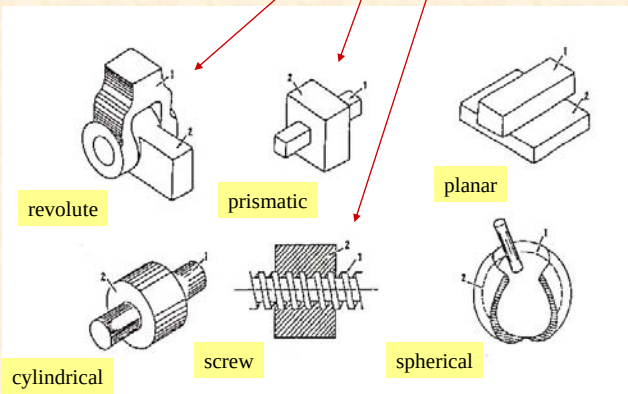
- **Unilateral constraints**

$$\boldsymbol{\Psi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \geq 0$$

Classical mechanical pairs

Only 3 lower pairs
- surface contact
- motion reversibility

- All other mechanical pairs are higher pairs



Other modes of classification:
- number of DOF (1 for all lower pairs)
- mode of closure

The algebraic constrained problem

$$\begin{cases} \min\limits_{q} \mathcal{F}(\boldsymbol{q}) \\ \text{subject to} \\ \boldsymbol{\Phi}(\boldsymbol{q}) = 0 \end{cases}$$

Possible methods of solution
- constraint elimination
- Lagrange multipliers
- penalty function
- augmented Lagrangian
- perturbed Lagrangian

Formulated in terms of

Residual vector

$$\boldsymbol{r}(\boldsymbol{q}) = \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}}$$

Jacobian matrix

$$\boldsymbol{B} \quad \text{such that} \quad B_{mj} = \frac{\partial \Phi_m}{\partial q_j}$$

Hessian matrix

$$\boldsymbol{H} \quad \text{such that} \quad H_{ij} = \frac{\partial^2 \mathcal{F}}{\partial q_i \partial q_j}$$

Constraint elimination method

- First variation of constraints

$$\delta \boldsymbol{\Phi} = \boldsymbol{B}\delta \boldsymbol{q} = 0$$

- Partitioning into independent and dependent variables

$$\boldsymbol{B}_I \delta \boldsymbol{q}_I + \boldsymbol{B}_D \delta \boldsymbol{q}_D = 0$$

- Elimination of dependent variables

$$\delta \boldsymbol{q} = \boldsymbol{R}\delta \boldsymbol{q}_I \quad \Rightarrow \quad \boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{B}_D^{-1}\boldsymbol{B}_I \end{bmatrix}$$

reduction matrix

- Projected residual equation

$$\boldsymbol{R}^T \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}} = \boldsymbol{R}^T \boldsymbol{r}(\boldsymbol{q}) = 0$$

- exact verification of constraints
- complex and computationally expensive

Augmented Lagrangian method

Augmented functional

$$\mathcal{F}_p^*(\boldsymbol{q}, \boldsymbol{\lambda}) = \mathcal{F}(\boldsymbol{q}) + (k\boldsymbol{\lambda}^T \boldsymbol{\Phi} + \frac{p}{2}\boldsymbol{\Phi}^T \boldsymbol{\Phi})$$

p penalty coefficient
k scaling factor

Second-order solution method: by Newton-Raphson

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{q}^\star + \Delta \boldsymbol{q} \\ \boldsymbol{\lambda} &= \boldsymbol{\lambda}^\star + \Delta \boldsymbol{\lambda} \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \boldsymbol{H} + p\boldsymbol{B}^T\boldsymbol{B} & k\boldsymbol{B}^T \\ k\boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{q} \\ \Delta \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{r}^\star - \boldsymbol{B}^T(k\boldsymbol{\lambda}^\star + p\boldsymbol{\Phi}) \\ k\boldsymbol{\Phi} \end{bmatrix}$$

- provides exact solution regardless of penalty and scaling factors
- quadratic term $p\boldsymbol{B}^T\boldsymbol{B}$ reinforces positive character of $\boldsymbol{H}$
- scaling factor required for pivoting

Constrained dynamic problems

Hamilton's principle

$$\int_{t_1}^{t_2} [\delta(\mathcal{L} - k\boldsymbol{\lambda}^T \boldsymbol{\Phi} - p\boldsymbol{\Phi}^T \boldsymbol{\Phi}) + \delta\mathcal{W}]dt = 0$$

Matrix form of constrained motion equations

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{B}^T(p\boldsymbol{\Phi} + k\boldsymbol{\lambda}) = \boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$$
$$\boldsymbol{\Phi}(\boldsymbol{q}, t) = 0$$

- Differential - Algebraic nature (DAE)
- vanishing of penalty term at equilibrium → exact
- scaling of constraints for numerical conditioning
- multipliers $\boldsymbol{\lambda}$ = forces needed to close constraints
- all remaining forces (internal, external, complementary inertia) in RHS

Constrained equations of motion

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{B}^T(p\boldsymbol{\Phi} + k\boldsymbol{\lambda}) = \boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$$
$$\boldsymbol{\Phi}(\boldsymbol{q}, t) = 0$$

Methods of solution

- Constraint reduction
- Constraint regularization → $m+n$ ODE + constraint stabilization
- second-order solution → linearization

But: specific problem of numerical stability of time integration

Linearization of motion equations

$$\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}\ddot{\boldsymbol{q}} \\ \boldsymbol{\Delta}\dot{\boldsymbol{\lambda}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}^t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}\dot{\boldsymbol{q}} \\ \boldsymbol{\Delta}\dot{\boldsymbol{\lambda}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}^t & k\boldsymbol{B}^T \\ 0 & k\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}\boldsymbol{q} \\ \boldsymbol{\Delta}\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^\star \\ -\boldsymbol{\Phi}^\star \end{bmatrix} + O(\Delta^2)$$

Residual vector of equilibrium

$$\boldsymbol{r} = \boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - \boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{B}^T(p\boldsymbol{\Phi} + k\boldsymbol{\lambda})$$

Tangent damping matrix

$$\boldsymbol{C}^t = -\frac{\partial \boldsymbol{g}}{\partial \dot{\boldsymbol{q}}}$$

Tangent stiffness matrix

$$\boldsymbol{K}^t = -\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} + \frac{\partial(\boldsymbol{M}\ddot{\boldsymbol{q}})}{\partial \boldsymbol{q}} + \frac{\partial(\boldsymbol{B}^T(p\boldsymbol{\Phi} + k\boldsymbol{\lambda}))}{\partial \boldsymbol{q}}$$

approx.

- plays central role for convergence in iteration
- quality of approximation needed depends on penalty factor

$$\boldsymbol{K}^t \simeq -\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} + p\boldsymbol{B}^T\boldsymbol{B} + k\frac{\partial(\boldsymbol{B}^T\boldsymbol{\lambda})}{\partial \boldsymbol{q}}$$

Integration of motion equations: Implicit Method

Motivations

filtering of high frequencies (elasticity)

- independence of algorithm stability on time step size
- one-step method → simplicity of software
- experience in structural dynamics.

Newmark algorithm

Application to multibody systems

- appropriate integration of rotation motion
- stability in presence of constraints (DAE)

Implicit integration

Differential - algebraic equations

$$\boldsymbol{r}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{\lambda}) = \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{B}^T\boldsymbol{\lambda} - \boldsymbol{g}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = \boldsymbol{0}$$
$$\boldsymbol{\Phi}(\boldsymbol{q},t) = \boldsymbol{0}$$

Correction of solution at time *t*

$$(\boldsymbol{q}^*,\ \dot{\boldsymbol{q}}^*,\ \ddot{\boldsymbol{q}}^*,\ \boldsymbol{\lambda}^*) \longrightarrow \begin{array}{l} \boldsymbol{q} = \boldsymbol{q}^* + \Delta\boldsymbol{q} \\ \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^* + \Delta\dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}^* + \Delta\ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} = \boldsymbol{\lambda}^* + \Delta\boldsymbol{\lambda} \end{array}$$

Linearized matrices

Tangent damping

$$\boldsymbol{C}_T = \frac{\partial \boldsymbol{r}}{\partial \dot{\boldsymbol{q}}} = -\frac{\partial \boldsymbol{g}}{\partial \dot{\boldsymbol{q}}}$$

Tangent stiffness

$$\boldsymbol{K}_T = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}} = -\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} + \frac{\partial}{\partial \boldsymbol{q}}(\boldsymbol{M}\ddot{\boldsymbol{q}}^* + \boldsymbol{B}^T\boldsymbol{\lambda}^*)$$

Linearized equations of motion

$$\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\ddot{\boldsymbol{q}} \\ \Delta\ddot{\boldsymbol{\lambda}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\dot{\boldsymbol{q}} \\ \Delta\dot{\boldsymbol{\lambda}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_T & \boldsymbol{B}^T \\ \boldsymbol{B} & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{q} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^* \\ -\boldsymbol{\Phi}^* \end{bmatrix}$$

Newmark formula

- Interpolation of displacements and velocities

$$\dot{\boldsymbol{q}}_{n+1} = \dot{\boldsymbol{q}}_n + (1-\gamma)h\ddot{\boldsymbol{q}}_n + \gamma h\ddot{\boldsymbol{q}}_{n+1}$$
$$\boldsymbol{q}_{n+1} = \boldsymbol{q}_n + h\dot{\boldsymbol{q}}_n + (\tfrac{1}{2} - \beta)h^2\ddot{\boldsymbol{q}}_n + +\beta h^2\ddot{\boldsymbol{q}}_{n+1}$$

- Free parameters: average constant acceleration

$$\gamma = \frac{1}{2} \qquad \text{and} \qquad \beta = \frac{1}{4}$$

- Introduction of numerical damping

$$\gamma = \frac{1}{2} + \alpha \qquad \text{and} \qquad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2 \qquad \alpha > 0$$

Integration procedure

- Prediction at time t

$$\ddot{\boldsymbol{q}}^0_{n+1} = 0$$
$$\dot{\boldsymbol{q}}^0_{n+1} = \dot{\boldsymbol{q}}_n + (1-\gamma)h\ddot{\boldsymbol{q}}_n$$
$$\boldsymbol{q}^0_{n+1} = \boldsymbol{q}_n + h\dot{\boldsymbol{q}}_n + (\tfrac{1}{2}-\beta)h^2\ddot{\boldsymbol{q}}_n$$

- Correction

$$\ddot{\boldsymbol{q}}^{k+1}_{n+1} = \ddot{\boldsymbol{q}}^k_{n+1} + \Delta\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}^k_{n+1} + \tfrac{1}{\beta h^2}\Delta\boldsymbol{q}$$
$$\dot{\boldsymbol{q}}^{k+1}_{n+1} = \dot{\boldsymbol{q}}^k_{n+1} + \Delta\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^k_{n+1} + \tfrac{\gamma}{\beta h}\Delta\boldsymbol{q}$$
$$\boldsymbol{q}^k_{n+1} = \boldsymbol{q}^k_{n+1} + \Delta\boldsymbol{q}$$

- Correction equation

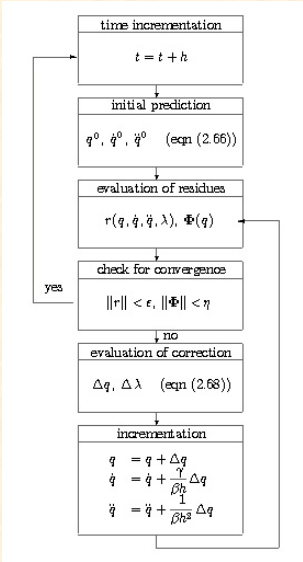
$$\begin{bmatrix} \boldsymbol{S}_T & \boldsymbol{B}^T \\ \boldsymbol{B} & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{q} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^* \\ -\boldsymbol{\Phi}^* \end{bmatrix}$$

Iteration matrix

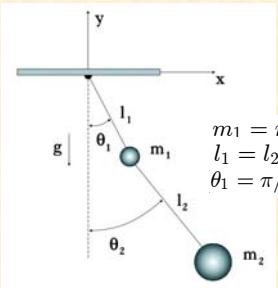
$$\boldsymbol{S}_T = \boldsymbol{K}_T + \frac{\gamma}{\beta h}\boldsymbol{C}_T + \frac{1}{\beta h^2}\boldsymbol{M}$$

- Convergence check

$$\|\boldsymbol{r}\| < \epsilon \quad \text{and} \quad \|\boldsymbol{\Phi}\| < \eta$$



The double pendulum



$m_1 = m_2 = 1.$
$l_1 = l_2 = 1.,$
$\theta_1 = \pi/2,\ \theta_2 = 0$

- Description by Cartesian coordinates (redundant)

$$\boldsymbol{q}^T = [\, x_1 \quad y_1 \quad \theta_1 \quad x_2 \quad y_2 \quad \theta_2 \,]$$

- Kinetic and potential energies

$$\mathcal{K} = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$
$$\mathcal{P} = m_1 g y_1 + m_2 g y_2$$

- Kinematic constraints

$$\boldsymbol{\Phi} = \begin{bmatrix} x_1 - \ell_1 \sin\theta_1 \\ y_1 + \ell_1 \cos\theta_1 \\ x_2 - x_1 - \ell_2 \sin\theta_2 \\ y_2 - y_1 + \ell_2 \cos\theta_2 \end{bmatrix}$$

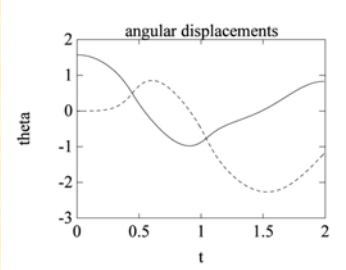
- Constraint matrix

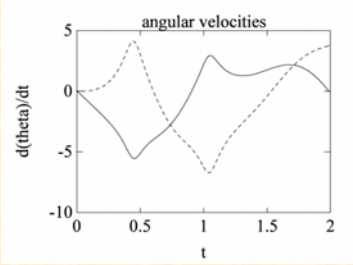
$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & -\ell_1\cos\theta_1 & 0 & 0 & 0 \\ 0 & 1 & -\ell_1\sin\theta_1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -\ell_2\cos\theta_2 \\ 0 & -1 & 0 & 0 & 1 & -\ell_2\sin\theta_2 \end{bmatrix}$$

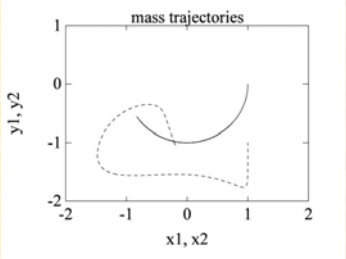
- External force vector and mass matrix are constant
- Tangent stiffness contains only geometric terms

$$\boldsymbol{K}_T = \text{diag}\ [\, 0 \quad 0 \quad \ell_1(\lambda_1\sin\theta_1 - \lambda_2\cos\theta_1) \quad 0 \quad 0 \quad \ell_2(\lambda_3\sin\theta_2 - \lambda_4\cos\theta_2) \,]$$

Newmark integration : undamped $(\gamma = \frac{1}{2},\ \beta = \frac{1}{4})$

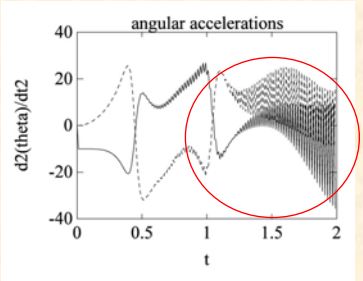


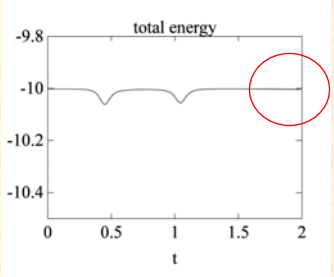


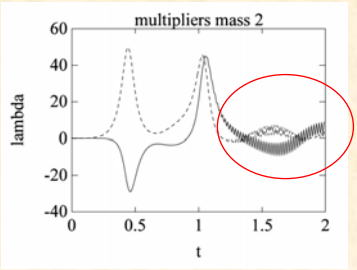


- normal evolution of angular displacements and velocities
- but: numerical instability detected after 2 sec.

Newmark integration : undamped $(\gamma = \frac{1}{2},\ \beta = \frac{1}{4})$







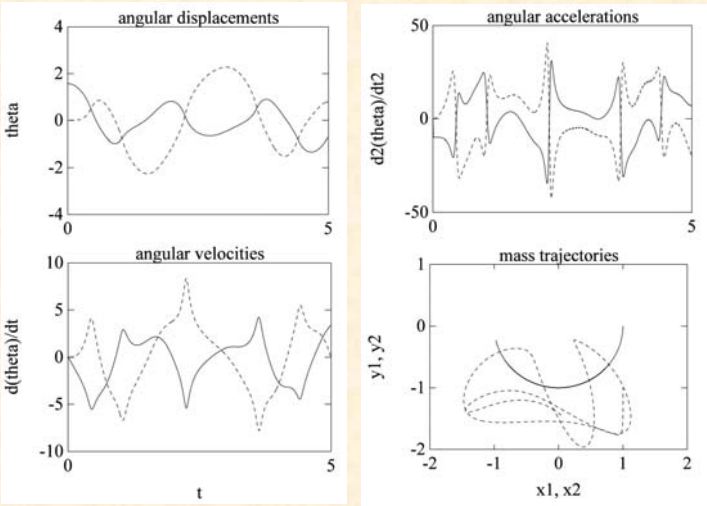
Weak instability: generated by kinematic constraints

⇩

Numerical damping

Newmark integration with damping

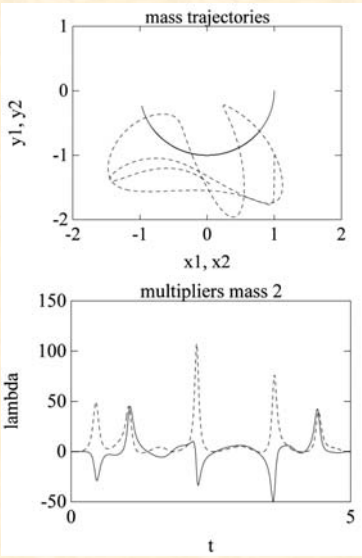
$(\gamma = \frac{1}{2} + \alpha,\ \beta = \frac{1}{4}(\frac{1}{2} + \gamma)^2,\ \alpha = 0.015)$

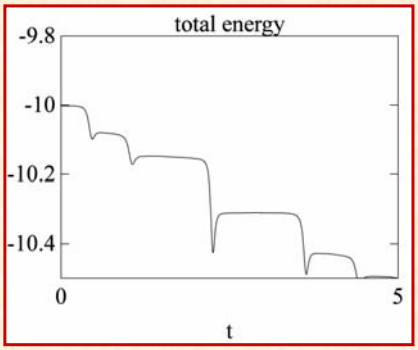


Normal aspect of response over 5 sec. period, but ...

Newmark integration with damping

$(\gamma = \frac{1}{2} + \alpha,\ \beta = \frac{1}{4}(\frac{1}{2} + \gamma)^2,\ \alpha = 0.015)$





Unacceptable amount of energy loss

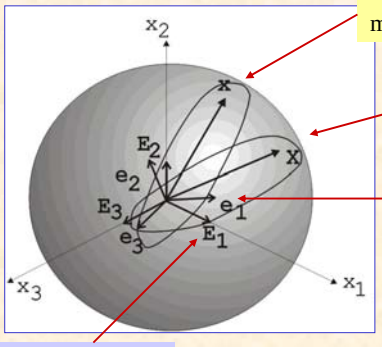
⇓

Search for better compromise between accuracy and stability

Kinematics of finite motion

- Spherical motion
  - Eigenvalue analysis of rotation operator
  - Euler theorem
  - Explicit expressions of rotation operator
  - Expression in terms of linear invariants
  - Exponential map
- General motion of rigid body
- Velocity analysis of spherical motion
- Explicit expression of angular velocities
  - Rotation parameterization
  - Cartesian rotation vector

Spherical motion



Current position: material configuration $\boldsymbol{x}$

Spatial coordinates

$\boldsymbol{x} = [x_1 \ x_2 \ x_3]^T$

Initial position: material configuration $\boldsymbol{X}$

material coordinates

$\boldsymbol{X} = [X_1 \ X_2 \ X_3]^T$

Base vectors of current configuration

$[\vec{\boldsymbol{e}}_1 \ \vec{\boldsymbol{e}}_2 \ \vec{\boldsymbol{e}}_3]$

Base vectors of initial configuration

$[\vec{\boldsymbol{E}}_1 \ \vec{\boldsymbol{E}}_2 \ \vec{\boldsymbol{E}}_3]$

Spherical motion such that:

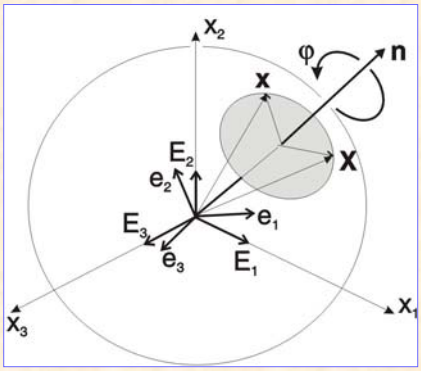
- length of position vector unaffected by pure rotation
- relative angle between 2 directions remain constant

Linear transformation

$$\boldsymbol{x} = \boldsymbol{R}\boldsymbol{X} \quad \text{with} \quad \boldsymbol{R}^T = \boldsymbol{R}^{-1} \quad \det(\boldsymbol{R}) = 1$$

Proper orthogonal

Euler theorem



If a rigid body undergoes a motion leaving fixed one of its points, then a set of points of the body lying on a line that passes through that point, remains fixed as well.

Explicit expressions of rotation operator

Orthonomality implies 6 constraints

$$\boldsymbol{R} = [\,\boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \boldsymbol{r}_3\,] \quad \Rightarrow \quad \boldsymbol{r}_i^T \boldsymbol{r}_j = \delta_{ij}$$

$$\Downarrow$$

3 independent rotation parameters

$$\boldsymbol{R} = \boldsymbol{R}(\alpha_1, \alpha_2, \alpha_3)$$

Outer product expression

$$\boldsymbol{R} = \sum_i \boldsymbol{e}_i \boldsymbol{E}_i^T$$

Inner product expression

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{E}_1^T \boldsymbol{e}_1 & \boldsymbol{E}_1^T \boldsymbol{e}_2 & \boldsymbol{E}_1^T \boldsymbol{e}_3 \\ \boldsymbol{E}_2^T \boldsymbol{e}_1 & \boldsymbol{E}_2^T \boldsymbol{e}_2 & \boldsymbol{E}_2^T \boldsymbol{e}_3 \\ \boldsymbol{E}_3^T \boldsymbol{e}_1 & \boldsymbol{E}_3^T \boldsymbol{e}_2 & \boldsymbol{E}_3^T \boldsymbol{e}_3 \end{bmatrix}$$

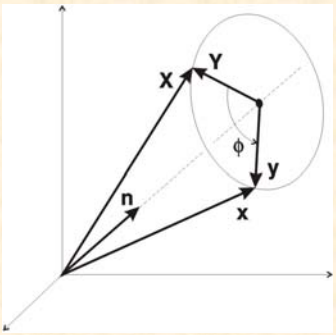
Expression in terms of linear invariants

Rotation operator

$$\boldsymbol{R} = \boldsymbol{I}\cos\phi + (1-\cos\phi)\boldsymbol{n}\boldsymbol{n}^T + \tilde{\boldsymbol{n}}\sin\phi$$

Invariants

$$\begin{aligned} \mathrm{tr}(\boldsymbol{R}) &= \lambda_1 + \lambda_2 + \lambda_3 &&= 1 + 2\cos\phi \\ \mathrm{vect}(\boldsymbol{R}) &&&= \boldsymbol{n}\sin\phi \end{aligned}$$



Exponential map

- Differentiate with respect to ϕ

$$\boxed{\boldsymbol{x} = \boldsymbol{R}\boldsymbol{X}}$$

- verify that

$$\frac{d\boldsymbol{R}}{d\phi}\boldsymbol{R}^T = \tilde{\boldsymbol{n}}$$

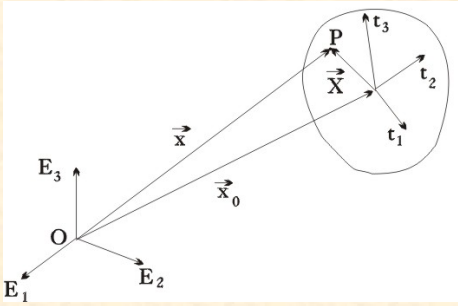
- differential motion

$$\frac{d\boldsymbol{x}}{d\phi} - \tilde{\boldsymbol{n}}\boldsymbol{x} = 0 \qquad \text{with} \qquad \boldsymbol{x}(0) = \boldsymbol{X}$$

- integration

$$\boxed{\boldsymbol{x} = \exp(\tilde{\boldsymbol{n}}\phi)\ \boldsymbol{X}} \quad \Rightarrow \quad \boxed{\boldsymbol{R} = \exp(\tilde{\boldsymbol{n}}\phi)}$$

General motion of rigid body



Combination of translation and rotation

$$\boldsymbol{x}_P = \boldsymbol{x}_0 + \boldsymbol{R}\boldsymbol{X}_P$$

spatial motion of origin

material coordinates of point *P*

Velocity analysis of spherical motion

Spatial description

Differentiate

$\boldsymbol{x}_P = \boldsymbol{R}\boldsymbol{X}_P$ with $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$

spatial velocities

$\boldsymbol{v}_P = \dot{\boldsymbol{x}}_P = \dot{\boldsymbol{R}}\boldsymbol{X}_P$

skew-symmetry

$\dot{\boldsymbol{R}}\boldsymbol{R}^T + (\dot{\boldsymbol{R}}\boldsymbol{R}^T)^T = 0$

$\boldsymbol{v}_P = \tilde{\boldsymbol{\omega}}\boldsymbol{x}_P$ with $\tilde{\boldsymbol{\omega}} = \dot{\boldsymbol{R}}\boldsymbol{R}^T$

spatial angular velocities

$\boldsymbol{\omega} = \mathrm{vect}(\dot{\boldsymbol{R}}\boldsymbol{R}^T)$

Material description

material velocities

$\boldsymbol{V}_P = \boldsymbol{R}^T\boldsymbol{v}_P$

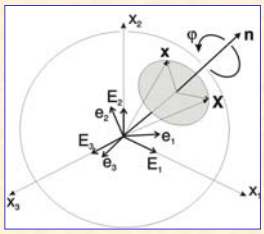
$\boldsymbol{V}_P = \tilde{\boldsymbol{\Omega}}\boldsymbol{X}_P$ with $\tilde{\boldsymbol{\Omega}} = \boldsymbol{R}^T\dot{\boldsymbol{R}}$

material angular velocities

$\boldsymbol{\Omega} = \mathrm{vect}(\boldsymbol{R}^T\dot{\boldsymbol{R}})$

Cartesian rotation vector

defined as $\boldsymbol{\Psi} = \boldsymbol{n}\phi$



Rotation operator

Trigonometric form

$$\boldsymbol{R} = \boldsymbol{I} + \frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}\tilde{\boldsymbol{\Psi}} + \frac{1-\cos\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|^2}\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Psi}}$$

Exponential map

$$\boldsymbol{R} = \exp(\tilde{\boldsymbol{\Psi}})$$

Angular velocities

$$\boldsymbol{\Omega} = \boldsymbol{T}(\boldsymbol{\Psi})\dot{\boldsymbol{\Psi}} \qquad \boldsymbol{\omega} = \boldsymbol{T}^T(\boldsymbol{\Psi})\dot{\boldsymbol{\Psi}}$$

Tangent operator

$$\boldsymbol{T}(\boldsymbol{\Psi}) = \boldsymbol{I} + \left(\frac{\cos\|\boldsymbol{\Psi}\|-1}{\|\boldsymbol{\Psi}\|^2}\right)\tilde{\boldsymbol{\Psi}} + \left(1-\frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}\right)\frac{\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Psi}}}{\|\boldsymbol{\Psi}\|^2}$$

Limit properties

$$\lim_{\|\boldsymbol{\Psi}\|\to 0}\boldsymbol{T}(\boldsymbol{\Psi}) = \lim_{\|\boldsymbol{\Psi}\|\to 0}\boldsymbol{R}(\boldsymbol{\Psi}) = \boldsymbol{I}$$